Higher Level Mathematics

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Introduction

This textbook was not originally designed as a textbook. It was composed from independent classroom materials, worksheets, classtests and presentations that I had prepared for my IB Mathematics Higher Level classes. Later on I added some new chapters. This is why when going through the textbook one may have a strong impression that the chapters are written in diverse styles.

I am still working on improving the content and I introduce minor changes on a daily basis. This is why, if you decide to use some of my concepts, I would like you to email me (krzyssikora@gmail.com) so that I will be able to notify you about major changes and new versions of the textbook. I would be happy to share some more of my materials with you. You will not have to pay anything for that. All I expect from you is that you accept the conditions of a free licence that are explained below.

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The followig colour-coding is used to emphasize the important parts of the textbook's content.

Definition definitions are not always strictly formal
Lemma
Theorem
Corollary to theorems or to simpler observations
Example
solutions of examples
Facts, remarks, observations
Properties
Notation

Part I

INTRODUCTORY TOPICS

contents:

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Chapter 1

Quadratics

1.1 Factorisation

Factorisation of a quadratic expression is expressing it as a product of two linear factors. In simple cases we do it by guessing the factors.

Example 1 (factorisation).
Factorise $x^2 + 5x + 6$.

$1\ x^2 + 5\ x + 6$

$pq = 1 \times 6$

$p + q = 5$

$$\begin{cases} p = 2 \\ q = 3 \end{cases}$$

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Example 2 (factorisation).
Factorise $x^2 - x - 6$.

$1\ x^2 + (-1)\ x + (-6)$

$pq = -6$

$p + q = -1$

$$\begin{cases} p = 2 \\ q = -3 \end{cases}$$

$$x^2 - x - 6 = (x + 2)(x - 3)$$

When the coefficient of x^2 is not equal 1 we can use the following method:

Example 3 (factorisation).
Factorise $2x^2 + 11x + 12$.

$2\ x^2 + 11\ x + 12$

$pq = 24$

$p + q = 11$

$$\begin{cases} p = 3 \\ q = 8 \end{cases}$$

$$\begin{aligned} 2x^2 + \mathbf{11}x + 12 &= 2x^2 + \mathbf{3}x + \mathbf{8}x + 12 = \\ &= (2x^2 + 3x) + (8x + 12) = \\ &= x(2x + 3) + 4(2x + 3) = \\ &= (x + 4)(2x + 3) \end{aligned}$$

Example 4 (factorisation).
Factorise $3x^2 + 2x - 8$.

$3\ x^2 + 2\ x + (-8)$

$pq = -24$

$\rightarrow p + q = 2$

$$\begin{cases} p = -4 \\ q = 6 \end{cases}$$

$$\begin{aligned} 3x^2 + \mathbf{2}x - 8 &= 3x^2 \mathbf{-4}x + \mathbf{6}x - 8 = \\ &= (3x^2 - 4x) + (6x - 8) = \\ &= x(3x - 4) + 2(3x - 4) = \\ &= (x + 2)(3x - 4) \end{aligned}$$

Factorisation of quadratic expressions can be used to solve quadratic equations.

Example 5 (solving quadratic equation by factorisation).
Solve $8x^2 + 10x - 3 = 0$.

$8\ x^2 + 10\ x + (-3)$

$pq = -24$

$\rightarrow p + q = 10$

$p = -2, q = 12$

$$\begin{aligned} 8x^2 + \mathbf{10}x - 3 &= 8x^2 \mathbf{-2}x + \mathbf{12}x - 3 = \\ &= (8x^2 - 2x) + (12x - 3) = \\ &= 2x(4x - 1) + 3(4x - 1) = \\ &= (2x + 3)(4x - 1) \end{aligned}$$

so $8x^2 + 10x - 3 = 0$ can be expressed as:

$$\begin{array}{rcl} 2x + 3 = 0 & \text{or} & 4x - 1 = 0 \\ 2x = -3 & \text{or} & 4x = 1 \\ x = -\frac{3}{2} & \text{or} & x = \frac{1}{4} \end{array}$$

Questions 1.1.

Q1. Solve the equations by factorising.

(1) $x^2 + 10x + 24 = 0$
(2) $x^2 + 8x + 15 = 0$
(3) $x^2 - x - 42 = 0$
(4) $x^2 - 8x + 15 = 0$
(5) $x^2 + 4x - 32 = 0$
(6) $x^2 - 14x - 32 = 0$
(7) $x^2 - 6x + 7 = 0$
(8) $x^2 - 5x = 0$
(9) $3x - 5x^2 = 0$
(10) $64 - 9x^2 = 0$
(11) $4x^2 - 5 = 0$
(12) $2x^2 = 5$
(13) $(2x + 3)^2 - 36 = 0$
(14) $25 - (3 - x)^2 = 0$
(15) $50 - 2(2x - 2)^2 = 0$
(16) $(5x - 1)^2 = 11^2$
(17) $(1 - 2x)^2 + 4 = 0$
(18) $x^4 - 5x^2 + 4 = 0$
(19) $4x^2 + 12x + 9 = 0$
(20) $x^2 + 5x + 6 = 0$
(21) $x^2 - 9x + 14 = 0$
(22) $-x^2 + 3x + 10 = 0$
(23) $x^2 + 3x - 4 = 0$
(24) $2x^2 - 18 = 0$
(25) $3x^2 + 12x + 12 = 0$
(26) $5x^2 - 10x + 5 = 0$
(27) $x^2 + 7x + 10 = 0$
(28) $x^2 - 8x + 15 = 0$
(29) $-x^2 + 4x + 21 = 0$
(30) $x^2 + 3x - 18 = 0$
(31) $3x^2 - 12 = 0$
(32) $2x^2 - 12x + 18 = 0$
(33) $7x^2 + 14x + 7 = 0$

Q2. Solve the equations by factorising.

(1) $2x^2 + 5x + 3 = 0$
(2) $2x^2 - x - 6 = 0$
(3) $2x^2 + 5x - 3 = 0$
(4) $4x^2 + 4x - 3 = 0$
(5) $3x + 5 - 2x^2 = 0$
(6) $2x^2 + 7x + 3 = 0$
(7) $3x^2 - 10x + 3 = 0$
(8) $5x^2 + 17x + 6 = 0$
(9) $2x^2 - 7x + 6 = 0$
(10) $2x^2 + 5x - 3 = 0$
(11) $-3x^2 + x + 2 = 0$
(12) $4x^2 - 20x + 9 = 0$
(13) $6x^2 + 10x - 4 = 0$
(14) $6x^2 + 3x - 3 = 0$
(15) $5x^2 + 80 = 0$
(16) $5x^2 + 8x + 3 = 0$
(17) $3x^2 - 11x + 6 = 0$
(18) $2x^2 - x - 3 = 0$
(19) $-3x^2 + 5x + 2 = 0$
(20) $9x^2 - 15x + 4 = 0$
(21) $6x^2 - 3x - 3 = 0$

(22) $6x^2 + 2x - 4 = 0$
(23) $2x^2 + 50 = 0$
(24) $9x^2 - 10x + 20 = 4(5x + 1)$
(25) $5x(x + 1) = 2(3 - 4x)$
(26) $x(x^2 - 16) = x(2x^2 - 25)$

(27) $(x - 2)(2x^2 - 5x + 7) = (x^2 + 2x - 5)(x - 2)$
(28) $(x + 2)^2(2x - 5) = (2x - 5)(3x^2 + x - 1)$
(29) $(x - 3)(x^2 + 2x) = (3 - x)(3x + 4)$

1.2 Completing the square

Let us firstly expand:

$$\begin{aligned}(x+3)^2 - 2 &= (x^2 + 6x + 9) - 2 = \\ &= x^2 + 6x + 7\end{aligned}$$

the reverse procedure is called "**completing the square**":

$$\begin{aligned}x^2 + 6x + 7 &= (x^2 + 6x + 9) - 2 = \\ &= (x+3)^2 - 2\end{aligned}$$

For completing the sqare we will use the formula:

$$(x \pm a)^2 = x^2 \pm 2ax + a^2$$

Example 6 (completing the square).
Complete the square in $x^2 - 8x + 10$.

$$\begin{aligned}x^2 - 8x + 10 &= x^2 - 2 \times 4x + 10 = \\ &= x^2 - 2 \times 4x + 16 - 6 = \\ &= (x-4)^2 - 6\end{aligned}$$

Example 7 (completing the square).
Complete the square in $x^2 + 5x - 2$.

$$\begin{aligned}x^2 + 5x - 2 &= x^2 + 2 \times 2.5x - 2 = \\ &= x^2 + 2 \times 2.5x + 6.25 - 8.25 = \\ &= (x + 2.5)^2 - 8.25\end{aligned}$$

Example 8 (completing the square).
Complete the square in $2x^2 - 6x + 9$.

$$\begin{aligned}2x^2 - 6x + 9 &= 2\left(x^2 - 3x + \tfrac{9}{2}\right) = \\ &= 2\left(x^2 - 2 \times \tfrac{3}{2}x + \tfrac{9}{2}\right) = \\ &= 2\left(x^2 - 2 \times \tfrac{3}{2}x + \tfrac{9}{4} - \tfrac{9}{4} + \tfrac{9}{2}\right) = \\ &= 2\left(x^2 - 2 \times \tfrac{3}{2}x + \tfrac{9}{4} + \tfrac{9}{4}\right) = \\ &= 2\left(\left(x - \tfrac{3}{2}\right)^2 + \tfrac{9}{4}\right) = \\ &= 2\left(x - \tfrac{3}{2}\right)^2 + \tfrac{9}{2}\end{aligned}$$

Completing the square can be used to solve quadratic equations. It is useful especially when a quadratic expression cannot be easily factorised.

Example 9 (solving quadratic equations by completing the square).
Solve the equation $3x^2 - 18x - 23 = 0$.

$$
\begin{aligned}
3x^2 - 18x - 23 &= 0 \\
3\left(x^2 - 6x - \tfrac{23}{3}\right) &= 0 \\
3\left(x^2 - 2\times 3x + 9 - 9 + \tfrac{23}{3}\right) &= 0 \\
3\left((x-3)^2 - \tfrac{4}{3}\right) &= 0 \\
(x-3)^2 - \tfrac{4}{3} &= 0 \\
(x-3)^2 &= \tfrac{4}{3} \\
x - 3 &= \pm\frac{2}{\sqrt{3}} \\
x &= 3 \pm \tfrac{2}{\sqrt{3}}
\end{aligned}
$$

Questions 1.2.

Q3. Use completing the square to solve the following equations.

(1) $x^2 + 4x - 1 = 0$
(2) $x^2 + 6x + 7 = 0$
(3) $x^2 - 9x + 15 = 0$
(4) $x^2 - 5x + 1.5 = 0$
(5) $3x^2 - 7x + 3 = 0$
(6) $5x^2 + 4x + 2 = 0$
(7) $7x^2 + 16x + 7 = 0$
(8) $2x^2 - 5x + 5 = 0$

1.3 Quadratic function

Quadratic function is defined by one of the following equivalent equations:

Definition 10 (general form of a quadratic function).
A quadratic function is expressed in a **general** form *when it is written as*

$$y = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$.
The discriminant *of the quadratic equation* $ax^2 + bx + c = 0$ *is*

$$\Delta = b^2 - 4ac$$

Definition 11 (vertex form of a quadratic function).
A quadratic function is expressed in a **vertex** form *when it is written as*

$$y = a(x - p)^2 + q,$$

where $a, p, q \in \mathbb{R}$, $a \neq 0$.
The parabola's vertex *is* $V = (-\frac{b}{2a}, -\frac{\Delta}{4a})$
The graph of a quadratic function is called a **parabola**.

Definition 12 (product form of a quadratic function).
A quadratic function is expressed in a product form *when it is written as*

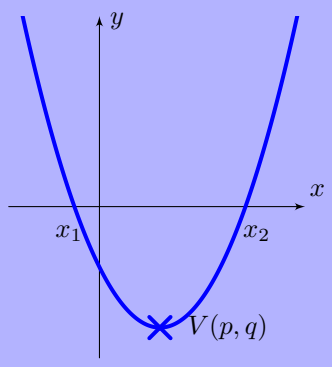
$$y = a(x - x_1)(x - x_2),$$

where $a, x_1, x_2 \in \mathbb{R}$, $a \neq 0$.
The roots *of the quadratic equation* $ax^2 + bx + c = 0$ *are given by:*

$$x_1/x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$$

EXAMPLE 13 (parabola).
The parabola shown in figure below is a graph of a quadratic function defined with equations as in definitions above, where:

- $a > 0$ *(as the parabola is open* upwards*),*
- $\Delta > 0$ *(as it has two distinct zeroes),*
- $b < 0$ *(why? conclude that from the vertex's position)*



QUESTIONS 1.3.

Q4. Solve the equations with use of the quadratic formula.

(1) $x^2 + 4x - 1 = 0$
(2) $x^2 + 6x + 7 = 0$
(3) $x^2 - 9x + 15 = 0$
(4) $x^2 - 5x + 1.5 = 0$
(5) $3x^2 - 7x + 3 = 0$
(6) $5x^2 + 4x + 2 = 0$
(7) $7x^2 + 16x + 7 = 0$
(8) $2x^2 - 5x + 5 = 0$

Q5. Solve the equations.

(1) $x^4 - x^2 - 12 = 0$
(2) $x^4 - 2x^2 - 24 = 0$
(3) $2x^4 + 13x^2 - 45 = 0$
(4) $3x^4 - 39x^2 + 108 = 0$
(5) $6x^4 - 5x^2 + 1 = 0$
(6) $3x^4 - 5x^2 - 2 = 0$
(7) $x^6 - 7x^3 - 8 = 0$
(8) $x^8 - 15x^4 - 16 = 0$

Q6. Find the points of intersection of parabola $y = x^2 - 2x - 4$ and line $y = x$.

Q7. Find the points of intersection of parabola $y = x^2 + 4x - 1$ and line $y = -2x - 6$.

Q8. Find the points of intersection of parabola $y = -x^2 + 4x - 2$ and line $y = 2 - x$.

Q9. Find the points of intersection of parabola $y = \frac{1}{2}x^2 + x - 3$ and line $y = -\frac{1}{2}x - 1$.

Q10. Find the points of intersection of the curves $y = x^2 - 4$ and $y = \frac{1}{2}x^2 - x$.

Q11. Find the points of intersection of the curves $y = -\frac{1}{2}x^2 - 2x + 3$ and $y = x^2 - 2x - 3$.

Q12. Find the points of intersection of the curves $y = -\frac{1}{2}x^2 - x + 1$ and $y = \frac{1}{4}x^2 - x - 2$.

Q13. A parabola intersects the coordinate axes at $(-4, 0)$,$(2, 0)$ and $(0, 4)$. Find its equation in the form $y = ax^2 + bx + c$.

Q14. A parabola has a vertex at $(-2, -3)$ and passes through $(0, 4)$. Find its equation in the form $y = ax^2+bx+c$.

Q15. For what values of m a parabola defined by $y = \frac{1}{2}x^2 - mx + 2m$ has exactly one point in common with the $x-$axes?

Q16. For what values of m a graph of $y = (m - 1)x^2 + 3mx + 4m + 1$ intersects the $x-$axes only once?

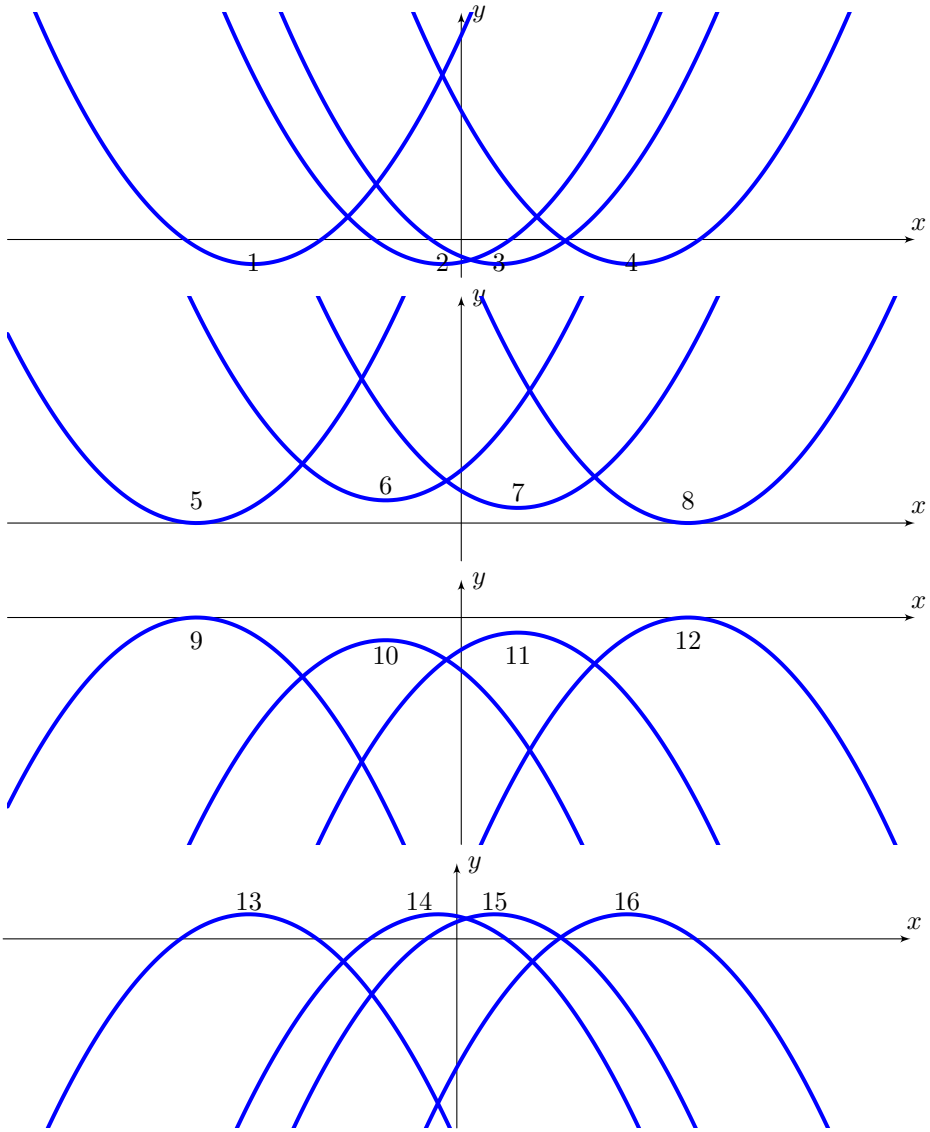
Q17. For what values of m the inequality $x^2 - (m + 2)x + (m + 3) > 0$ is true for all values of x?

Q18. The range of the function $y = 2x^2 - 6x + c$ is $[-\frac{3}{2}, +\infty[$. Find the value of c and the zeroes of the function.

Q19. Complete the table.

	GENERAL form $(y = ax^2 + bx + c)$	CANONICAL form (vertex form) $(y = a(x-p)^2 + q)$	FACTOR form $(y = a(x - x_1)(x - x_2))$	zeroes x_1 and x_2	vertex (p, q)	discriminant $\Delta = b^2 - 4ac$
(1)	$y = x^2 - 2x - 15$					
(2)			$y = 2(x-1)(x-5)$			
(3)		$y = 2(x + \frac{5}{4})^2 - \frac{49}{8}$				
(4)	$y = x^2 ...$				$(-2, -3)$	
(5)	$y = -2x^2 ...$			-2 and -7		
(6)	$y = -4x^2 + 28x - 49$					
(7)	$y = -\frac{1}{2}x^2 - 2x - 3$					
(8)		$y = -4(x-1)^2 + 2$				
(9)	$y = -2x^2 + 2x\sqrt{2} + 7$					

Q20. Which of the graphs **1-16** fits each of the equations below? Why?
(consider coefficient of x^2, determinant, $x-$coordinate of a vertex, $y-$intercept).



(1) $y = 2x^2 - 5x - 4$
(2) $y = -3x^2 + 9x - 4$
(3) $y = 4x^2 - 28x + 49$
(4) $y = -5x^2 + 3x - 7$
(5) $y = 7x^2 + 9x + 4$
(6) $y = 2x^2 + 9x - 3$
(7) $y = 3x^2 - 2x + 5$
(8) $y = -2x^2 - 3x + 4$
(9) $y = -3x^2 - x - 2$
(10) $y = -x^2 + 6x - 9$

Q21. Sketch the graphs of the following functions showing clearly:

- x-intercepts,
- y-intercept,
- axis of symmetry,
- coordinates of a vertex.

(1) $y = x^2 - 2x - 3$
(2) $y = \frac{1}{2}x^2 - \frac{1}{2}x - \frac{15}{8}$
(3) $y = \frac{x^2-x-6}{2}$
(4) $y = -\frac{1}{2}x^2 - x + 4$
(5) $y = 2x^2 - 6x + 4$
(6) $y = 4x^2 + 4x - 3$

1.4 Quadratic inequalities

Questions 1.4.

Q22. Solve the inequalities.

(1) $20 + x^2 < 0$
(2) $x^2 + 2x > 0$
(3) $x^2 + 3 \geq 0$
(4) $x^2 - 2x - 3 > 0$
(5) $6x - 5 - x^2 \geq 0$
(6) $x^2 + 10x + 24 < 0$
(7) $x^2 - 2x - 24 \leq 0$
(8) $-(x + 3)(2x - 3) < 0$
(9) $(2x + 5)(3x - 2) \geq 0$
(10) $(3 - x)(5 - 3x) \leq 0$
(11) $-(2 + x)(4 + x) \leq 0$
(12) $-(5 - x)(1 - 3x) > 0$

Q23. Solve the inequalities.

(1) $(x - 1)(x + 2) > 0$
(2) $-(x + 1)(x + 4) \geq 0$
(3) $(2x - 3)(3x + 2) < 0$
(4) $(2x - 5)(3x - 3) > 0$
(5) $x^2 + 3x \leq 0$
(6) $5x - x^2 < 0$
(7) $(2 - x)(3 + x) \geq 0$
(8) $(2 - x)(3 - x) \geq 0$
(9) $(x + 3)(x + 3) > 0$
(10) $(2x - 3)(2x - 3) \geq 0$
(11) $3(x + 2)^2 < 0$
(12) $(1 - x)^2 > 0$
(13) $-2(x - \frac{1}{2})(2x - 1) \geq 0$
(14) $x(x + 1) + x(2x - 3) < 0$
(15) $(x - 1)(2x + 3) - (x - 1)(x + 1) > 0$
(16) $(x + 2)(x - 3) + (x + 2)(x + 3) \leq 0$
(17) $(2x - 3)(x - 1) - (2x - 3)(2x + 1) < 0$
(18) $(x - 3)(3x - 2) \geq (x - 3)(x - 1)$
(19) $(3 - 2x)(1 - x) > (3 - 2x)(x - 2)$
(20) $(5x - 2)(x + 1) < (5x - 2)(2x + 1)$

1.5 Vieta's formulae

Theorem 14 (Vieta's formulae for quadratics).
For a function $y = ax^2 + bx + c$ with zeroes $x_1/x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$

1. ***SUM** of zeroes is $x_1 + x_2 = -\frac{b}{a}$*
2. ***PRODUCT** of zeroes is $x_1 x_2 = \frac{c}{a}$*

Questions 1.5.

Q24. Fill in the table with "+", "−" or "?".

	a	b	c	$\frac{c}{a}$	$-\frac{b}{a}$	x_1	x_2
1.	+					+	+
2.	+					+	−
3.	+					−	+
4.	+					−	−
5.	−					+	+
6.	−					+	−
7.	−					−	+
8.	−					−	−

Q25. For each of the following equations state whether their roots are:

- both positive,
- both negative,
- of different signs.

Do not evaluate the roots.

(1) $x^2 - 13x - 40 = 0$
(2) $x^2 - 13x + 40 = 0$
(3) $x^2 + 17x - 60 = 0$
(4) $x^2 - x - 140 = 0$
(5) $x^2 + 111x + 54 = 0$
(6) $x^2 + 3x - 204 = 0$
(7) $-3x^2 - 31x + 14 = 0$
(8) $-2x^2 + 29x - 64 = 0$
(9) $-x^2 - 13x + 40 = 0$
(10) $-5x^2 + 21x + 12 = 0$
(11) $x^2 - 11x - 5 = 0$
(12) $-4x^2 + 19x - 11 = 0$

Q26. For what values of m the equation has two distinct, positive roots?

$$x^2 + mx + 9 = 0$$

Q27. For what values of m the equation has two distinct, negative roots?

$$x^2 + (m+1)x + m = 0$$

Q28. For what values of m the equation has two distinct roots of opposite signs?

$$(m-3)x^2 + (3-m)x + 1 = 0$$

Q29. State what are the signs of the solutions of the following equations without solving them.

(1) $y = 13x^2 - 21x - 18$
(2) $y = 23x^2 - 42x + 16$
(3) $y = -17x^2 - 23x + 15$
(4) $y = -19x^2 + 31x - 11$
(5) $y = 21x^2 + 29x + 8$
(6) $y = -13x^2 - 338 - 2132$

Q30. The equation $3x^2 - 5x - 6 = 0$ has the roots α and β. Without evaluating the roots find the value of:

(1) $\alpha + \beta$
(2) $\alpha\beta$
(3) $\alpha^2 + \beta^2$
(4) $\alpha^3 + \beta^3$
(5) $\frac{1}{\alpha} + \frac{1}{\beta}$
(6) $|\alpha - \beta|$
(7) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
(8) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$
(9) $|\alpha^2 - \beta^2|$

Q31. The equation $5x^2 + 8x + 2 = 0$ has the roots α and β. Without evaluating the roots find the value of:

(1) $\alpha + \beta$
(2) $\alpha\beta$
(3) $\alpha^2 + \beta^2$
(4) $\alpha^3 + \beta^3$
(5) $\frac{1}{\alpha} + \frac{1}{\beta}$
(6) $|\alpha - \beta|$
(7) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
(8) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$
(9) $|\alpha^2 - \beta^2|$

Q32. The equation $4x^2 - 7x - 8 = 0$ has the roots α and β. Without evaluating the roots find a quadratic equation with integer coefficients whose roots are:

(1) $-\alpha$ and $-\beta$
(2) 2α and 2β
(3) α^2 and β^2
(4) $\frac{1}{2}\alpha^2$ and $\frac{1}{2}\beta^2$
(5) α^3 and β^3
(6) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$
(7) $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$
(8) $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$
(9) $\alpha - \frac{2}{\beta}$ and $\beta - \frac{2}{\alpha}$

Q33. The equation $-5x^2 + 3x + 9 = 0$ has the roots α and β. Without evaluating the roots find a quadratic equation with integer coefficients whose roots are:

(1) $-\alpha$ and $-\beta$
(2) 2α and 2β
(3) α^2 and β^2
(4) $\frac{1}{2}\alpha^2$ and $\frac{1}{2}\beta^2$
(5) α^3 and β^3
(6) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$
(7) $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$
(8) $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$
(9) $\alpha - \frac{2}{\beta}$ and $\beta - \frac{2}{\alpha}$

1.6 Applications of quadratics

Questions 1.6.

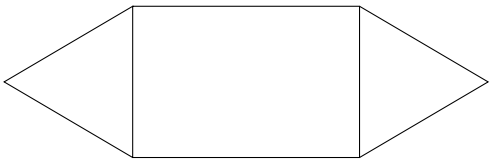
Q34. Prove that among the rectangles with perimeter 40 the one with largest area is a square.

Q35. The longest side of a right angled triangle is $2cm$ longer than one of the shorter sides and $16cm$ longer than the other. Find the lengths of the sides.

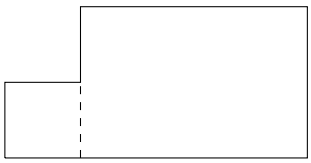
Q36. The length of the room is 2 metres larger than its width. Knowing that the area of the room is $48m^2$ find the rooms dimensions.

Q37. The plot of land is in the shape of a rectangle. Its one side is adjacent to a river while along the other three a fence is built. Knowing that the lengths of the fence is $60m$, find the dimensions of the plot that has the greatest area.

Q38. The plot of land is in the shape of a rectangle finished at two ends with equilateral triangles (see diagram). Knowing that the perimeter of the plot is $80m$, find the dimensions of the rectangle for which the area of the plot is the greatest. Give your answer correct to $10cm$.



Q39. The plot of land is in the shape of a rectangle with a square attached to one of rectangles sides (see diagram). Length of side of the square is equal to a half of length of side of the rectangle to which the square is attached. Knowing that the perimeter of the plot is $60m$, find the dimensions of the rectangle for which the area of the plot is the greatest.



Q40. A ball is thrown from the 5^{th} floor upwards. Its height in metres above the ground level is given by the formula $h(t) = \frac{1}{3}(-4t^2 + 16t + 48)$, where $t \geq 0$ is time given in seconds.

a) What is the initial height of the ball?

b) When does the ball reach the ground level?

c) When is the ball's height largest?

d) What is the largest height of the ball?

Q41. One side of a rectangle is $2cm$ longer than the other. The area of the rectangle is $35cm^2$. Form and solve a quadratic equation to find the lengths of the sides.

Q42. In a trapezium one of its parallel sides is of the same length as height. The other is $6cm$ shorter. The area of the trapezium is $88cm^2$. Form and solve a quadratic equation to find the lengths of the parallel sides.

Q43. The area of a rectangle is $91cm^2$ and its perimeter is $40cm$. Form and solve a quadratic equation to find the lengths of the sides.

Q44. The area of a right-angled triangle is $60cm^2$. The length of the hypotenuse is $17cm$. Form and solve an equation to find the lengths of the perpendicular sides.

Q45. Show that the line $y = 3x - 4$ is tangent to the parabola $y = \frac{1}{3}x^2 + x - 1$.

Q46. Show that the line $x-3y-10 = 0$ is normal (i.e. perpendicular to tangent) to the parabola $y = -\frac{1}{2}x^2+x+2$.

Q47. A parabola is defined with the equation $y = x^2 + 3x - 2$. Find the coordinates of the point of intersection of the tangent to the parabola at its y-intercept with the x-axis.

Q48. A parabola is defined with the equation $y = -x^2 + 3x + 2$. The normal to the parabola at point $(2, 3)$ intersects the parabola again at point P. Find the coordinates of P.

1.6 Summary of key points

$y = ax^2 + bx + c$	general form discriminant: $\Delta = b^2 - 4ac$
	$\Delta > 0$: 2 different real roots $\Delta = 0$: 2 equal real roots (1 root) $\Delta < 0$: no real roots (2 complex roots)
$y = a(x - p)^2 + q$	vertex form where parabola's vertex is $V = \left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$
$y = a(x - x_1)(x - x_2)$	product form $x_1/x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$

Chapter 2

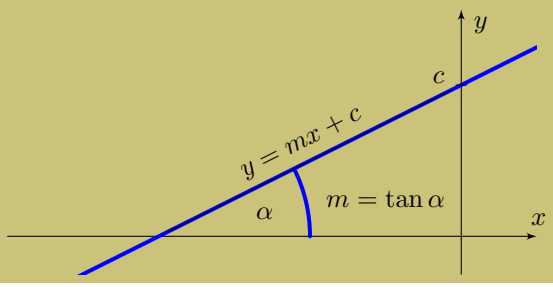
Lines, circles and vectors

2.1 Equations of lines

Definition 15 (slope-intercept equation of a straight line).
A **slope-intercept** *form of an equation of a line is*

$$y = mx + c$$

where: $m \in \mathbb{R}$ *is the* gradient slope,
$c \in \mathbb{R}$ *is the* y*-intercept.*



Definition 16 (general form of an equation of a line).
A **general** *form of an equation of a line is*

$$Ax + By + C = 0$$

where $A, B, C \in \mathbb{R}$.

Example 17 (slope-intercept vs general equation of a straight line).
Find the equation of the line passing through the points $A = (-2, 3)$ *and* $B = (4, -1)$. *Give your answer in general form.*

The slope of the line is

$$m_{AB} = \frac{-1-3}{4-(-2)} = -\frac{2}{3}$$

It passes through A, so its slope-intercept equation is

$$y - 3 = -\frac{2}{3}(x - (-2))$$

i.e.

$$y = -\frac{2}{3}x + \frac{5}{3}$$

which can be easily expressed as

$$2x + 3y - 5 = 0$$

PROPERTIES OF EQUATIONS OF LINES.

1. ***[vector normal]*** *A vector* $\begin{pmatrix} A \\ B \end{pmatrix}$ *is* ***normal*** *(perpendicular) to the line* $Ax + By + C = 0$.

2. ***[parallel lines]***

- $y = mx + b$ *is parallel to* $y = mx + c$
  (gradients are the same)
- $Ax + By + C_1 = 0$ *is parallel to*
  $(kA)x + (kB)y + C_2 = 0,\ k \in \mathbb{R}$
  (coefficients of x *and* y *must be proportional)*

3. ***[perpendicular lines]***

- $y = mx + b$ *is perpendicular to*
  $y = -\frac{1}{m}x + c$
  (gradient of one line must be **negative reciprocal** *of the gradient of the other)*
- *equivalently*
  $y = m_1x + b$ *is perpendicular to*
  $y = m_2x + c$
  whenever $m_1m_2 = -1$
- $A_1x + B_1y + C_1 = 0$ *is perpendicular to*
  $A_2x + B_2y + C_2 = 0$
  whenever $A_1A_2 + B_1B_2 = 0$
- *equivalently*
  $Ax + By + C = 0$ *is perpendicular to*
  $Bx - Ay + sth = 0$

2.2 CIRCLES

DEFINITION 18 (equation of a circle).
Equation of a circle with radius R *and:*

- *centre at the origin* $(0, 0)$ *is* $x^2 + y^2 = R^2$
- *centre at the point* (a, b) *is* $(x - a)^2 + (y - b)^2 = R^2$

QUESTIONS 2.1.

Q1. Find the area of the triangle ABC , where $A = (-4, 1)$, $B = (8, -5)$, $C = (2, 7)$.

Q2. Find the equation of a line that passes through the point A and is perpendicular to the line k.

(1) $A = (-4, 2), k : y = 2x - 5$
(2) $A = (-4.5, 2), k : y = 1.5x + 3$
(3) $A = (4, -2), k : y = -\frac{2}{3}x - 1$
(4) $A = (\frac{1}{5}, -1), k : 2x - 5y + 2 = 0$
(5) $A = (-\frac{1}{3}, \frac{5}{2}), k : 4x + 3y = 0$
(6) $A = (2, -1), k : 1.5x - 3y - 5 = 0$

Q3. Find the centre and the radius of the circle.

(1) $x^2 + y^2 - 8x - 6y - 11 = 0$
(2) $x^2 + y^2 + 6x - 2y - 1 = 0$
(3) $x^2 + y^2 + 16x - 12y = 0$
(4) $x^2 + y^2 + 10x + 7 = 0$
(5) $x^2 + y^2 + 4x - y - 2 = 0$
(6) $x^2 + y^2 - 3x + 8y - 2 = 0$
(7) $x^2 + y^2 + 7x - 5y + 2.5 = 0$

Q4. Find the points of intersection of the line k and the circle O.

(1) $k : x - 7y - 3 = 0$, $\quad O : (x+1)^2 + (y-3)^2 = 25$

(2) $k : y = x + 7$, $\quad O : x^2 - 4x + y^2 + 16y - 101 = 0$

Q5. Find the distance between the point A and the circle O.

(1) $A = (5, 4)$, $\quad O : (x+1)^2 + (y-5)^2 = 13$

(2) $A = (-1, 3)$, $\quad O : (x-2)^2 + (y+5)^2 = 20$

Q6. Find the value of p such that the points $A(-3, 0)$, $B(7, 5)$ and $C(5, p)$ are collinear.

Q7. Find the value of p such that the points $A(-7, 3)$, $B(-1, -1)$ and $C(8, p)$ are collinear.

Q8. Find the set of values of p and q such that the points $A(-8, -3)$, $B(p, 1)$ and $C(q, 7)$ are collinear.

Q9. For what value(-s) of parameter c the line $x - 3y = c$ is tangent to the circle $(x+4)^2 + (y-4)^2 = 10$? Find the coordinates of the points of tangency.

Q10. For what value(-s) of parameter m the line $y = mx - 5$ is tangent to the circle $(x-1)^2 + (y-2)^2 = 25$?

Q11. For what value(-s) of parameter m the line $y = mx - 1$ is tangent to the circle $\left(x + \frac{5}{3}\right)^2 + \left(y - \frac{2}{3}\right)^2 = 5$?

Q12. For what value(-s) of parameter m the line $y = m(x+2) - 1$ is tangent to the circle $(x+3)^2 + (y-4)^2 = 5$?

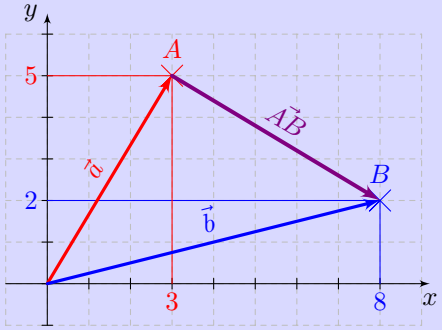
Q13. For what value of parameter R the circles $(x-1)^2 + (y+3)^2 = 25$ and $(x+7)^2 + (y-12)^2 = R^2$ are tangent? Find the coordinates of the points of tangency.

2.3 Vectors

NOTATION.
Let O be the origin. *The following notation and terminology will be used.*

- $\vec{a}$, $\vec{b}$, $\vec{c}$ *or simply* a, b, c — *position vectors of points A, B, C respectively, i.e. vectors $\vec{OA}$, $\vec{OB}$, $\vec{OC}$*
- $\vec{AB}$ — *vector that starts at A and ends at B* $\vec{AB} = \vec{AO} + \vec{OB} = -\vec{OA} + \vec{OB} = b - a$
- $|\vec{a}|$ — ***magnitude*** *of vector $\vec{a}$* For $A = (x_A, y_A)$ and $a = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$, $|a| = \sqrt{(x_A)^2 + (y_A)^2}$
- $\boldsymbol{i} = \binom{1}{0}$, $\boldsymbol{j} = \binom{0}{1}$ — base *vectors*, $\binom{x}{y} = x\boldsymbol{i} + y\boldsymbol{j}$
- unit *vector* — *vector of magnitude 1*

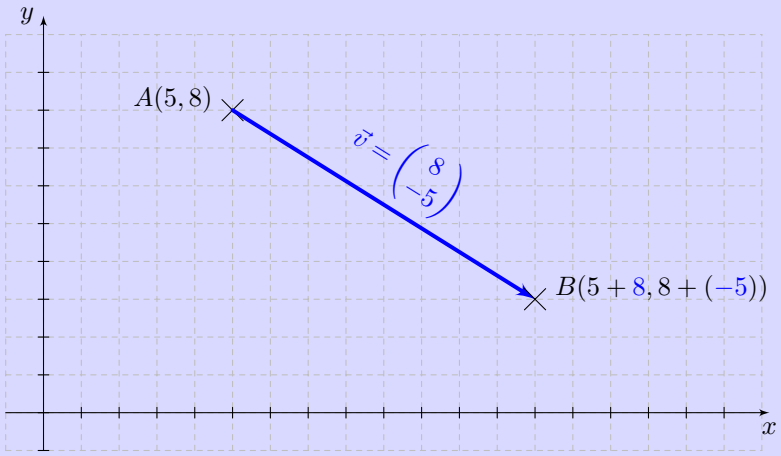
EXAMPLE 19 (vectors).
For points $A(3, 5)$ and $B(8, 2)$ find vector $\vec{AB}$.



$\vec{AB} =$
$=$(END)$-$(START)$=$
$= \vec{b} - \vec{a} =$
$= \begin{pmatrix} 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix} =$
$= \begin{pmatrix} 8-3 \\ 2-5 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

EXAMPLE 20 (vectors).

Given point $A(5,8)$ *and vector* $\vec{AB} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$ *find the coordinates of point* B.



$\vec{OB} = \vec{OA} + \vec{AB} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} + \begin{pmatrix} 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \end{pmatrix}$, so B is $(13, 3)$.

QUESTIONS 2.2.

Q14. Let D, E, F be the midpoints of the sides BC, AC, AB of the triangle ABC respectively. Let M be the midpoint of DE. Express in terms of a, b and c the following vectors:

(1) $\vec{AB}$
(2) $\vec{CB}$
(3) $\vec{BC}$
(4) $\vec{AC}$
(5) $\vec{AD}$
(6) $\vec{AM}$
(7) $\vec{DF}$
(8) $\vec{MB}$
(9) $\vec{MF}$

Q15. Write down three different vectors that move a line so that it becomes tangent to a parabola:

(1) $y = -2$; $y = 2x^2 - 4x + 5$
(2) $y = 7$; $y = -\frac{1}{2}x^2 + 2x + 5$
(3) $x = 1$; $y^2 + 2y - x - 4 = 0$

Q16. Write down three different vectors that move line k onto line m.

(1) $k : y = 2x - 5$; $m : y = 2x + 1$
(2) $k : y = -\frac{2}{3}x - 2$; $m : y = -\frac{2}{3}x + 3$
(3) $k : 3x - 2y = 5$; $m : 3x - 2y + 2 = 0$

Q17. Find a unit vector in the direction of:

(1) $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$
(2) $\begin{pmatrix} -12 \\ 5 \end{pmatrix}$
(3) $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$
(4) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$
(5) $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$
(6) $\begin{pmatrix} 2.487 \\ -3.097 \end{pmatrix}$ $(3s.f.)$

Q18. Find a vector in the direction of the one given with magnitude d.

(1) $\begin{pmatrix} 3 \\ -4 \end{pmatrix}, d = 3$
(2) $\begin{pmatrix} -12 \\ 5 \end{pmatrix}, d = 26$
(3) $\begin{pmatrix} -1 \\ -2 \end{pmatrix}, d = \sqrt{15}$
(4) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}, d = 2\sqrt{15}$
(5) $\begin{pmatrix} -3 \\ 6 \end{pmatrix}, d = 2\sqrt{15}$
(6) $\begin{pmatrix} 2.487 \\ -3.097 \end{pmatrix}, d = 5$

Q19. Find a vector $\vec{v}$ normal (perpendicular) to the line k and $\vec{u}$ parallel to k, both with magnitude d.

(1) $k : 3x + 4y = 8; d = 10$
(2) $k : y = 2x + 5; d = 10$
(3) $k : x - 3y = 6; d = 10$
(4) $k : y = \frac{1}{2}x + 1; d = 10$

Q20. Find a vector in the direction of $\vec{v}$ that moves point A onto line k.

(1) $\vec{v} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, $A(2, -1)$, $k : 2x + 3y = 10$
(2) $\vec{v} = 3i + 2j$, $A(-3, 1)$, $k : y = \frac{3}{2}x - \frac{3}{4}$

Q21. Find the distance between the line k and the point A.

(1) $k : 2x + 3y - 4 = 0$, $A = (3, 8)$
(2) $k : y = 2x + 2$, $A = (11, -1)$

Q22. Find the distance between line $k : x + 3y - 25 = 0$ and circle $O : (x + 1)^2 + (y + 3)^2 = 10$.

2.4 Angle between vectors

Theorem 21 (angle between vectors).
Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$. Let θ be the angle between vectors $\vec{a} = \vec{OA}$ and $\vec{b} = \vec{OB}$, where O is the origin. Then

$$\cos\theta = \frac{x_A x_B + y_A y_B}{|\vec{a}||\vec{b}|}$$

Proof. The proof can be easily conducted with the use of the cosine rule (theorem 125). □

Questions 2.3.

Q23. Find the angle between vectors $\vec{u}$ and $\vec{v}$, where:

(1) $\vec{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$
(2) $\vec{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$
(3) $\vec{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$
(4) $\vec{u} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
(5) $\vec{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$
(6) $\vec{u} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$
(7) $\vec{u} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$
(8) $\vec{u} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$
(9) $\vec{u} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$
(10) $\vec{u} = \begin{pmatrix} -5 \\ -6 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Q24. Find the angle A in triangle ABC where $A = (-5, -1)$, $B = (2, -2)$ and $C = (1, 4)$. Hence find the area of the triangle.

Q25. Find the acute angle between the lines $x - 2y + 5 = 0$ and $2x + 3y - 4 = 0$.

Q26. Find the angle between line $3x + 5y - 14 = 0$ and the y-axis.

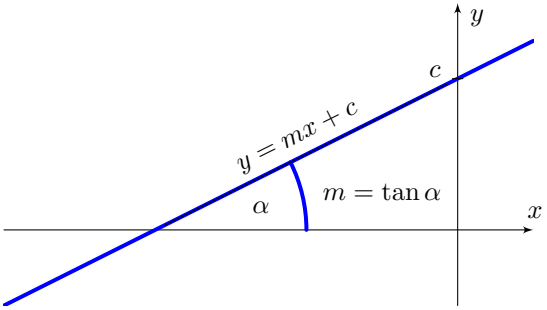
Q27. Point $P = (x, y)$ lies on a circle with diameter AB, where $A = (-3, 1)$ and $B = (5, -5)$. Use the angle between vectors $\vec{PA}$ and $\vec{PB}$ to find the equation of the circle.

Q28. Prove that diagonals of a rhombus intersect at a right angle.

2.4 Summary of key points

A ***slope-intercept*** form of equation of a line is: $y = mx + c$

where: m is the *gradient (slope)*, b is the *y-intercept*.



A ***general linear*** form of equation of a line is: $Ax + By + C = 0$

where $\begin{pmatrix} A \\ B \end{pmatrix}$ is a vector *normal* (perpendicular) to the line.

parallel lines

- $y = mx + b$ is parallel to $y = mx + c$ (gradients must be the same)
- $Ax + By + C_1 = 0$ is parallel to $(kA)x + (kB)y + C_2 = 0$, $k \in \mathbb{R}$ (coefficients of x and y must be proportional)

perpendicular lines

- $y = mx + b$ is perpendicular to $y = -\frac{1}{m}x + c$ (the gradient of one line must be ***negative reciprocal*** of the gradient of the other)
- equivalently $y = m_1x + b$ is perpendicular to $y = m_2x + c$ whenever $m_1m_2 = -1$
- $A_1x + B_1y + C_1 = 0$ is perpendicular to $A_2x + B_2y + C_2 = 0$ whenever $A_1A_2 + B_1B_2 = 0$
- equivalently $Ax + By + C = 0$ is perpendicular to $Bx - Ay + sth = 0$

equation of a circle with *radius* R and

centre at the *origin* $(0, 0)$ $\quad x^2 + y^2 = R^2$

centre at the point (a, b) $\quad (x - a)^2 + (y - b)^2 = R^2$

If θ is an angle between vectors $\vec{a} = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} x_B \\ y_B \end{pmatrix}$ then

$$\cos\theta = \frac{x_Ax_B + y_Ay_B}{|\vec{a}||\vec{b}|}$$

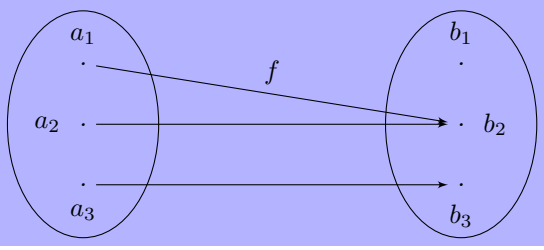
Chapter 3

Functions

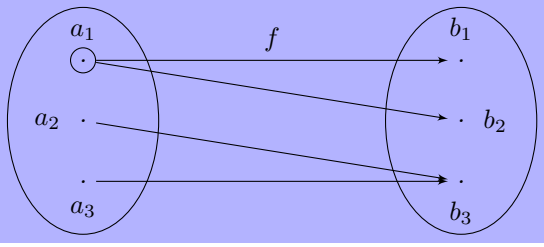
3.1 Basics

Definition 22 (function).
Let A and B be the two sets.
Informally [1] *speaking, a* ***function*** *f from A to B (which is denoted as $f : A \to B$) is an* assignment *such that* ***for every*** *$a \in A$ there exists* ***exactly one*** *$b \in B$ assigned to a.*
This particular b is called the ***value*** *of function f for an* ***argument*** *a.*

Example 23 (function).
A figure below shows a simple example of function.

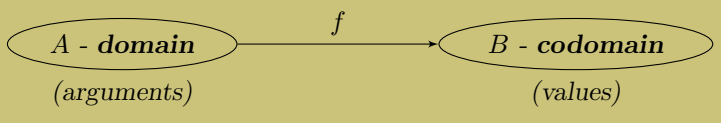


Example 24 (not a function).
A figure below shows an example of a relation that is not a function.



[1] More formally, a function f is a set of *ordered pairs* (a, b) such that
(1) $a \in A, b \in B$
(2) if $a \in A$ then there exists $b \in B$ such that (a, b) is in f
(3) if (a, b_1) and (a, b_2) are in f then $b_1 = b_2$.

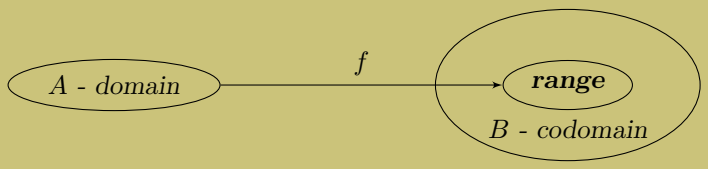
Definition 25 (domain, codomain).
*For function $f : A \to B$ set A is called the **domain** of function f. Set B is called the **codomain** of f.*



In a number of functions there are elements of codomain B that are not values of f.
These elements of B that are values of f form a set called the **range** of f.

Definition 26 (range).
*For a function $f : A \to B$ the **range** of f is a subset of B defined as:*

$$f(A) = \{y \in B | \ \exists\ x \in A \text{ such that } y = f(x)\}.$$



Example 27 (range vs codomain).
Let $f : \mathbb{R} \longmapsto \mathbb{R}$ be defined by

$$f(x) = x^2$$

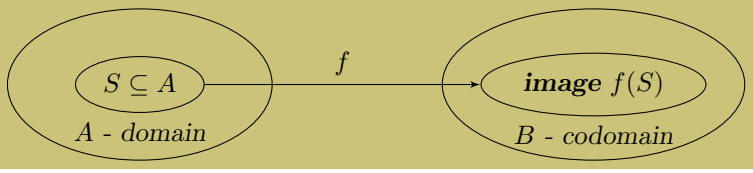
Find the range of f.

For every real number x its square $x^2 \geq 0$.
Hence the range of f is a set of non-negative values of y, i.e. $[0, +\infty[$.

Definition 28 (image).
*For any subset $S \subseteq A$ by **image** of S (denoted $f(S)$) we understand the set*

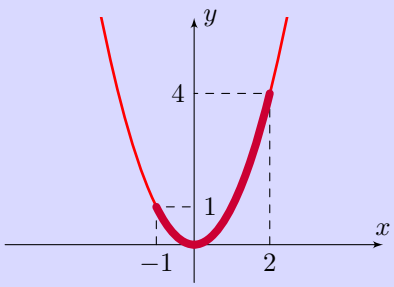
$$f(S) = \{b \in B | \exists a \in S : f(a) = b\}$$

i.e. $f(S)$ is the set of values for all arguments that are in S.



Example 29 (image).
For $f : \mathbb{R} \longmapsto \mathbb{R}$ defined by $f(x) = x^2$ find the image of $A = [-1, 2]$.

As can be seen in the diagram:



$f(-1) = 1$, $f(2) = 4$ and the minimum value is $f(0) = 0$.
Hence $f(A) = [0, 4]$.

Note. The **range** of f is the **image** of the **domain**, A.
We are going to deal with real functions mostly, i.e. functions $f : A \to B$ such that $A, B \subseteq \mathbb{R}$.

3.2 "Many-to-one", "one-to-one" and "onto" functions

Definition 30 (many-to-one function).
A function $f : A \to B$ is called ***"many-to-one"*** if and only if
there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.

Example 31 (many-to-one function).
Show that the function $f : \mathbb{R} \longmapsto \mathbb{R}$ defined by $f(x) = x^2$ is a many-to-one function.

$f(-1) = f(1)$, so f is many-to-one.

Definition 32 (injection).
A function $f : A \to B$ is called ***"one-to-one"*** *(or an* ***injection****)* if and only if *it fulfils one of the following equivalent conditions:*

1. *Each argument $a \in A$ of function f has a* ***unique*** *value $b \in B$ assigned to a.*
2. *For all $a_1, a_2 \in A$: $(a_1 \neq a_2) \Rightarrow (f(a_1) \neq f(a_2))$*
3. *For all $a_1, a_2 \in A$: $(f(a_1) = f(a_2)) \Rightarrow (a_1 = a_2)$*

Example 33 (injection).
Show that functions:

1. *$f : [1, +\infty[\longmapsto \mathbb{R}$ defined by $f(x) = x^2$,*
2. *$g : \mathbb{R} \longmapsto \mathbb{R}$ defined by $g(x) = 2x - 1$*

are injections.

1. Suppose otherwise, that there are two arguments $a_1, a_2 \in [1, +\infty[$ such that

$$a_1 \neq a_2 \text{ and } f(a_1) = f(a_2)$$

Hence

$$\begin{aligned} (a_1)^2 &= (a_2)^2 \\ a_1 &= \pm a_2 \end{aligned}$$

However, a_1 and a_2 are both positive, so $a_1 = a_2$. The contradiction shows that f is an injection.

2. Suppose otherwise, that there are two arguments $a_1, a_2 \in \mathbb{R}$ such that

$$a_1 \neq a_2 \text{ and } g(a_1) = f(a_2)$$

Hence

$$\begin{aligned} 2a_1 - 1 &= 2a_2 - 1 \\ 2a_1 &= 2a_2 \\ a_1 &= a_2 \end{aligned}$$

The contradiction shows that g is an injection.

Facts, remarks, observations- injections.

1. *If a function is increasing or decreasing, it is injective.*
2. *[horizontal line test] If no horizontal line intersects the graph of a function more than once, the function is injective.*

Note. Using the definition of a function both implications ($\Rightarrow$) in above conditions can be changed to equivalences ($\Leftrightarrow$).

Definition 34 (surjection).
A function $f : A \to B$ is called ***"onto"*** *(or a* ***surjection****)* if and only if *it fulfils one of the following equivalent conditions:*

1. *Each element $b \in B$ of codomain is a value for at least one $a \in A$.*
2. *If $b \in B$ then there exists $a \in A$ such that $f(a) = b$.*

Example 35 (surjection).
Consider the functions:

1. $f : \mathbb{R} \longmapsto \mathbb{R}$ *defined by* $f(x) = x^2$,
2. $g : \mathbb{R} \longmapsto [0, +\infty[$ *defined by* $g(x) = x^2$

Show that f is not a surjection while g is a surjection.

1. Let us consider an element of codomain of f, $y = -1$.
   If it is a value of f then there must be an argument x such that
   $$f(x) = -1$$
   i.e.
   $$x^2 = -1$$
   This equation has no solution in $\mathbb{R}$, so f is not a surjection.
2. For every non-negative value y the equation
   $$g(x) = y$$
   i.e.
   $$x^2 = y$$
   has a solution (even two) in $\mathbb{R}$. Hence g is a surjection.

Notation.
Let $f : A \to B$. The following notation will be used:

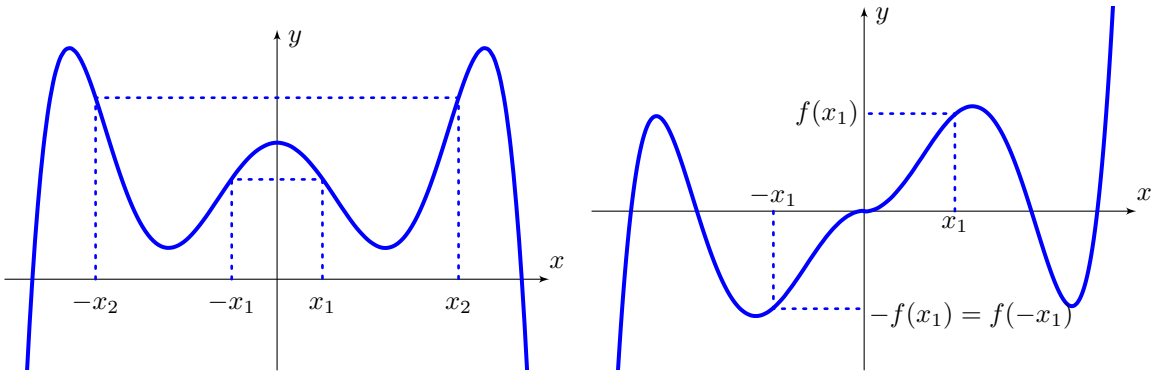
$f : A \xrightarrow{1-1} B$	f *is* $1-1$ *(or* injective*)*
$f : A \xrightarrow{onto} B$	f *is "onto" (or* surjective*)*
$f : A \xrightarrow[onto]{1-1} B$	f *is both* $1-1$ *and "onto"* such f *is called a* ***bijection***

3.3 Even and odd functions

Definition 36 (even function).
Let $f : X \to Y$, where $X, Y \subseteq \mathbb{R}$.
f is ***even*** $\Leftrightarrow \forall x \in X \quad f(-x) = f(x)$

Definition 37 (odd function).
Let $f : X \to Y$, where $X, Y \subseteq \mathbb{R}$.
f is ***odd*** $\Leftrightarrow \forall x \in X \quad f(-x) = -f(x)$

Example 38 (even and odd functions).
Let $f, g : \mathbb{R} \longmapsto \mathbb{R}$ be defined by $f(x) = x^2$ and $g(x) = x^3$.
Then f is even while g is odd.



(a) **Even** function. (b) **Odd** function.

Figure 3.1: Even and odd functions.

PROPERTIES OF EVEN AND ODD FUNCTIONS.

1. *Graph of an* ***even*** *function is symmetrical in y-axis.*
2. *Graph of an* ***odd*** *function is symmetrical in the* origin.
3. *Any function defined by* $y = x^n$, *where* n *is an* ***even*** *number is* ***even****.*
4. *Any function defined by* $y = x^n$, *where* n *is an* ***odd*** *number is* ***odd****.*

Let $f, f_1, f_2 : X \to Y$ *be even functions and* $g, g_1, g_2 : X \to Y$ *be odd functions,* $X, Y \subseteq \mathbb{R}$. *Let* $k \in \mathbb{R}$ *be a constant. Then:*

5. *the following functions are* ***even****:*

   - $x \longmapsto kf(x)$
   - $x \longmapsto (f_1(x) + f_2(x))$
   - $x \longmapsto (f_1(x) - f_2(x))$
   - $x \longmapsto (f_1(x) \times f_2(x))$
   - $x \longmapsto (g_1(x) \times g_2(x))$
   - $x \longmapsto \left(\frac{1}{f(x)}\right)$

6. *the following functions are* ***odd****:*

   - $x \longmapsto kg(x)$
   - $x \longmapsto (g_1(x) + g_2(x))$
   - $x \longmapsto (g_1(x) - g_2(x))$
   - $x \longmapsto (f(x) \times g(x))$
   - $x \longmapsto \left(\frac{1}{g(x)}\right)$

7. $x \longmapsto \sin x$ *is* ***odd*** *while* $x \longmapsto \cos x$ *is* ***even****.*

Note. *In case of* $x \longmapsto \left(\frac{1}{f(x)}\right)$ *the domain will be different than* X.

3.4 INCREASING AND DECREASING FUNCTIONS

DEFINITION 39 (increasing function).
Let $f : X \to Y$, *where* $X, Y \subseteq \mathbb{R}$. *Let* $A \subseteq X$.
f *is* ***increasing*** *on* $A \Leftrightarrow \forall x_1, x_2 \in A \quad ((x_1 < x_2) \Rightarrow (f(x_1) < f(x_2)))$

DEFINITION 40 (decreasing function).
Let $f : X \to Y$, *where* $X, Y \subseteq \mathbb{R}$. *Let* $A \subseteq X$.
f *is* ***decreasing*** *on* $A \Leftrightarrow \forall x_1, x_2 \in A \quad ((x_1 < x_2) \Rightarrow (f(x_1) > f(x_2)))$

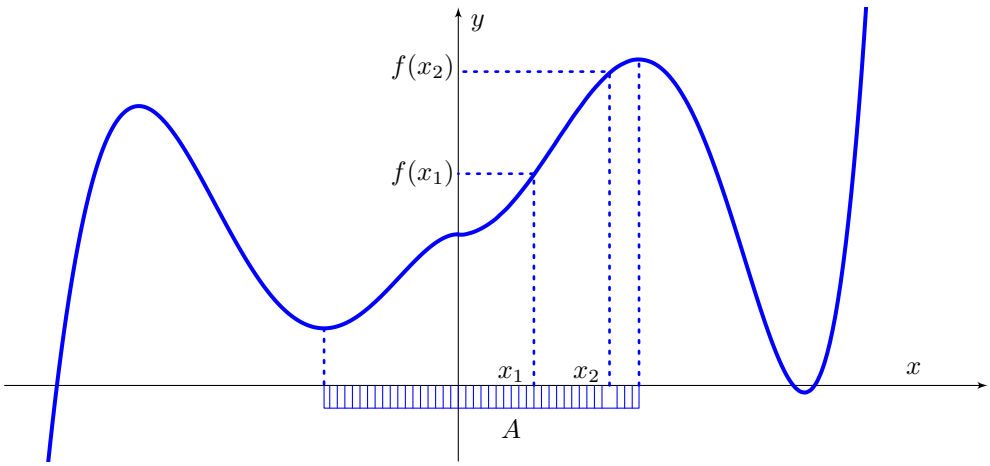


Figure 3.2: Graph of function increasing on set A.

Questions 3.1.

Q1. Is a function $f : \mathbb{R} \longmapsto \mathbb{R}$ injective, surjective and / or bijective? Justify your answers.

(i) $f(x) = x^2$ (ii) $f(x) = x^3$ (iii) $f(x) = x^3 - x$ (iv) $f(x) = x^3 + x$ (v) $f(x) = e^x$

Q2. The function $f : \mathbb{R} \longmapsto \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} -x^2 - 2x + 1 & \text{for } x < a \\ \frac{1}{5}x^2 + \frac{8}{5}x + 1 & \text{for } x \geq a \end{cases}$$

Find the values of a for which f is:

(a) injective, (b) surjective, (c) bijective.

Q3. The function $f : \mathbb{R} \longmapsto \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} -x^2 - 2x + 1 & \text{for } x < a \\ x^2 + 8x + 13 & \text{for } x \geq a \end{cases}$$

Find the values of a for which f is:

(a) injective, (b) surjective, (c) bijective.

Part II

ALGEBRA

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Chapter 4

Sequences

Definition 41 (sequence).
Formally, a sequence $\{a_n\}$ *is a function* $a : \mathbb{Z}^+ \mapsto \mathbb{R}$ *(or* $a : \mathbb{N} \mapsto \mathbb{R}$*).*
For simplification instead of $a(n)$ *we write* a_n*.*

Definition 42 (series).
A series $\{S_n\}$ *is a sequence of sums of consecutive terms of a given sequence:*

$$S_n = a_1 + a_2 + a_3 + ... + a_n$$

Sometimes S_n *is convergent to* S *- a sum of all terms of a sequence.*

$$S = a_1 + a_2 + a_3 + ...$$

4.1 Arithmetic sequences

Definition 43 (arithmetic sequence).
A sequence $\{a_n\}$ *is called* ***arithmetic*** *when the* ***difference*** *between any two consecutive terms is the same.*

$$a_{n+1} - a_n = d = const.$$

d is called a ***common difference****.*

Useful formulae:

n^{th} term	a_n	$= a_1 + (n-1)d$
	a_n	$= a_k + (n-k)d$
common difference		$d = \frac{a_n - a_k}{n-k}$
sum of n terms	S_n	$= (\frac{a_1+a_n}{2})n$
		$= na_1 + \frac{n(n-1)}{2}d$
		$= \frac{n}{2}(2a_1 + (n-1)d)$

Questions 4.1.

Q1. For the sequence 4, 9, 14, 19, 24,..., find the 21^{st} term.

Q2. An arithmetic sequence has the first term of 12 and the common difference of -2. Find the 17^{th} term.

Q3. An arithmetic sequence has the first term of 41 and the 12^{th} term of 57.5. Find the 34^{th} term.

Q4. An arithmetic sequence has the 13^{th} term of 20 and the 18^{th} term of 32.5. Find the first term.

Q5. An arithmetic sequence has the 5^{th} term of 17 and the 12^{th} term of 3. Find the 15^{th} term.

Q6. Copy and complete the table (a_n is an arithmetic sequence).

	n	k	a_n	a_k	$\frac{n+k}{2}$	$a_{\frac{n+k}{2}}$
(1)	1	5	7	12		
(2)	2	10	3	15		
(3)	6	14	28	3		
(4)	13	25	7	11		
(5)	7	15	8	-2		
(6)	12	18	-3	28		

Write down the formula for $a_{\frac{n+k}{2}}$ in terms of a_n and a_k.

Q7. Find the sum of 20 terms of the series $-7 - 3 + 1 + 5 + 9 + ...$

Q8. Find the sum of 30 terms of the series $-\frac{6}{7} - \frac{1}{7} + \frac{4}{7} + \frac{9}{7} + ...$

Q9. An arithmetic sequence has the fourth term of 2. The sum of the first 20 terms is -350. Find the first term and the sum of the first 13 terms.

Q10. Find each sum.

(1) $\sum_{k=1}^{50}(2k)$ (3) $\sum_{k=1}^{50}(2k-5)$ (5) $\sum_{k=1}^{70}(2k+1)$ (7) $\sum_{k=21}^{50}(3k-4)$

(2) $\sum_{k=1}^{50}(k-1)$ (4) $\sum_{k=1}^{50}(3k+2)$ (6) $\sum_{k=20}^{50}(3k-4)$ (8) $\sum_{k=15}^{45}(3-2k)$

Q11. An arithmetic sequence has the common difference of 3 and the 11^{th} term equal 32. Find the 7^{th} term.

Q12. The 15^{th} term of an arithmetic sequence is 25 and the 8^{th} term is 11. Find the 1^{st} term and the common difference.

Q13. Find the sum of 11 terms of an arithmetic sequence with the first term of 3 and the common difference of 4.

Q14. The first term of an arithmetic sequence is 50 and the 20th term is -7. Find the sum of first 11 terms.

Q15. The first term of an arithmetic sequence is 11, the common difference is 0.5 and the sum of the first n terms is 182. Find the value of n.

Q16. The 11^{th} term of an arithmetic sequence is 36, the common difference is -0.5 and the sum of the first n terms is 1250. Find the possible values of n.

Q17. The 16^{th} term of an arithmetic sequence is 25 and the 10^{th} term is 23. Find the sum of the first 16 terms.

Q18. The sum of first 8 terms of an arithmetic sequence is 107 and the sum of the first 13 terms is 182. Find the first term and the common difference.

Q19. The first term of an arithmetic sequence is 8 and the sum of terms from 11^{th} to 21^{st} (including) is 55. Find the common difference.

Q20. The 4^{th} term of an arithmetic sequence is 9 and the sum of terms from 4^{th} to 13^{th} (including) is 75. Find the first term and the common difference.

Q21. For the sequence 3, 6.5, 10, 13.5,..., find the 23^{rd} term.

Q22. An arithmetic sequence has a 24^{th} term of 2 and a 29^{th} term of -13. Find the 19^{th} term.

Q23. Find the sum of 20 terms of the series $-\frac{3}{7} + \frac{1}{7} + \frac{5}{7} + \frac{9}{7} + ...$

Q24. An arithmetic sequence has a third term of 7. The sum of the first 3 terms is 15. Find the first term and the sum of the first 12 terms.

Q25. The sum of three consecutive terms of an arithmetic sequence is 12 while the sum of their squares is 66. Find the terms.

4.2 Geometric sequences

Definition 44 (geometric sequence).
*A sequence $\{a_n\}$ is called **geometric** if each term of the sequence but first may be produced as a product of the previous term and a constant real number:*

$$a_{n+1} = a_n \times r$$

*The above can be understood in a way that the **ratio** between any two consecutive terms is the same:*

$$\frac{a_{n+1}}{a_n} = r = const.^a$$

*r is called a **common ratio**.*

[a] However, this definition is less general. Why?

Useful formulae:

n^{th} term	a_n	$= a_1 r^{n-1}$	
	a_n	$= a_k r^{n-k}$	
common ratio			$r^{n-k} = \frac{a_n}{a_k}$
sum of n terms	S_n	$= a_1 \frac{1-r^n}{1-r}$	
		$= a_1 \frac{r^n-1}{r-1}$,	$r \neq 1$
sum to infinity	S	$= \frac{a_1}{1-r}$,	$\lvert r \rvert < 1$

Questions 4.2.

Q26. For the sequence $4, 6, 9, ...$ find the 9^{th} term.

Q27. A geometric sequence has the first term of 0.5 and the common ratio of -2. Find the 12^{th} term.

Q28. A geometric sequence has the first term of 41 and the 8^{th} term of 5248. Find the 11^{th} term.

Q29. A geometric sequence has the 13^{th} term of 8 and the 19^{th} term of $\frac{1}{8}$. Find the first term.

Q30. Copy and complete the table (a_n is a geometric sequence).

	n	k	m	a_n	a_k	a_m	R	S_m
(1)	1	5	8	4	0.25			
(2)	2	6	7			1	-0.5	
(3)	3	5	8		6	162		
(4)	2	3	6		3		1.5	
(5)	4	7	9			162	-3	
(6)	1	6	8		96	864		
(7)	2	5	10				-2	-1023

Q31. The sum of first two terms of a geometric sequence is 2 while the sum of the first four terms of the same sequence is 20. Find the first term and the common ratio.

Q32. The sum of first three terms of a geometric sequence is 2 while the sum of the first six terms of the same sequence is 18. Find the first term and the common ratio.

Q33. The sum of first four terms of a geometric sequence is 80 while the sum of the next four (i.e. from the fifth to the eighth) is 6480. Find the first term and the common ratio.

Q34. The first term of an infinite geometric series exceeds the second term by 2. The sum of the series is 18. Find the common ratio and the first term.

Q35. The fourth term of a geometric sequence is 6.75 while the sum of the first four terms is 16.25. Find the first term and the common ratio of the sequence.

Q36. The numbers x, a and y are three consecutive terms of a sequence.
Express a in terms of x and y given that the sequence is:

(1) arithmetic

(2) geometric

Q37. 800 PLN was deposited in an account at the rate of 3% per annum compounded yearly.

(i) What are the savings after 5 years?

(ii) When the savings exceed 5000 PLN?

Q38. 800 PLN was deposited in an account at the rate of 3% per annum compounded monthly.

(i) What are the savings after 5 years?

(ii) When the savings exceed 5000 PLN?

Q39. Adili considers two savings plans.
In plan A he saves 200 PLN in the first month and each next month he saves 10 PLN more than in the previous month.
In plan B he saves 50 PLN in the first month and each next month he saves 10% more than in the previous month.

(i) What are the monthly savings in the 10^{th} month in both plans?

(ii) When monthly savings in plan B exceed monthly savings in plan A?

(iii) When total savings in plan B exceed monthly savings in plan A?

Q40. $8, x$ and y are consecutive terms of an arithmetic sequence.

x, y and $2\frac{2}{3}$ are consecutive terms of a geometric sequence.

Find the values of x and y.

Q41. The sum to infinity of a geometric sequence is 1.8.
The first three terms of the geometric sequence are the fourth, first and third terms respectively of an arithmetic sequence.
Find the first term and the common ratio of the geometric sequence.

4.2 Summary of key points

Arithmetic sequences:

n^{th} term	a_n	$= a_1 + (n-1)d$	
	a_n	$= a_k + (n-k)d$	
common difference			$d = \frac{a_n - a_k}{n-k}$
sum of n terms	S_n	$= (\frac{a_1+a_n}{2})n$	
		$= na_1 + \frac{n(n-1)}{2}d$	
		$= \frac{n}{2}(2a_1 + (n-1)d)$	

Geometric sequences:

n^{th} term	a_n	$= a_1 r^{n-1}$	
	a_n	$= a_k r^{n-k}$	
common ratio			$r^{n-k} = \frac{a_n}{a_k}$
sum of n terms	S_n	$= a_1 \frac{1-r^n}{1-r}$	
		$= a_1 \frac{r^n-1}{r-1}$,	$r \neq 1$
sum to infinity	S	$= \frac{a_1}{1-r}$,	$\lvert r \rvert < 1$

Chapter 5

Binomial theorem

5.1 Factorials

Definition 45 (factorial).
For any positive integer n ($n \in \mathbb{Z}^+$) we define $n!$ (read: "n factorial") as a product of all consecutive integers from 1 to n:

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$$

To extend the definition to all non-negative integers we have to additionally define:

$$0! = 1$$

Factorials can be defined also in the following recursive way:

$$n! = \begin{cases} 1 & \text{for } n = 0 \\ n \times (n-1)! & \text{for } n \in \mathbb{Z}^+ \end{cases}$$

Example 46 (factorial).
Evaluate 7!

$$\begin{aligned} 7! &= 7 \times 6! = \\ &= 7 \times 6 \times 5! = \\ &= 7 \times 6 \times 5 \times 4! = \\ &= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \\ &= 5040 \end{aligned}$$

Example 47 (simplifying factorials).
Evaluate $\frac{9!}{6!}$

$$\begin{aligned} \frac{9!}{6!} &= \frac{9 \times 8 \times 7 \times 6!}{6!} = \\ &= 9 \times 8 \times 7 = 504 \end{aligned}$$

5.2 Binomial coefficients, Pascal's triangle

Definition 48 (binomial coefficient).
*The **binomial coefficient** is defined for $n \in \mathbb{Z}^+$ and $r \in \{0, 1, 2, 3, \ldots, n\}$ as follows:*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\binom{n}{r}$ is read as "n-c-r" or "n-choose-r" and sometimes denoted as ${}^nC_r$.
The binomial coefficients can be arranged in a graphical pattern called the **Pascal's triangle** .

$$\begin{array}{ccccccccc}
 & & & \binom{0}{0} & & & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & 1 & & & & & \\
 & & 1 & & 1 & & & & \\
 & 1 & & 2 & & 1 & & & \\
1 & & 3 & & 3 & & 1 & & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Properties of binomial coefficients and Pascal's triangle.

$\binom{n}{r} = \binom{n}{n-r}$	*(symmetry of Pascal's triangle)*
$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$	*(each number is a sum of the two above it)*
$\binom{n}{0} = \binom{n}{n} = 1$	
$\binom{n}{1} = \binom{n}{n-1} = n$	
$\binom{n}{2} = \binom{n}{n-2} = \frac{n\times(n-1)}{2}$	
$\binom{n}{3} = \binom{n}{n-3} = \frac{n\times(n-1)\times(n-2)}{1\times2\times3}$	*etc.*

5.3 Binomial theorem

Theorem 49 (binomial theorem).
For any numbers a and b and a positive integer n:
$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$
or

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

Questions 5.1.

Q1. Evaluate without a calculator.

(1) 3! (2) 4! (3) 5! (4) 6! (5) 7!

Q2. Simplify not using a calculator.

(1) $\frac{7!}{6!}$ (2) $\frac{8!}{7!}$ (3) $\frac{12!}{11!}$ (4) $\frac{11!}{9!}$ (5) $\frac{20!}{18!}$

Q3. Solve the equations without a calculator.

(1) $\frac{n!}{(n-1)!} = 7$ (2) $\frac{n!}{(n-2)!} = 20$ (3) $\frac{(n+1)!}{(n-1)!} = 72$

Q4. Evaluate without a calculator.

(1) $\binom{3}{2}$ (2) $\binom{4}{1}$ (3) $\binom{4}{2}$ (4) $\binom{5}{2}$ (5) $\binom{5}{3}$ (6) $\binom{5}{4}$ (7) $\binom{6}{1}$ (8) $\binom{6}{2}$ (9) $\binom{6}{3}$ (10) $\binom{6}{4}$ (11) $\binom{7}{2}$ (12) $\binom{7}{3}$ (13) $\binom{7}{4}$ (14) $\binom{8}{5}$ (15) $\binom{8}{6}$

Q5. Simplify not using a calculator.

(1) $\binom{n+1}{2}$
(2) $\binom{3n}{2}$
(3) $\binom{n}{n-2}$
(4) $\binom{n+1}{n-1}$
(5) $\binom{2n}{2n-2}$
(6) $\binom{n+2}{3}$

Q6. Expand.

(1) $(x+1)^4$
(2) $(x-1)^4$
(3) $(x+2)^5$
(4) $(2x-1)^5$
(5) $(2a-b)^5$
(6) $(a^2-3b)^4$
(7) $(x^2-2xy)^3$
(8) $(\sqrt{3}-\sqrt{2})^4$
(9) $(a\sqrt{2}-b)^3$

Q7. Find the coefficient of x^2 in the expansion of $(2x+\frac{1}{x^2})^5$.

Q8. Find the coefficient of x^5 in the expansion of $(x^2-\frac{3}{x})^7$.

Q9. Find the term independent of x in the expansion of $(\frac{2}{x}+x)^6$.

Q10. Find the term independent of x in the expansion of $(x^2-\frac{2}{x})^9$.

Q11. Determine the first 3 terms in the expansion of $(x+2)^3(2x-1)^4$ in ascending powers of x.

Q12. Determine the first 3 terms in the expansion of $(2-\frac{1}{2}x)^4(2x+1)^7$ in ascending powers of x.

Q13. The coefficient of $\frac{1}{x^2}$ in the expansion of $(2x+\frac{1}{ax^2})^7$ is -70. Find the possible values of a.

Q14. Show that the sum of all terms in the n^{th} row of Pascal's triangle is equal to 2^n.

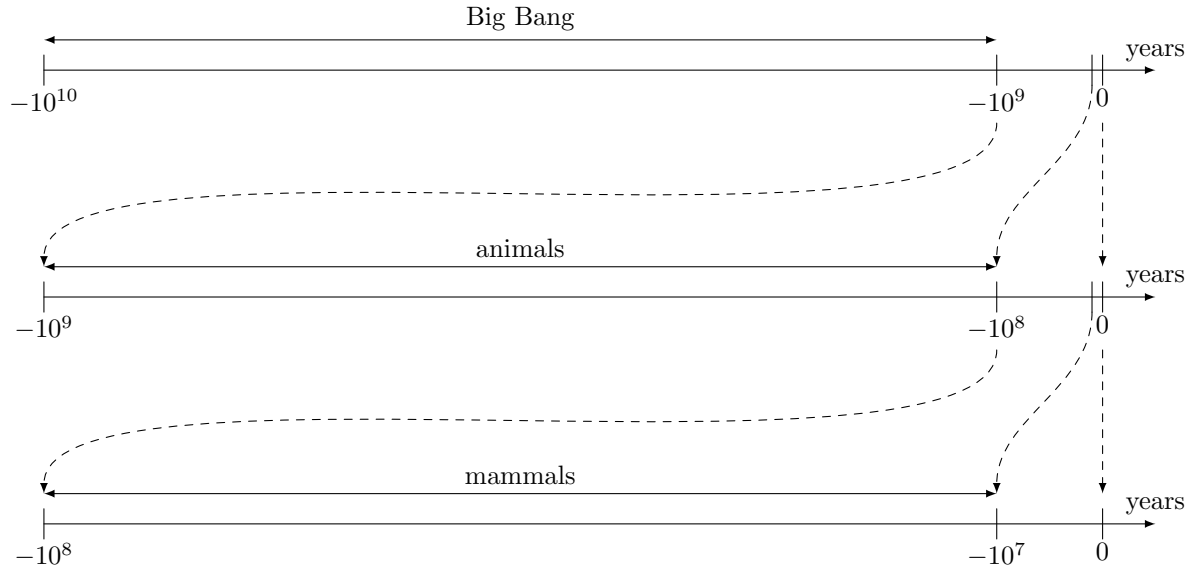
Chapter 6

Logarithms and exponents

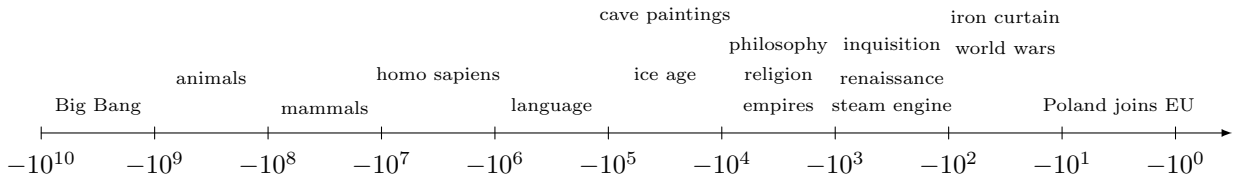
The table below shows a non-linear time line with examples of various events. Some of them are of huge importance for us, people living in the world, in Europe, in Poland. Some of them are of smaller significance. The table shows natural tendency to place recent events more accurately than the older ones. We can observe easily that the accuracy of description of time decreases with the increase of amount of time that has passed since an event.

years ago	**i.e. approximately**	**actions**
10^{-6} to 10^{-5}	a few minutes ago	you are looking at this handout
10^{-5} to 10^{-4}	less than an hour ago	...
10^{-4} to 10^{-3}	a few hours ago	...
10^{-3} to 10^{-2}	a few days ago	...
10^{-2} to 10^{-1}	a month ago	...
10^{-1} to 10^{0}	a few months ago	...
10^{0} to 10^{1}	last decade	Poland joins EU
10^{1} to 10^{2}	last century	world wars, iron curtain
10^{2} to 10^{3}	last millenium, middle ages	inquisition, renaissance, steam engine
10^{3} to 10^{4}	...	religion, philosophy, empires
10^{4} to 10^{5}	...	ice age, cave paintings
10^{5} to 10^{6}	...	language
10^{6} to 10^{7}	...	homo sapiens
10^{7} to 10^{8}	...	mammals
10^{8} to 10^{9}	...	animals
10^{9} to 10^{10}	...	Big Bang

If we wanted to show all these actions graphically on a time line it would not be easy, as each of the periods considered is 10 times longer than the previous one.



etc.
We can create a different time line instead:



The scale on the axis above is obviously non-standard, not **linear**. It is **logarithmic**.

6.1 Algebra of logarithms

Properties - rules of powers.
For $a, b, m, k \in \mathbb{R}$ and $a, b \neq 0$:

- $a^m \times a^k = a^{m+k}$
- $a^k \times b^k = (ab)^k$
- $a^m \div a^k = a^{m-k}$
- $a^k \div b^k = (\frac{a}{b})^k$
- $(a^m)^k = a^{mk}$

Definition 50 (logarithm).
For $a, b, x \in \mathbb{R}$ such that $a \neq 1$ and $a, b > 0$
*a **logarithm** of b to base a is the exponent x to which a must be raised to produce b, i.e.*

$$\log_a b = x \;\Leftrightarrow\; a^x = b$$

Definition 51 (Euler's number e).

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828$$

Notation.

- $\log x = \log_{10} x$
- $\ln x = \log_e x$ *(natural logarithm)*

Properties - rules of logarithms.
For $a, b, c, s, x \in \mathbb{R}$, $a, b, c, s > 0$, $a, s \neq 1$:

- $\log_a b + \log_a c = \log_a bc$
- $\log_a b - \log_a c = \log_a(\frac{b}{c})$
- $\log_a n^s = s \log_a n$
- $\log_a b = \frac{\log_s b}{\log_s a}$ *(change of base)*
- $\log_b b^a = a$ $\quad b^{\log_b a} = a$
- $\ln e^a = a$ $\quad e^{\ln a} = a$
- $e^{x \ln a} = a^x$

Questions 6.1.

Q1. Evaluate. Do not use a calculator.

(1) $\frac{3^5 9^4}{27^4}$
(2) $\frac{16^6 8^{-4}}{4^7}$
(3) $\frac{(4\sqrt{2})^5 8^{-3}}{4^2}$
(4) $\frac{(8^6+4^{10})^2}{2^{37}}$
(5) $4^5 + 4^5 + 4^5 + 2^{10}$
(6) $3^{n+1} + 3^{n+2} - 3^n$

Q2. Evaluate. Do not use a calculator.

(1) $\log_2 8$
(2) $\log_4 8$
(3) $\log_8 4$
(4) $\log_2 2\sqrt{2}$
(5) $\log_4 \sqrt{2}$
(6) $\log_{\frac{1}{2}} \sqrt{2}$
(7) $\log_{\frac{1}{2}} \sqrt{8}$
(8) $\log_{\frac{1}{3}} 27$
(9) $\log_4 \frac{\sqrt{2}}{16}$
(10) $\log_{\frac{1}{2}} \frac{\sqrt{8}}{32}$
(11) $\log_3 \frac{27}{\sqrt{3}}$
(12) $\log_9 \frac{81}{\sqrt{27}}$
(13) $\log_{\sqrt{3}} \frac{\sqrt{3}}{27}$
(14) $\log_{\frac{1}{\sqrt{3}}} \frac{\sqrt{27}}{81}$
(15) $\log_{0.2} 25$
(16) $\log_4 0.125$
(17) $\log_{4\sqrt{2}} \sqrt{8}$
(18) $\log_{32} 4\sqrt{2}$
(19) $\log_{2\sqrt{2}}(\frac{1}{16})$
(20) $\log_{27} 9\sqrt{3}$
(21) $\log_{25} 5\sqrt{5}$
(22) $\log_{0.25} 4\sqrt{8}$
(23) $0.01 \times 0.001^{\log 5}$
(24) $(9\sqrt{3})^{\frac{\log_3 4 - 1}{5}}$
(25) $(2\sqrt{2})^{\frac{\log_4 3 + 2}{3}}$

Q3. Knowing that $a = \log_2 3$ and $b = \log_2 5$ express the following in terms of a and b.

(1) $\log_2 25$
(2) $\log_2 15$
(3) $\log_2 45$
(4) $\log_2 40$
(5) $\log_2 60$
(6) $\log_2 3000$
(7) $\log_2 27$
(8) $\log_2 72$

Q4. Knowing that $a = \log_3 5$ and $b = \log_9 2$ express the following in terms of a and b.

(1) $\log_3 10$
(2) $\log_3 100$
(3) $\log_9 10$
(4) $\log_9 100$
(5) $\log_3 40$
(6) $\log_3 8$
(7) $\log_9 80$
(8) $\log_9 25$

Q5. Knowing that $a = \log_2 b$ express the following in terms of a.

(1) $\log_2 \frac{1}{b}$
(2) $\log_2(b^2)$
(3) $\log_b 2$
(4) $\log_b 8b$
(5) $\log_4 b$
(6) $\log_8 b$
(7) $\log_8 4b^5$
(8) $\log_{b^2} 32$

Q6. Let $a = \ln x$ and $b = \ln y$. Express the following in terms of a and b.

(1) $\ln(\frac{1}{xy^3})$
(2) $\ln(\frac{\sqrt{x}}{y})$
(3) $\ln(\frac{x}{\sqrt{y}})$
(4) $\ln(\frac{e}{x^4\sqrt{y^3}})$

Q7. Let $a = \log x$, $b = \log y$ and $c = \log z$. Express the following in terms of a, b and c.

(1) $\log(\frac{x}{y^2\sqrt{z}})$ (2) $\log(\frac{x^3\sqrt{y}}{z^2})$ (3) $\log(\frac{x^5}{zy\sqrt{y}}\frac{1}{x^2\sqrt{x}})$ (4) $\log(\frac{1}{x^4\sqrt{y^5}\sqrt[3]{z^4}})$

Q8. Find integer values of a and b such that $\log 400 = a\log 2 + b$.

Q9. Find integer values of a and b such that $\log 75 = a\log 5 + b\log 3$.

Q10. Find integer values of a and b such that $\log 160 = a\log 2 + b$.

Q11. Find integer values of a and b such that $\log_2 1000 = a\log_2 5 + b$.

Q12. Find integer values of a and b such that $\log_2 250 = a\log_2 5 + b$.

Q13. Find integer values of a and b such that $\log_3 144 = a\log_3 2 + b$.

Q14. (1) Knowing that $\log_4 98 = a$ express $\log_4 343$ in terms of a.
(2) Knowing that $\log_{49} 28 = a$ express $\log_4 98$ in terms of a.
(3) Knowing that $\log_{25} 75 = a$ express $\log_{15} 125$ in terms of a.
(4) Knowing that $\log_{40} 160 = a$ express $\log_8 250$ in terms of a.

Q15. Simplify.

(1) $2^{\log_4 3}$ (3) $e^{3\ln 2}$ (5) $e^{\frac{1}{2}\ln 8}$ (7) $3^{-\log_9 16}$
(2) $4^{\log_2 5}$ (4) $4^{2\log_8 3}$ (6) $(\frac{1}{e})^{2\ln 3}$ (8) $1000^{2\log\sqrt{2}}$

Q16. Let $a = 2^x$, $b = 3^x$ and $c = 5^x$. Express the following in terms of a, b and c.

(1) 2^{x+1} (7) 8^x (13) 4^{x-2} (19) $\frac{1}{16^{3x-2}}$
(2) 2^{x-1} (8) 9^{-x} (14) 5^{-x} (20) $\frac{81}{27^{3-2x}}$
(3) 2^{x+5} (9) 25^x (15) 2^{4-x} (21) $\frac{3\times 9^{2x+1}}{81^{1+x}}$
(4) 2^{2x+3} (10) 16^x (16) 9^{3-x} (22) $\frac{625^{x-1}}{125^{3x-2}}$
(5) 3^{3x-1} (11) 16^{-x} (17) 25^{2x} (23) 10^x
(6) 4^x (12) 27^x (18) 8^{2x-1} (24) 18^{2x-1}

6.2 Logarithmic equations

Questions 6.2.

Q17. Solve the equations.

(1) $\log_{\frac{\sqrt{3}}{2}} x = 6$
(2) $\log_{3\sqrt{3}} x = \log 100$
(3) $\log_8 x + \log_4 x = 2.5$
(4) $\log_{27} x + \log_9 x - \frac{5}{3} = 0$
(5) $\log_{25\sqrt{5}} x^2 + 3\log_5 x = 19$
(6) $\log_4 \log_{\frac{1}{\sqrt{3}}} \log_2 x = 0.5$
(7) $\log_{x^2} 4 + \log_{x^3} 64 - 9 = 0$
(8) $\log_{x^2} 64 = \log_{x^8} 256 - 2$

Q18. Solve the equations.

(1) $\log_4 x^2 - 2\log_x 2 = 1$
(2) $\log_5 x^2 + \log_x 125 = 7$
(3) $\log^2_{\frac{1}{2}} x^3 = \frac{9}{2}\log_{2\sqrt{2}}(\frac{1}{x}) + 2$
(4) $5\log^2_{9\sqrt{3}} x + 0.8 = 3.2\log_9 x$
(5) $\frac{1}{\log^2_{x^3} 3} + \log_{243} x^{75} = 6$
(6) $4\log^2_{x^2} 5 + \log_3 7 = \log_9 4$
(7) $4^x - 2^{x+1} + 1 = 0$
(8) $2^{2x+1} + 2^2 = 2^x + 8^{\frac{3+x}{3}}$
(9) $4^{x-1} + 2^{x-1}(1 + 2^{x-1}) = = 2^{1-x} + 2$
(10) $8^x + 4 = 4^x + 2^{x+2}$
(11) $3^{3x+2} + 6 = 9^{x+1} + 2(3^{x+1})$
(12) $27^x - 1 = \frac{9^{x+1} - 3^{x+2}}{3}$
(13) $\log^2_{\sqrt{x}} 3\sqrt{3} + 36\log_{x^2} 3 = 4.5\log_{\sqrt{3}} x^2 + 9$

Q19. Solve the equations.

(1) $\frac{e^{x^2+3}}{e^{4x}} = 1$

(2) $\ln(2x+3) - 2\ln x = 0$

(3) $\ln^2 x - \ln x^2 = \ln \frac{1}{e}$

(4) $2\ln^2 x + \frac{9}{\ln x} = 3(\ln x^2 + 1)$

Q20. Solve the equations.

(1) $\log_2 x + \log_2 \frac{1}{x^{0.4}} + \log_2 x^{0.16} + \log_2 \frac{1}{x^{0.064}} + \cdots = 5$

(2) $\log x - \frac{1}{2}\log^2 x + \frac{1}{4}\log^3 x - \frac{1}{8}\log^4 x + \cdots = 1$

Q21. Solve the equations.

(1) $\frac{(16^{5-2x})^{2x+3} \times (16\sqrt{2})^{2x-3}}{8^{19-3x}} = \frac{16}{4^{1-2x}}$

(2) $16 \times 2^{x-3} \times (\frac{1}{4})^{3-2x} = 256^5$

(3) $\left(\frac{4}{9}\right)^{4-6x} \times 1.5^{8x} = \frac{64}{729}$

(4) $\frac{243^{(2-x)(2+x)} \times (9\sqrt{3})^{2x-1}}{3^{10}} = \frac{\sqrt{3}}{27^{x-1}}$

(5) $(0.3\sqrt{10})^{2x} \times 0.9^{x^2} = 0.81$

(6) $12.5^x \div (0.2\sqrt{2})^{x^2} = 0$

(7) $12.5^x \div (0.2\sqrt{2})^{x^2} = 31.25\sqrt{2}$

(8) $8x^{\log_{0.5} x + 0.5} = 1$

(9) $x^{2\log x + 5} = 1000$

(10) $x^{2\log_{0.5} x + 7} = 8$

Q22. The half-life of a radioactive element is modelled by the formula

$$A(t) = t_0 e^{kt}, t \geq 0$$

where A is the amount of radioactive element in grams after t years and t_0 is a constant.

(i) Given that when $t = 0$ the amount of radioactive element is 6.6 grams find t_0.

(ii) Find the value of k if after 10 years there is 4.95 grams of the radioactive element left.

(iii) Find the amount of the radioactive element left after 100 years.

(iv) Find the half-life (i.e. the time taken to decay to half of its original amount).

Q23. Solve the inequalities.

(1) $\log_{\frac{\pi}{4}} x + \log_{\frac{\pi}{4}} x^2 + 9 > 0$

(2) $\log_{\frac{\sqrt{2}}{2}} x \leq 6$

(3) $\log^2_{\frac{1}{2}} x^2 < 3\log^2_2 x$

(4) $4^{x-5} \geq 64^{x-2} \times 128$

(5) $(\frac{2\sqrt{3}}{3\sqrt{2}})^{2x} \div 1.5^{x^2} > 2.25$

(6) $(\frac{1}{4})^{x-1} < \frac{6}{2^{2x-1}}$

(7) $\log^2_5 x \leq 4\log_{x^2} \sqrt{x}$

(8) $\log_{5^2} x^2 + 2\log_{\sqrt{5^3}} x \geq \geq \frac{5}{2}\log_5 64$

Q24. Consider the function $f(x) = 2\ln|3\log x|$.

(i) Find the domain of f.

(ii) Find the product of zeroes of f.

Chapter 7

Proof by mathematical induction

Mathematical induction is a technique of a mathematical proof that can be used for a set of natural numbers or similar. It consists of two steps:

1. **(base case)** in which we proove that the statement considered is true for initial number (usually $n = 0$ or $n = 1$)
2. **(inductive step)** in which we show that if the statement holds for a certain number (k) then it holds for the next number ($k + 1$), too.

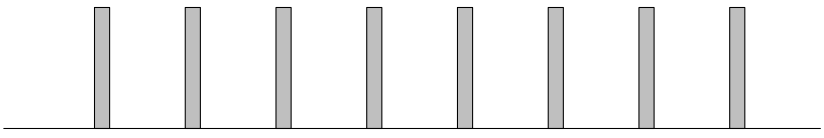
Let $S(n)$ denote the statement considered for a natural number n. If we want to prove by mathematical induction that the statement holds e.g. for all $n \in \mathbb{N}$, $n \geq 1$ we have to prove that:

1. **(base case)** the statement is true for $n = 1$, i.e. $S(1)$
2. **(inductive step)** if the statement is true for k then it is true for $k + 1$, too, i.e. $S(k) \Rightarrow S(k + 1)$

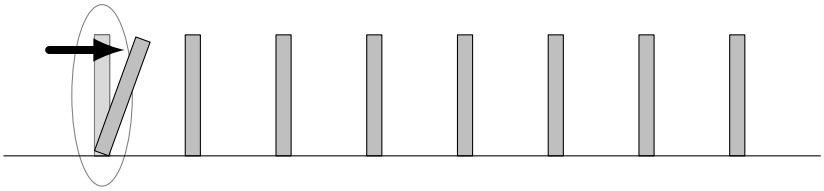
The **inductive step** gives us the following sequence of implications:

$$S(1) \Rightarrow S(2)$$
$$S(2) \Rightarrow S(3)$$
$$S(3) \Rightarrow S(4)$$

... and so on, so by mathematical induction the statement is true for $n \in \{1, 2, 3, ...\}$.
Mathematical induction can be illustrated by a sequence of falling dominoes.

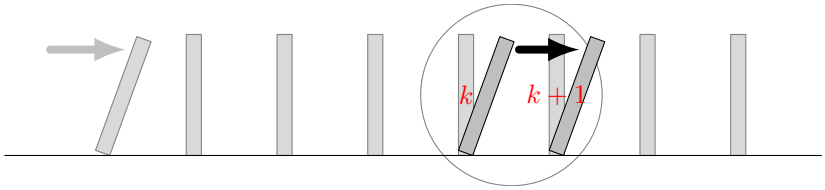


If we want to have all blacks fallen down, we have to ensure that:
1. **(base case)** the first block falls, i.e. $S(1)$

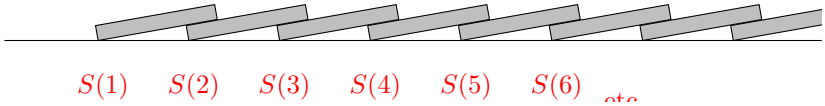


and

2. (inductive step) if block k falls, it pushes block $k+1$ so that it falls, too, i.e. $S(k) \Rightarrow S(k+1)$



On the basis of the two steps we can be sure that eventually all dominoes will fall.



It is good to know, what should be written to make the proof formally fully correct.

Let $S(n)$ be {*the statement in terms of* n}.
~~Let us show that $S(1)$ is true.~~
{*proof that* $S(1)$ *is true goes here*}
therefore $S(1)$ is true.
~~Let us now assume that $S(k)$ is true for an integer k.~~ (*)
We will show that $S(k+1)$ is true, i.e. {*the statement*}
{*proof that* $S(k+1)$ *is true goes here;*}
{*in the proof the assumption* (*) *must be used*}
~~Thus $S(k+1)$ is true whenever $S(k)$ is true.~~
As $S(1)$ is true, too,
by **MI** the statement $S(n)$ is true for $n \in \mathbb{Z}^+$.

Questions 7.1.

Prove the following statements using mathematical induction.

(1) $1+2+3+...+n = \frac{n(n+1)}{2}, n \in \mathbb{N}^+$

(2) $1+3+5+...+(2n-1) = n^2, n \in \mathbb{N}^+$

(3) $2+5+8+...+(3n-1) = \frac{3}{2}n^2 + \frac{1}{2}n, n \in \mathbb{N}^+$

(4) $3+7+11+...+(4n-1) = 2n^2 + n, n \in \mathbb{N}^+$

(5) $4+6.5+9+11.5+...+(\frac{5n+3}{2}) = \frac{5}{4}n^2 + \frac{11}{4}n, n \in \mathbb{N}^+$

(6) $3+9+27+...+3^n = \frac{3^{n+1}}{2} - \frac{3}{2}, n \in \mathbb{N}^+$

(7) $1+2+4+8+...+2^n = 2^{n+1} - 1, n \in \mathbb{N}$

(8) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}, n \in \mathbb{N}^+$

(9) $3+2+\frac{4}{3}+...+\frac{2^n}{3^{n-1}} = 9 - \frac{2^{n+1}}{3^{n-1}}, n \in \mathbb{N}$

(10) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1$

(11) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}, n \geq 1$

(12) $\sum_{k=1}^{n} (2k-1)^2 = \frac{n(2n-1)(2n+1)}{3}, n \geq 1$

(13) $2^n > n^2, n \in \mathbb{N}, n \geq 5$

(14) $n! > 2^n, n > 4$

(15) 3 divides $n^3 + 2n, n \in \mathbb{N}^+$

(16) 3 divides $5^n + 2^{n+1}, n \in \mathbb{N}^+$

(17) 80 divides $3^{4n} - 1, n \in \mathbb{N}^+$

(18) 4 divides $7^{2n} - 1, n \in \mathbb{N}^+$

(19) 4 divides $7^{2n+1} + 1, n \in \mathbb{N}^+$

(20) 5 divides $8^n - 3^n, n \in \mathbb{N}^+$

(21) 4 divides $7^n - 3^n, n \in \mathbb{N}^+$

(22) 5 divides $7^{2n} - 3^{2n}, n \in \mathbb{N}^+$

(23) 3 divides $5^n + 2 \times 11^n, n \in \mathbb{N}^+$

(24) 9 divides $5^{2n+1} + 4^{2n+1}, n \in \mathbb{N}^+$

(25) if $P_0(x) = 1$, $P_1(x) = 2x + 1$ and $P_n(x) = xP_{n-1}(x) + P_{n-2}(x)$ for $n > 1$ then $P_n(x)$ is a polynomial of degree n for $n \in \mathbb{N}$

(26) $(x - 1)$ is a factor of $(x^n - 1)$, $n \in \mathbb{N}^+$

(27) $\frac{1}{x(x+1)} + \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \cdots + \frac{1}{(x+n-1)(x+n)} = \frac{n}{x(x+n)}$, $x > 0, n \in \mathbb{N}^+$

(28) $\frac{1}{x-1} + \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \cdots + \frac{2^n}{x^{(2^n)}+1} = \frac{2^{n+1}}{x^{(2^{n+1})}+1}$, $x \in \mathbb{R} \setminus \{-1, 1\}, n \in \mathbb{N}$

(29) Let H_n be the minimum number of moves needed to solve the *towers of Hanoi* problem for n discs. Prove by mathematical induction that $H_n = 2^n - 1$.

(30) The *Fibonacci sequence* is defined as follows:

$$\begin{cases} F_0 = & 0 \\ F_1 = & 1 \\ F_n = & F_{n-1} + F_{n-2} \text{ for } n \geq 2 \end{cases}$$

Prove by mathematical induction that

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right) \text{ for } n \in \mathbb{N}$$

(31) Let $f(x) = \frac{2x}{2x-1}$. Let

$$f^n(x) = \underbrace{(f \circ f \circ \cdots \circ f)}_{n \text{ times}}(x)$$

Show by mathematical induction that $f^n(x) = \frac{3 \times 2^n x}{2^{n+1}x + 2(-1)^{n+1}x + (-1)^n}$ for $n \in \mathbb{Z}^+$.

(32) Consider $f(x) = \sin 2x + \cos 3x, x \in \mathbb{R}$. Show by mathematical induction that:

$$f^{(2n+1)}(x) = (-1)^n (2^{(2n+1)} \cos 2x - 3^{(2n+1)} \sin 3x),$$

where $f^{(n)}$ denotes n^{th} derivative of f.

(33) Prove by mathematical induction that for $n \in \mathbb{Z}^+$:

$$(n + 1) + (n + 2) + (n + 3) + \cdots + (2n) = \frac{n(3n + 1)}{2}$$

Part III

FUNCTIONS AND EQUATIONS

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Chapter 8

Graphs of rational functions

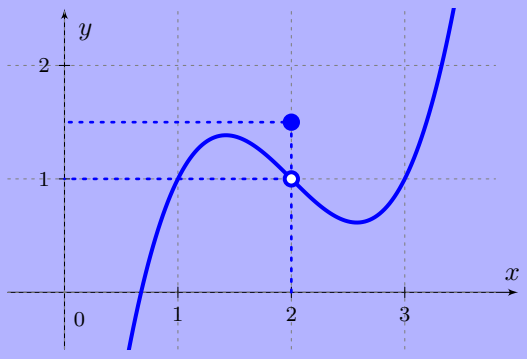
8.1 Limit of a function

Let $f : A \longrightarrow B$ be a function, where $A, B \subseteq \mathbb{R}$.

Limit at a point

Limit of function f at a point $a \in A$ is understood as a value to which values of f tend when x's apporach a. For *continuous* [1] functions limit of a function f at a is simply $f(a)$.

Example 52 (limit of a function at a point for an incontinuous function).
When the function is not continuous, *the situation can be slightly different, e.g. as shown in the figure below.*



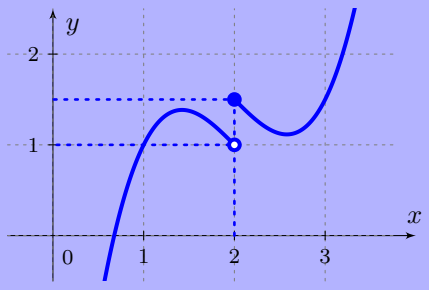
The limit of the function at $x = 2$ is 1, while the value is $f(2) = 1.5$.
We write it as:

$$\lim_{x \to 2} f(x) = 1.$$

Sometimes the limit of function f at $x = a$ is different, when x's approach a from the left and from the right.

[1] *Continuous* function will not be defined here. We are to understand the term intuitively.

EXAMPLE 53 (left limit and right limit of a function).
The limit of the function with a graph shown below at $x = 2$ from the left differs from the limit from the right.



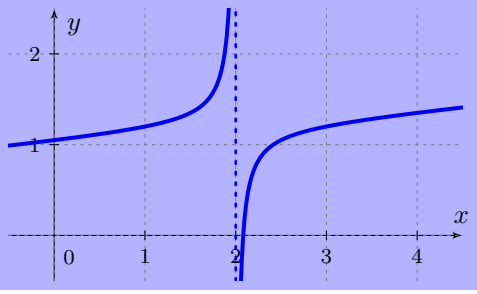
The limit of the function at $x = 2$ from the left is 1, while the limit from the right is equal to the value of the function, i.e. $f(2) = 1.5$.

NOTATION.
We write it as:

$$\lim_{x\to 2^-} f(x) = 1, \qquad \lim_{x\to 2^+} f(x) = 1.5, \qquad f(2) = 1.5.$$

Sometimes the limit of a function at a point is not a finite value.

EXAMPLE 54 (infinite limit of a function at a point).
The limit of the function at $x = 2$ is infinite *both from the left and from the right.*



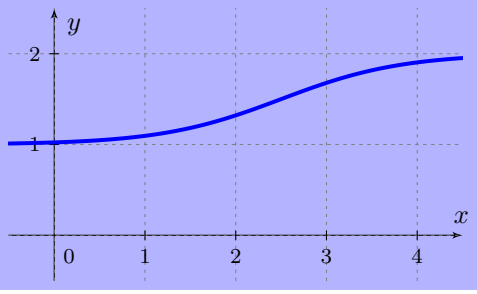
NOTATION.
We write it as:

$$\lim_{x\to 2^-} f(x) = +\infty, \qquad \text{and} \qquad \lim_{x\to 2^+} f(x) = -\infty.$$

Limit at infinity

Limit of function f at infinity (either $x \to +\infty$ or $x \to -\infty$) is understood as a value to which values of f tend when x's approach $+\infty$ or $-\infty$ respectively.

EXAMPLE 55 (limit of a function at infinity).
The limit of the function at $-\infty$ is 1 *while at $+\infty$ it is* 2.

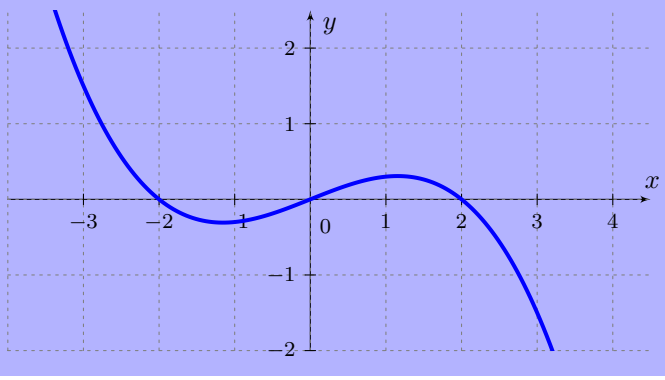


NOTATION.
The figure above shows a graph of function f such that

$$\lim_{x\to -\infty} f(x) = 1 \text{ and } \lim_{x\to +\infty} f(x) = 2.$$

It is also possible that a limit of a function at infinity is infinite.

EXAMPLE 56 (infinite limit of a function at infinity).
The limits of the function at both $-\infty$ and $+\infty$ are infinite.



NOTATION.
In the situation above

$$\lim_{x\to-\infty} f(x) = +\infty \text{ and } \lim_{x\to+\infty} f(x) = -\infty.$$

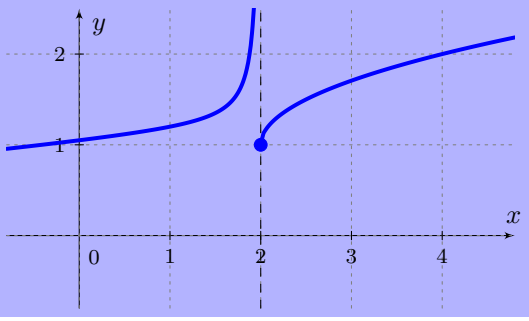
8.2 ASYMPTOTES

Let $f : A \longrightarrow B$ be a function, where $A, B \subseteq \mathbb{R}$.
An asymptote of the graph of f is a line such that the distance between the graph and the line approaches zero as they both tend to infinity.

Vertical asymptotes

Whenever limit of f at a point a is infinite, a graph of f has a **vertical** asymptote given by $x = a$. Figure **??** shows a vertical asymptote given by $x = 2$.
When a limit from one side is infinite while a limit from the other side is finite, the line $x = a$ is one-sided asymptote of the graph.

EXAMPLE 57 (vertical asymptote).
One-sided vertical asymptote. $\left(\lim_{x\to2^-} f(x) = +\infty\right)$



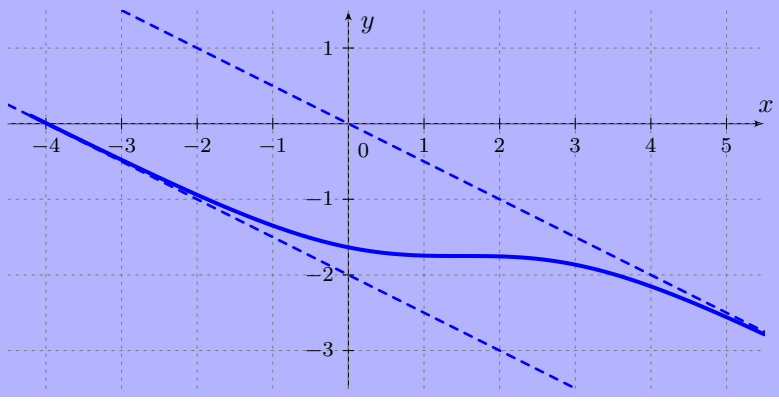
Horizontal asymptote

Whenever limit of f at $-\infty$ **or** at $+\infty$ is finite (i.e. $\lim_{x\to\pm\infty} f(x) = b \in \mathbb{R}$), a graph of f has a **horizontal** asymptote $y = b$. In example 55 there are two different horizontal asymptotes: $y = 1$ at $-\infty$ and $y = 2$ at $+\infty$.

Oblique asymptote

When a limit of f at $-\infty$ **or** at $+\infty$ is infinite (i.e. $\lim_{x\to\pm\infty} f(x) = \pm\infty$), a graph of f may have an **oblique** asymptote.

EXAMPLE 58 (oblique asymptote).
A graph of function with two oblique asymptotes: $y = -\frac{1}{2}x - 2$ at $-\infty$ and $y = -\frac{1}{2}x$ at $+\infty$



However, the condition $\lim_{x\to\pm\infty} f(x) = \pm\infty$ does not imply the existence of an oblique asymptote, as can be seen in example 56.

8.3 Rectangular hyperbola

In this section we are going to investigate graphs of functions defined by

$$y = \frac{ax+b}{cx+d},$$

where a, b, c, $d \in \mathbb{R}$ are constant parameters and $x \in \mathbb{R}$, $x \neq -\frac{d}{c}$.

EXAMPLE 59 (graph of a rational function).
Sketch the graph of a function $f(x) = \frac{2x+3}{x+4}, x \neq -4$.

We are going to investigate the shape of graph of f by considering the following factors.

Factor 1. zero of the function (x-intercept of the graph)

Factor 2. y-intercept of the graph

Factor 3. vertical asymptote

Factor 4. horizontal asymptote

For $f(x) = \frac{2x+3}{x+4}$:

Factor 1. zero of the function

In our particular case:

$$\begin{aligned} f(x) = 0 &\Leftrightarrow \frac{2x+3}{x+4} = 0 \\ &\Leftrightarrow 2x+3 = 0 \\ &\Leftrightarrow x = -\frac{3}{2} \end{aligned}$$

or, more generally:

$$\begin{aligned} f(x) = 0 &\Leftrightarrow \frac{ax+b}{cx+d} = 0 \\ &\Leftrightarrow ax+b = 0 \\ &\Leftrightarrow x = -\frac{b}{a} \end{aligned}$$

Thus, the x-intercept of the graph is $\left(-\frac{b}{a}, 0\right) = \left(-\frac{3}{2}, 0\right)$.

Factor 2. y-intercept of the graph

In our particular case:

$$\begin{aligned} f(0) &= \frac{2\times 0+3}{0+4} = \\ &= \frac{3}{4} \end{aligned}$$

or, more generally:

$$\begin{aligned} f(0) &= \frac{a\times 0+b}{c\times 0+d} = \\ &= \frac{b}{d} \end{aligned}$$

Thus, the y-intercept of the graph is $\left(0, \frac{b}{d}\right) = \left(0, \frac{3}{4}\right)$.

Factor 3. vertical asymptote As there is a *break* in the domain of the function at $x = -4$ (or $x = -\frac{d}{c}$ more generally), we will investigate the *limit* $\lim_{x\to -4} f(x)$. Actually, we will investigate two limits separately:

- $\lim_{x\to -4^-} f(x)$
  Let us find the values of f for a sequence of x's approaching -4 from the left, e.g.

$$-4.1, -4.01, -4.001, -4.0001...$$

$$\begin{aligned} f(-4.1) &= 52 \\ f(-4.01) &= 502 \\ f(-4.001) &= 5002 \\ f(-4.0001) &= 50002 \\ &\vdots \end{aligned}$$

  It is easy to realize that $\lim_{x\to -4^-} f(x) = +\infty$.

- $\lim_{x\to -4^+} f(x)$
  Let us find now the values of f for a sequence of x's approaching -4 from the right, e.g.

$$-3.9, -3.99, -3.999, -3.9999...$$

$$\begin{aligned} f(-3.9) &= -48 \\ f(-3.99) &= -498 \\ f(-3.999) &= -4998 \\ f(-3.9999) &= -49998 \\ &\vdots \end{aligned}$$

  It is easy to realize that $\lim_{x\to -4^+} f(x) = -\infty$.
  Finally, line $x = -4$ is both-sided vertical asymptote.

Factor 4. horizontal asymptote To find a horizontal asymptote we will investigate the tendency in values of f for x's approaching $-\infty$ and $+\infty$.

- $\lim_{x\to -\infty} f(x)$
  Let us find the values of f for a sequence of x's approaching $-\infty$, e.g.

$$-10,\ -10^2,\ -10^3,\ -10^4,\ -10^5,\ -10^6 \dots$$

(values in the table below are given correct to 5 d.p.)

x	-10	-10^2	-10^3	-10^4	-10^5	-10^6
$f(x)$	2.83333	2.05208	2.00502	2.00050	2.00005	2.00000

  From above values we can see that $\lim_{x\to -\infty} f(x) = 2$.

- $\lim_{x\to +\infty} f(x)$
  Let us now find the values of f for a sequence of x's approaching $+\infty$, e.g.

$$10, 10^2, 10^3, 10^4, 10^5, 10^6 \dots$$

(values in the table below are given correct to 5 d.p.)

x	10	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.64286	1.95192	1.99502	1.99950	1.99995	2.00000

  From above values we can see that $\lim_{x\to -\infty} f(x) = 2$, too.
  Finally, line $y = 2$ is a horizontal asymptote at both $-\infty$ and $+\infty$.

Thus:

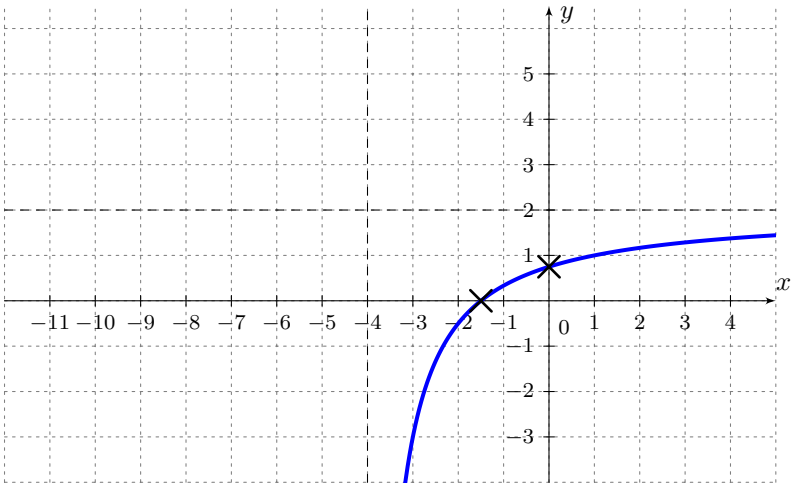
Factor 1. x-intercept of the graph: $\left(-\frac{3}{2}, 0\right)$

Factor 2. y-intercept of the graph: $\left(0, \frac{3}{4}\right)$

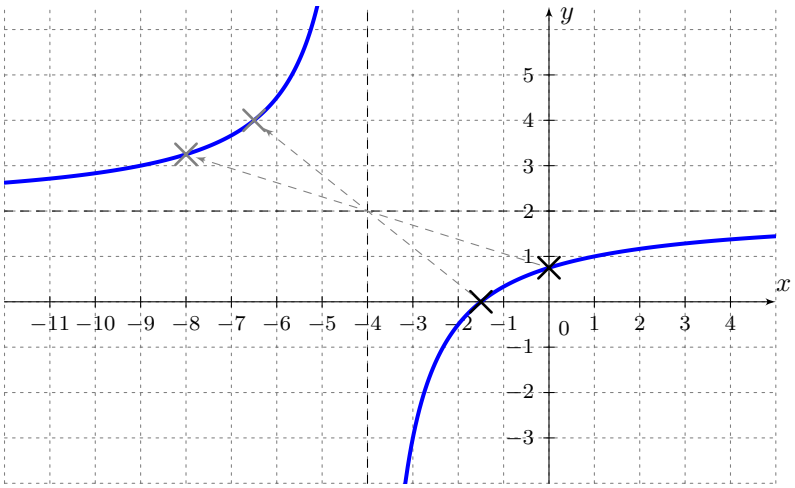
Factor 3. vertical asymptote: $x = -4$

Factor 4. horizontal asymptote: $y = 2$

Let us place both x- and y-intercept in the grid. When we join the two points and draw the curve on towards the asymptotes we will get:



To complete the graph we have got to make a symmetry of the curve in $(-4, 2)$ - the intersection of asymptotes:



To make the investigation complete, we should, however, ask ourselves a question:
Why an equation of the type $y = \frac{ax+b}{cx+d}$ *describes a hyperbola?*
Rearranging the equation we'll get:

$$\begin{aligned} y = \frac{ax+b}{cx+d} &= \frac{ax + \frac{ad}{c} - \frac{ad}{c} + b}{cx+d} = \\ &= \frac{\frac{a}{c}(cx+d) - \frac{ad}{c} + b}{cx+d} = \\ &= \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{c(x - \frac{d}{c})} = \\ &= \frac{a}{c} + \frac{k}{x - \frac{d}{c}} \end{aligned}$$

where $k = \frac{b - \frac{ad}{c}}{c}$.
This form looks a bit more complicated. However, it may be easily achieved from the basic hyperbola $y = \frac{1}{x}$ by a sequence of the following transformations:

Transformation 1. vertical dilation with a scale factor k $\longrightarrow y = \frac{k}{x}$

Transformation 2. translation by $\begin{pmatrix} \frac{d}{c} \\ \frac{a}{c} \end{pmatrix}$ $\longrightarrow y = \frac{k}{x-\frac{d}{c}} + \frac{a}{c}$

The meaning of the above mentioned transformations will be explained in chapter 9.

8.4 Reciprocal function

For function $f : A \to B$, A, $B \subseteq \mathbb{R}$ by *reciprocal* of $x \mapsto f(x)$ we understand the function $x \mapsto \frac{1}{f(x)}$. The domain of the reciprocal function is $A - A_0$, where A_0 is the set of zeroes of f.

The graph of $x \mapsto \frac{1}{f(x)}$ should be created according to the following rules:

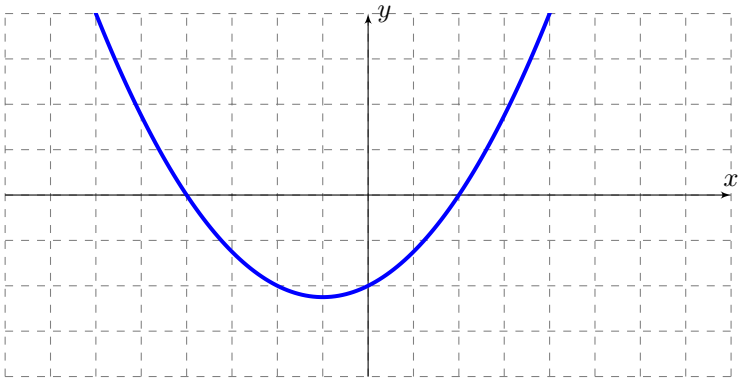
$f(x)$	$\frac{1}{f(x)}$
point $(x, f(x))$	point $\left(x, \frac{1}{f(x)}\right)$
zero $x = a$	vertical asymptote $x = a$
vertical asymptote $x = a$	zero $x = a$
	(Actually, the graph looks as if there was a zero at $x = a$. However, the point is not in the domain.)
increase	decrease
decrease	increase
minimum	maximum
maximum	minimum
horizontal asymptote $y = c$, $c \neq 0$	horizontal asymptote $y = \frac{1}{c}$
horizontal asymptote $y = 0$	graph tending to $\pm\infty$
graph tending to $\pm\infty$	horizontal asymptote $y = 0$
negative values	negative values
positive values	positive values
point $(x, \pm 1)$	point $(x, \pm 1)$

The horizontal line in the table above separates the factors that are to be changed from these that are to be kept unchanged.

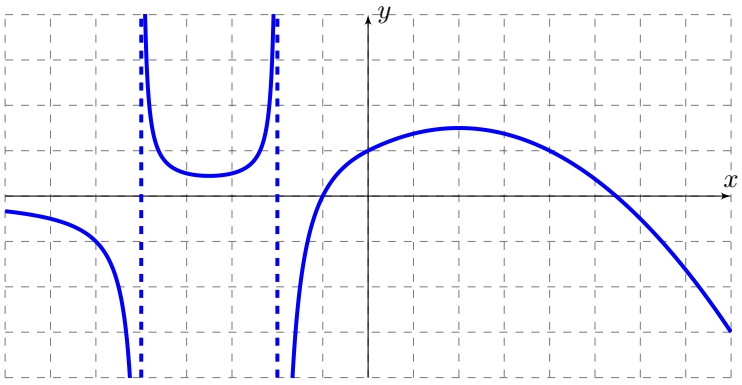
Questions 8.1.

Q1. The diagrams below shows the graphs of $y = f(x)$. In the same diagram sketch the graphs of $y = \frac{1}{f(x)}$.

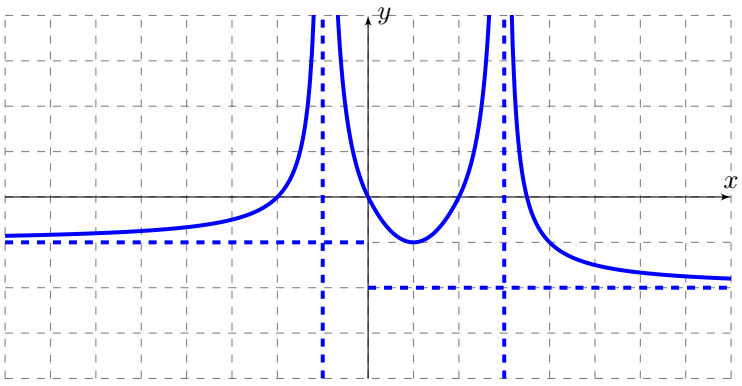
(1)



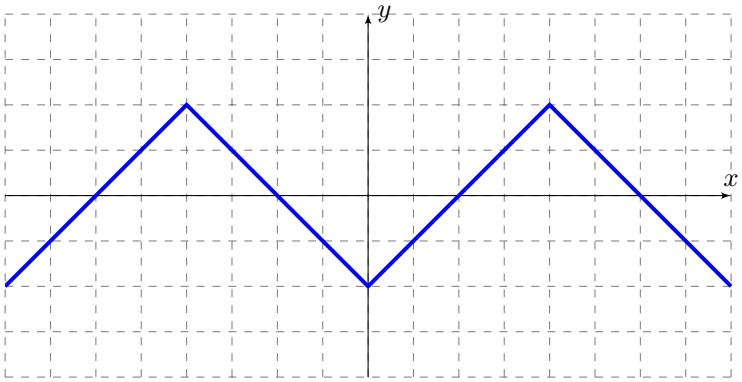
(2)



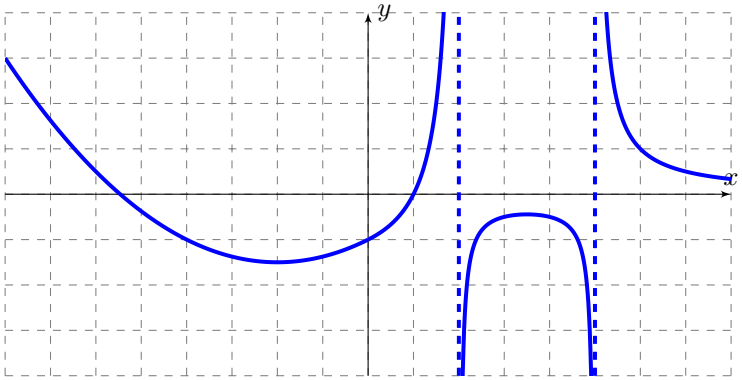
(3)



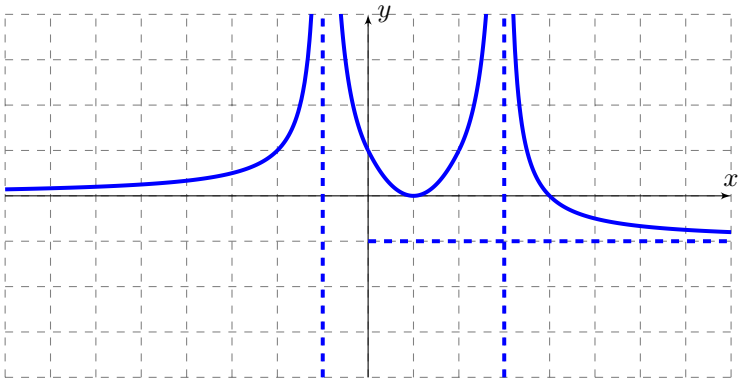
(4)



(5)



(6)



Q2.

- For each of the following functions consider its:
  - domain
  - zeroes
  - y-intercept
  - asymptotes (limits)
- Sketch the graph of each function.
- In the same diagram sketch the graph of $x \mapsto \frac{1}{y}$.

(1) $y = \frac{x-2}{x+1}$
(2) $y = \frac{2x-4}{x+3}$
(3) $y = \frac{2-x}{x+2}$
(4) $y = \frac{4-2x}{2x+3}$
(5) $y = \frac{2}{x^2-1}$
(6) $y = \frac{1}{x^2}$
(7) $y = \frac{x-2}{(x-1)(x+2)}$
(8) $y = \frac{(x-2)(x-1)}{(x+1)(x+2)}$
(9) $y = \frac{(x-2)^2}{(x+1)(x-2)}$
(10) $y = \frac{(x+1)(x-2)}{(x-2)^2}$
(11) $y = \frac{(x^2-4)(x-1)}{(x^2-1)(x+2)}$
(12) $y = \frac{(x+2)(x-1)}{x-2}$
(13) $y = \frac{x^2-1}{x}$
(14) $y = \frac{x^2}{x+1}$
(15) $y = \frac{x^2}{x^2-1}$

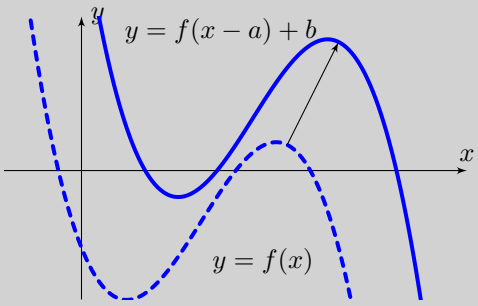
Chapter 9

Basic transformations of graphs of functions

9.1 Vector translation

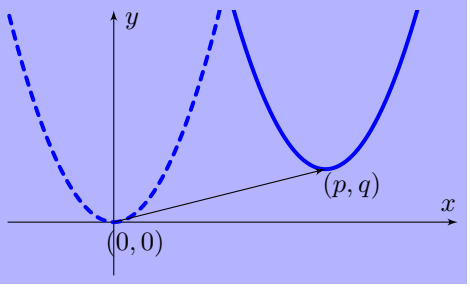
Theorem 60 (translation of a graph of a function).

When a graph of the function given by the equation $y = f(x)$ *is translated along the vector* $\begin{pmatrix} a \\ b \end{pmatrix}$ *the equation describing the transformed graph is given by the equation* $\boxed{y = f(x-a)+b}$ *or* $\boxed{y - b = f(x-a)}$.



Example 61 (translation of a graph).

When a parabola defined by $y = ax^2$ *(with a vertex at the* origin*) is moved by* $\begin{pmatrix} p \\ q \end{pmatrix}$*, the equation of the new parabola (with vertex at* (p, q)*) is* $\boxed{y = a(x-p)^2 + q}$.



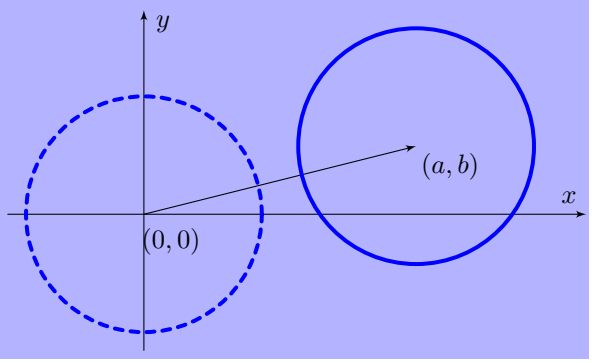
Corollary 62 (translation of a graph).

The above rule may be applied to the graph of any curve given by the equation $F(x, y) = 0$.

After translation by vector $\begin{pmatrix} a \\ b \end{pmatrix}$ *the equation describing the transformed curve is* $\boxed{F(x-a, y-b) = 0}$.

EXAMPLE 63 (translation of a circle).

When a circle defined by $x^2+y^2 = R^2$ (having centre at origin) is moved by $\begin{pmatrix} a \\ b \end{pmatrix}$, the equation of the new circle (with centre at (a, b)) is $\boxed{(x-a)^2 + (y-b)^2 = R^2}$.

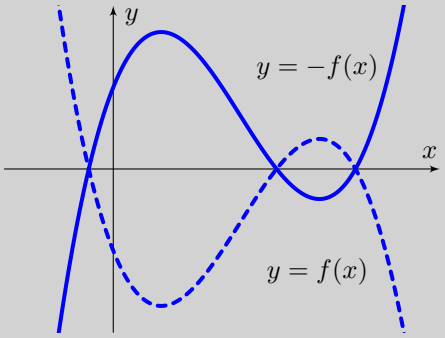


9.2 REFLECTIONS IN COORDINATE AXES

Reflection in x-axis

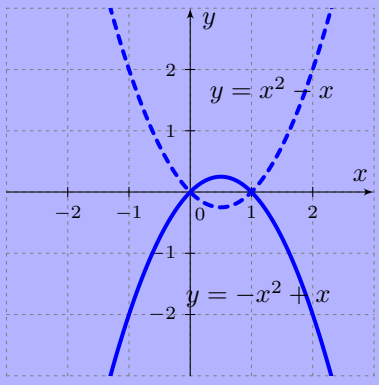
THEOREM 64 (reflection in x-axis).

When a graph of the function given by the equation $y = f(x)$ is reflected in the x-axis the equation describing the reflected graph is $\boxed{y = -f(x)}$.



EXAMPLE 65 (reflection in x-axis).

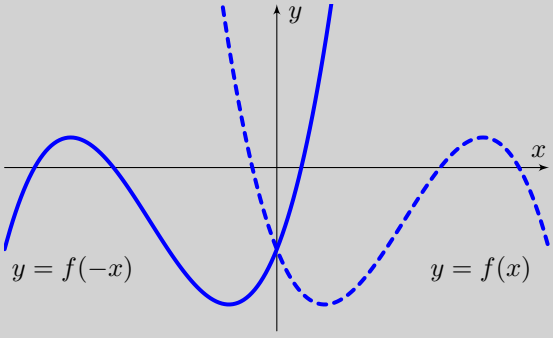
When a parabola defined by $y = x^2 - x$ is reflected in x-axis, the equation of the reflected parabola is $\boxed{y = -(x^2 - x)}$ *i.e.* $\boxed{y = -x^2 + x}$.



Reflection in y-axis

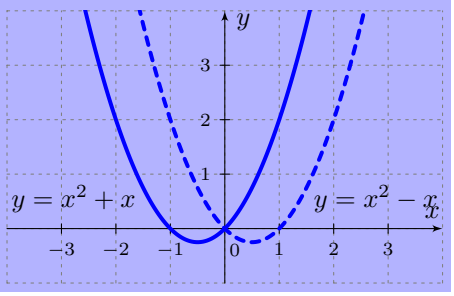
THEOREM 66 (reflection in y-axis).

When a graph of the function given by the equation $y = f(x)$ is reflected in they-axis the equation describing the reflected graph is $\boxed{y = f(-x)}$



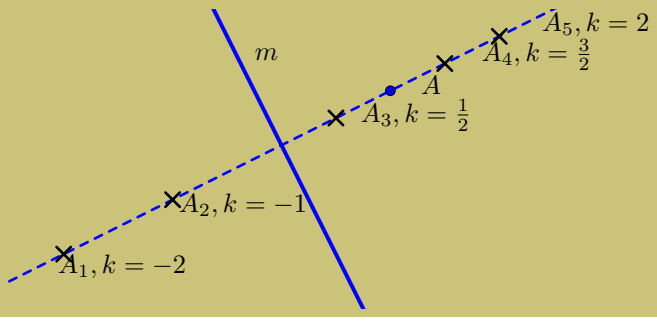
Example 67 (reflection in y-axis).

When a parabola defined by $y = x^2 - x$ is reflected in y-axis, the equation of the reflected parabola is $\boxed{y = (-x)^2 - (-x)}$ *i.e.* $\boxed{y = x^2 + x}$.



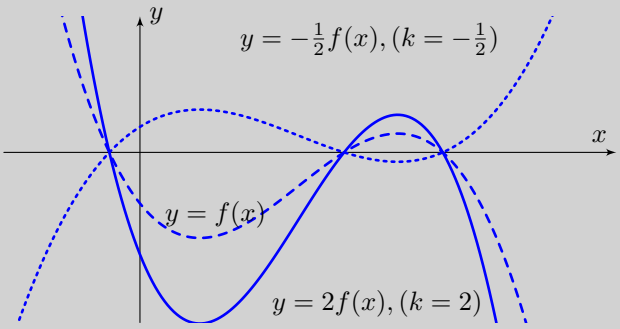
9.3 Dilations

Definition 68 (dilation).
We say that point A is dilated with respect to line m and with a scale factor k when the distance of the image of point A to line m is equal to k times the distance of A to line m.
⋆ If $k > 0$ the image of A and A itself lie on the same side of line m.
⋆ If $k < 0$ the image of A lies on the opposite side of line m than A.



Vertical dilation

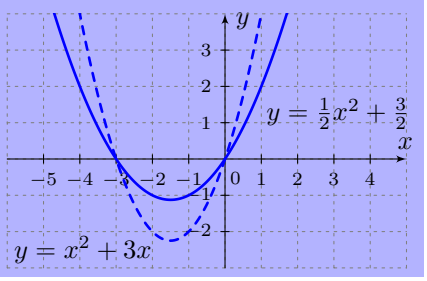
Theorem 69 (vertical dilation).
When a graph of the function given by the equation $y = f(x)$ is dilated vertically (with respect to the$x$-axis with a scale factor k the equation describing the transformed graph is $\boxed{y = kf(x)}$ *or* $\boxed{\frac{y}{k} = f(x)}$



Corollary 70 (vertical dilation with scale factor -1).
Vertical dilation with a scale factor -1 is the same as reflection in x-axis

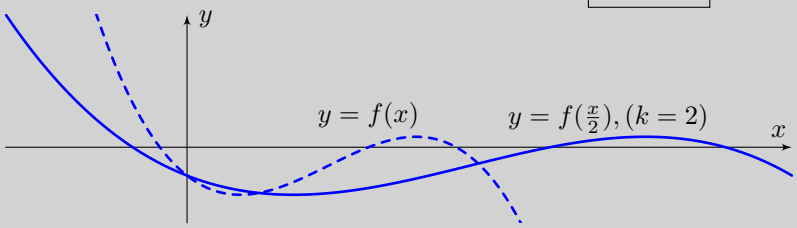
Example 71 (vertical dilation).

When a parabola defined by $y = x^2 + 3x$ is dilated vertically with a scale factor $\frac{1}{2}$, the equation of the transformed parabola is $\boxed{y = \frac{1}{2}(x^2 + 3x)}$ *i.e.* $\boxed{y = \frac{1}{2}x^2 + \frac{3}{2}x}$.



Horizontal dilation

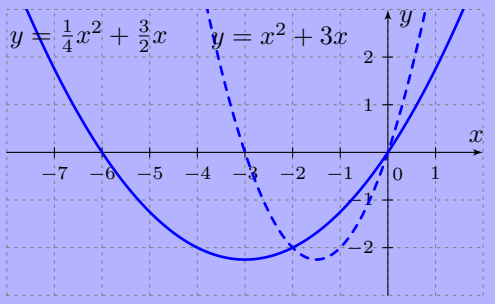
Theorem 72 (horizontal dilation).
When a graph of the function given by the equation $y = f(x)$ is dilated horizontally with respect to the y-axis with a scale factor k the equation describing the transformed graph is $\boxed{y = f(\frac{x}{k})}$.



Corollary 73 (horizontal dilation with scale factor -1).
Horizontal dilation with a scale factor -1 is the same as reflection in y-axis

Example 74 (horizontal dilation).

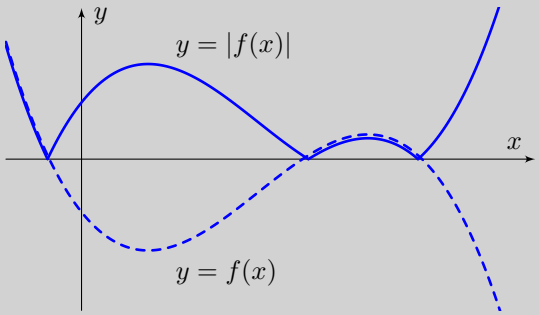
When a parabola defined by $y = x^2 + 3x$ is dilated horizontally with a scale factor 2, the equation of the transformed parabola is $\boxed{y = (\frac{x}{2})^2 + 3(\frac{x}{2})}$ *i.e.* $\boxed{y = \frac{1}{4}x^2 + \frac{3}{2}x}$.



9.4 Absolute value $y = |f(x)|$

Theorem 75 (absolute value transformation $y = |f(x)|$).

When the equation of the function $y = f(x)$ is changed into $\boxed{y = |f(x)|}$
the part of the graph below the x-axis is reflected in x-axis, while
the part of the graph above the x-axis stays unchanged.



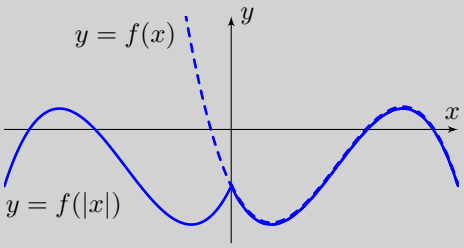
Corollary 76 (about the transformation $y = |f(x)|$).

- *When $f(x) \geq 0$ for all x, the transformation $y = f(x) \rightarrow$* $\boxed{y = |f(x)|}$ *does not affect the graph.*
- *When $f(x) \leq 0$ for all x, the transformation $y = f(x) \rightarrow$* $\boxed{y = |f(x)|}$ *is the same as reflection in x-axis.*

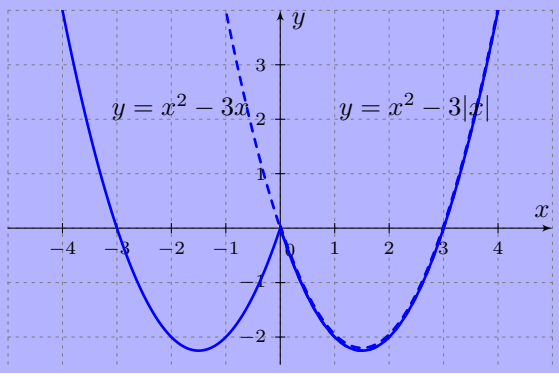
9.5 Absolute value $y = f(|x|)$

Theorem 77 (absolute value transformation $y = f(|x|)$).

When the equation of the function $y = f(x)$ is changed into $\boxed{y = f(|x|)}$
the part of the graph right from the y-axis stays unchanged and is reflected in y-axis, while
the part of the graph left from the y-axis disappears.



EXAMPLE 78 ($y = f(|x|)$).
When a quadratic function $y = x^2 - 3x$ is changed into $\boxed{y = |x|^2 - 3|x|}$ *(i.e.* $\boxed{y = x^2 - 3|x|}$*) its graph is changed as follows:*



COROLLARY 79 (about transformation $y = f(|x|)$).
- *When f is an **even** function, the transformation $y = f(x) \to \boxed{y = f(|x|)}$ does not affect the graph.*
- *When f is an **odd** function such that $f(x) \geq 0$ for $x > 0$, the transformations $y = f(x) \to \boxed{y = f|(x)|}$ and $y = f(x) \to \boxed{y = f(|x|)}$ give the same result.*

9.6 COMPOSITIONS OF TRANSFORMATIONS

When a graph of function $y = g(x)$ can be obtained from the graph of $y = f(x)$ by a sequence of *more* than one basic transformations, this usually may be done in more than one way.

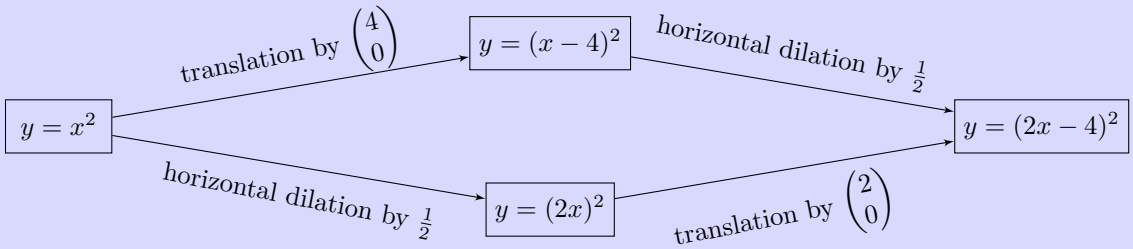
EXAMPLE 80 (composition of transformations).
Consider $f(x) = x^2$ and $g(x) = (2x - 4)^2$.
What is the sequence of transformations that change the graph of f into the graph of g?

Function g can be obtained from function f by:

A. translation followed by **horizontal dilation** or

B. horizontal dilation followed by **translation**

This is shown in the diagram below.



Note. While the dilation scale factor is the same ($= \frac{1}{2}$) in cases **A** and **B**, the translation vectors are different ($\begin{pmatrix}4\\0\end{pmatrix}$ and $\begin{pmatrix}2\\0\end{pmatrix}$ respectively). Let us investigate both cases separately.

A. translation by $\begin{pmatrix}4\\0\end{pmatrix}$ *followed by* **horizontal dilation** with scale factor $\frac{1}{2}$

(Step 1) As translation by vector $\begin{pmatrix}a\\b\end{pmatrix}$ changes the graph of $y = f(x)$ into $y = f(x - a) + b$,
the graph of $\boxed{y = x^2}$ when translated by $\begin{pmatrix}4\\0\end{pmatrix}$
will change into the graph of $y = (x - 4)^2 + 0$, i.e. $\boxed{y = (x - 4)^2}$.

(Step 2) As horizontal dilation with a scale factor k changes the graph of $y = f(x)$ into $y = f(\frac{x}{k})$,
the graph of $\boxed{y = (x-4)^2}$ when dilated horizontally by $\frac{1}{2}$
will change into the graph of $y = (\frac{x}{(\frac{1}{2})} - 4)^2$, i.e. $\boxed{y = (2x-4)^2}$.

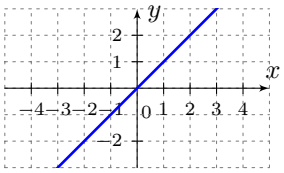
B. horizontal dilation with scale factor $\frac{1}{2}$ *followed by* **translation** by $\begin{pmatrix}2\\0\end{pmatrix}$

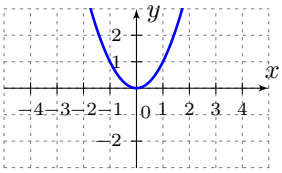
(Step 1) As horizontal dilation with a scale factor k changes the graph of $y = f(x)$ into $y = f(\frac{x}{k})$,
the graph of $\boxed{y = x^2}$ when dilated horizontally by $\frac{1}{2}$
will change into the graph of $y = (\frac{x}{(\frac{1}{2})})^2$, i.e. $\boxed{y = (2x)^2}$.

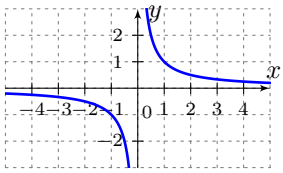
(Step 2) As translation by vector $\begin{pmatrix}a\\b\end{pmatrix}$ changes the graph of $y = f(x)$ into $y = f(x-a)+b$,
the graph of $\boxed{y = (2x)^2}$ when translated by $\begin{pmatrix}2\\0\end{pmatrix}$
will change into the graph of $y = (2(x-2))^2 + 0$, i.e. $\boxed{y = (2x-4)^2}$.

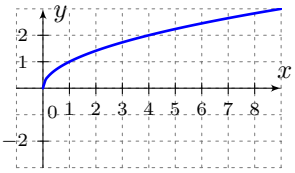
Basic parent functions

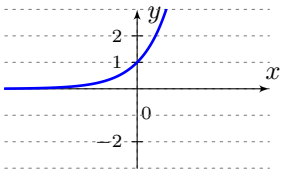
The following diagrams show examples of graphs of the *so-called* **parent functions**. We will use them as starting functions for further transformations.

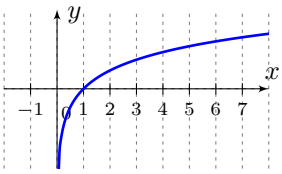

linear
$y = x$

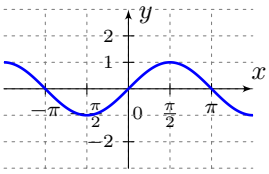

quadratic
$y = x^2$

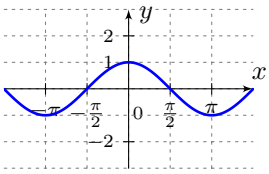

hyperbola
$y = \frac{1}{x}$


square root
$y = \sqrt{x}$


exponential
$y = 2^x$ or $y = e^x$

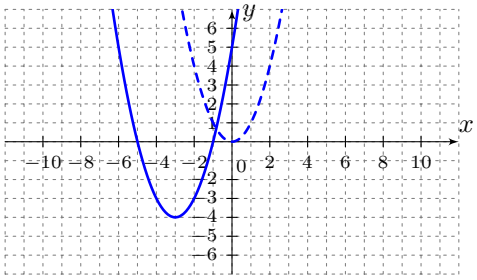

logarithmic
$y = \log_2 x$ or $y = \ln x$

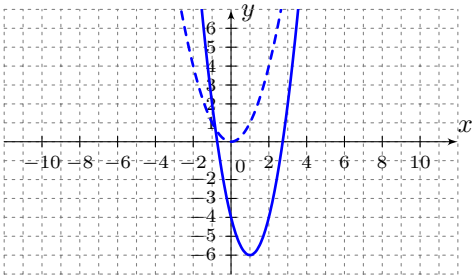

sine
$y = \sin x$

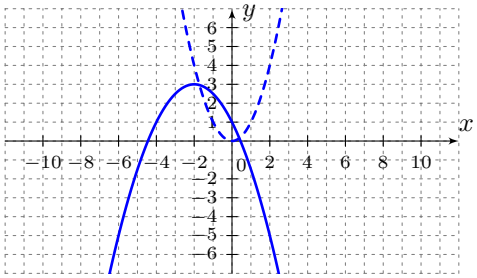

cosine
$y = \cos x$

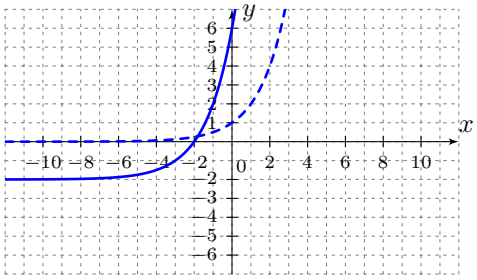
Questions 9.1.

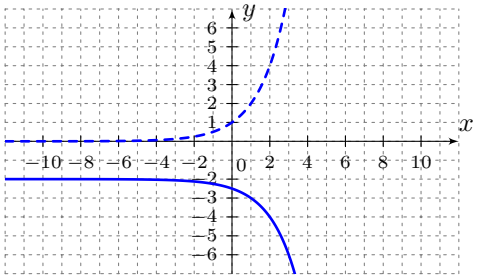
Q1. For each diagram separately write down the equation of the parent function (the dotted graph).
Find a sequence of transformations changing it into the solid graph. Hence write down its equation.

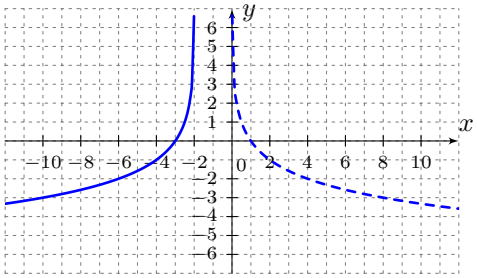
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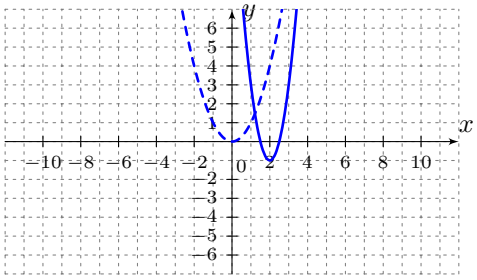
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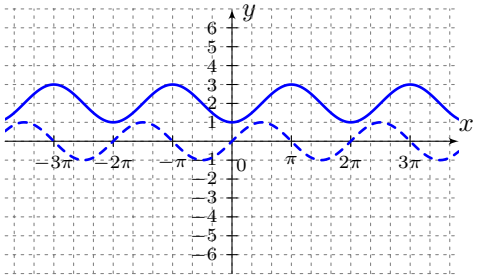
(3) 

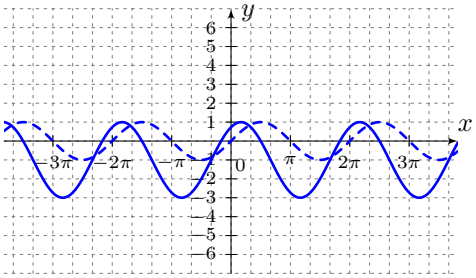
(4) 

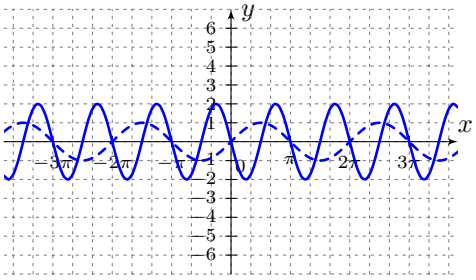
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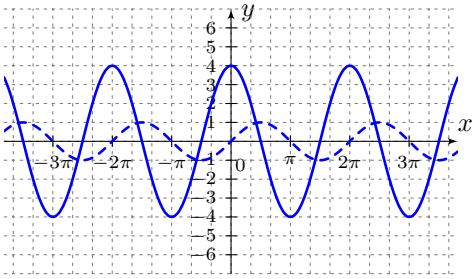
(6) 

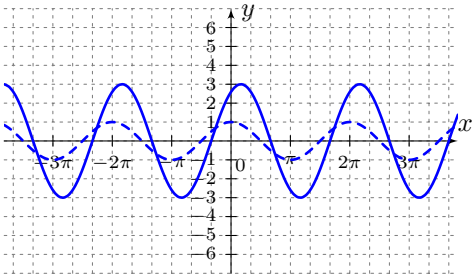
(7) 

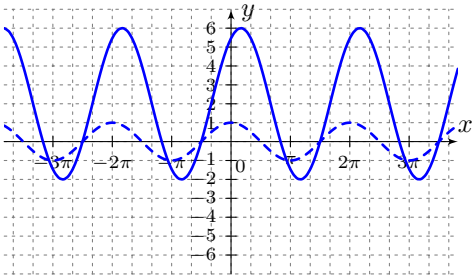
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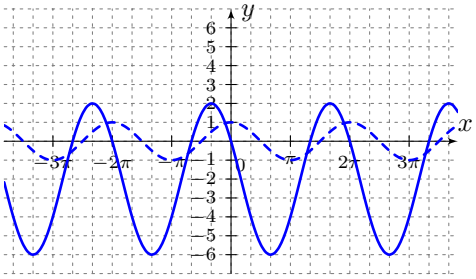
(9) 

(10) 

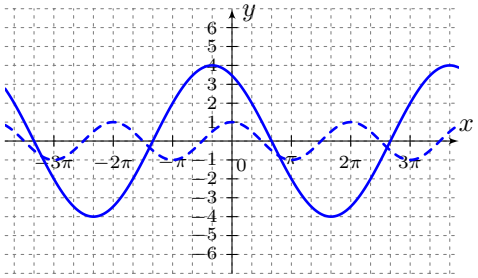
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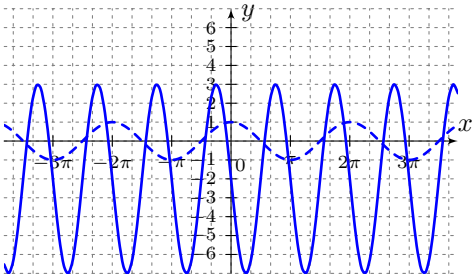
(13) 

(14) 

(15)



(16)



Q2. Which of the basic transformations was applied to the graph of $y = x^2 - 3x + 2$ if the resultant graph was defined by the equation:

(1) $y = |x^2 - 3x + 2|$
(2) $y = x^2 + 3x + 2$
(3) $y = \frac{1}{4}x^2 - \frac{3}{2}x + 2$
(4) $y = (x + 2)^2 - 3x - 2$
(5) $y = \frac{1}{2}x^2 - \frac{3}{2}x + 1$
(6) $y = -x^2 + 3x - 2$
(7) $y = x^2 - 3|x| + 2$

Q3. Find the vector that translates the graph of $y = f(x)$ onto the graph of $y = g(x)$.

(1)	$f(x) = x^2$	$g(x) = (x - 4)^2$
(2)	$f(x) = (2x)^2$	$g(x) = (2x - 4)^2$
(3)	$f(x) = \ln x$	$g(x) = \ln(x - 2) - 2$
(4)	$f(x) = \ln(-x)$	$g(x) = \ln(-x - 2)$
(5)	$f(x) = \sin x$	$g(x) = \sin(x + \frac{\pi}{2}) + 1$
(6)	$f(x) = \sin(-2x)$	$g(x) = \sin(\frac{\pi}{2} - 2x)$
(7)	$f(x) = e^{2x-3}$	$g(x) = e^{2x+1}$
(8)	$f(x) = 1 - \frac{2}{3x+2}$	$g(x) = 2 - \frac{2}{3x-1}$

Q4. The graph of $y = f(x)$ is transformed into the graph of $y = g(x)$ under vertical dilation with a scale factor k. Find the value of k.

(1)	$f(x) = x^2 - 2x + 3$	$g(x) = 2x^2 - 4x + 6$
(2)	$f(x) = x^3 - 3x$	$g(x) = 6x - 2x^3$
(3)	$f(x) = \ln x$	$g(x) = \ln x^3$
(4)	$f(x) = 2\sin x$	$g(x) = -\frac{1}{2}\sin x$
(5)	$f(x) = 1 - \frac{2}{3x-2}$	$g(x) = -2 - \frac{4}{2-3x}$

Q5. The graph of $y = f(x)$ is transformed into the graph of $y = g(x)$ under horizontal dilation with a scale factor k. Find the value of k.

(1)	$f(x) = x^2$	$g(x) = 4x^2$
(2)	$f(x) = (2x)^2$	$g(x) = (\frac{1}{2}x)^2$
(3)	$f(x) = \ln x$	$g(x) = \ln x - \ln 3$
(4)	$f(x) = \ln(-x)$	$g(x) = \ln x + \ln 2$
(5)	$f(x) = \sin x$	$g(x) = \sin(-\frac{2}{3}x)$
(6)	$f(x) = \sin(-2x)$	$g(x) = \sin(3x)$
(7)	$f(x) = e^{2x}$	$g(x) = \sqrt{e^x}$
(8)	$f(x) = 1 - \frac{2}{3x+2}$	$g(x) = 1 - \frac{1}{3x+1}$

Q6. Sketch the graphs of the following functions.

(1) $f(x) = x^2 - 2|x| - 3$
(2) $f(x) = x^2 + 2|x| - 3$
(3) $f(x) = |4 - \sqrt{x + 4}|$
(4) $f(x) = -|4 - \sqrt{4 - x}|$
(5) $f(x) = |1 - \sqrt{x + 4}|$
(6) $f(x) = |2 - \sqrt{x + 1}|$
(7) $f(x) = -|1 - \sqrt{4 - x}|$
(8) $f(x) = 2^{|x|+2}$
(9) $f(x) = e^{|x|-2}$
(10) $f(x) = \log_2(|x| + 2)$

Q7. Sketch the graphs of the following functions.

(1) $f(x) = \sin(x - \frac{\pi}{3})$
(2) $f(x) = \tan(x + \frac{\pi}{4})$
(3) $f(x) = \sin(x - \frac{\pi}{2}) - 1$
(4) $f(x) = -\sin(0.5x)$
(5) $f(x) = -\cos(0.5x)$
(6) $f(x) = -2\sin(x - \frac{\pi}{2})$
(7) $f(x) = -3\cos(x + \frac{\pi}{2})$
(8) $f(x) = \sin(2x) - \frac{1}{2}$
(9) $f(x) = \cos(2x) + \frac{1}{2}$
(10) $f(x) = \cos(2x - \pi)$
(11) $f(x) = \sin(2x - \pi)$
(12) $f(x) = \cos(2x + \pi)$

(13) $f(x) = \sin(2x + \pi)$
(14) $f(x) = \sin(\pi - x)$
(15) $f(x) = \sin(\frac{\pi}{2} - x)$
(16) $f(x) = \cos(\frac{\pi}{2} - x)$
(17) $f(x) = \tan(\frac{\pi}{2} - x)$
(18) $f(x) = 2\sin(2x - \frac{\pi}{2}) + 1$

Q8. How to transform the graph of $y = f(x)$ to get the graph of $y = g(x)$ using translations and symmetries only?

(1) $f(x) = \tan x$, $g(x) = \cot x$
(2) $f(x) = \cot x$, $g(x) = \tan x$
(3) $f(x) = \ln(x - 2)$, $g(x) = e^{x-2}$
(4) $f(x) = \ln x - 2$, $g(x) = e^x - 2$

Q9. Sketch in the same set of axes graphs of the function $f(x)$ and $\frac{1}{f(x)}$.

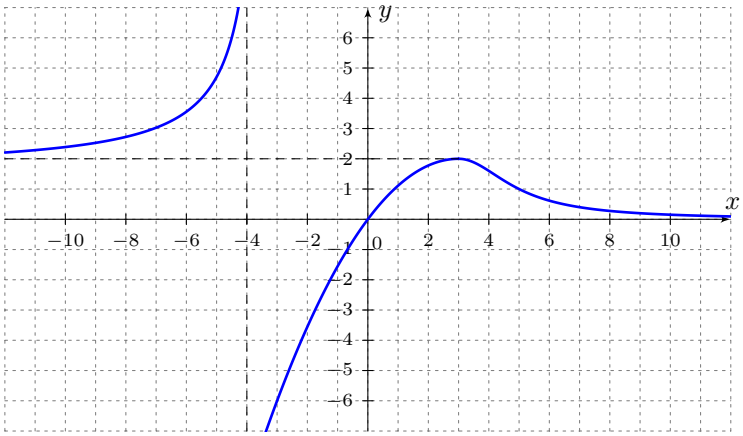
(1) $f(x) = x^2$
(2) $f(x) = \frac{x+1}{x-1}$
(3) $f(x) = x + 3$
(4) $f(x) = \frac{1}{2}x^2 - 2$
(5) $f(x) = x^2 - x - 6$
(6) $f(x) = \frac{x-2}{x+2}$
(7) $f(x) = \frac{x^2-1}{x^2}$
(8) $f(x) = \frac{x+2}{x^2-1}$
(9) $f(x) = \cos 2x$
(10) $f(x) = \tan 2x$

Q10. Sketch the graphs of the following functions.

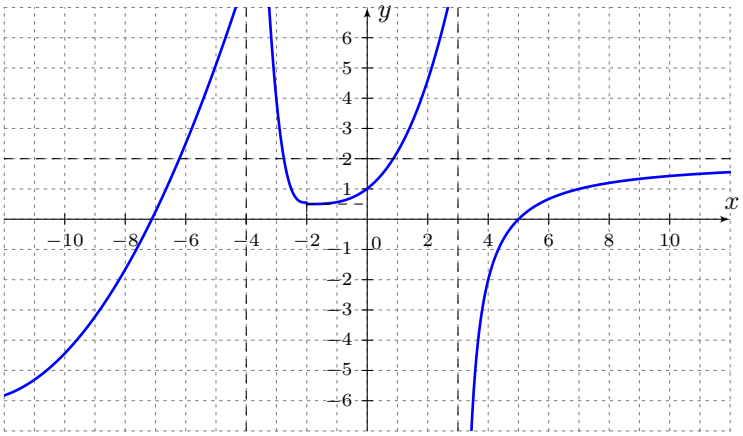
(1) $g(x) = |2 - \sqrt{|x|}|$
(2) $f(x) = |x^2 - 2x - 3| - 2$
(3) $g(x) = (|x| - 1)^3$
(4) $h(x) = \frac{(2x-4)(x+2)}{(x-1)(x+1)}$
(5) $h(x) = \frac{(x-2)(x+4)}{(x-3)(x+3)}$

Q11. (1) Sketch in the same set of axes graphs of the functions $x \mapsto f(x)$ and $x \mapsto \frac{1}{f(x)}$, where $f(x) = \frac{(x-3)(x+2)}{2}$.

(2) The diagram below shows the graph of the function $x \mapsto f(x)$. On the same diagram sketch the graph of $x \mapsto \frac{1}{f(x)}$.



Q12. The diagram below shows the graph of the function $x \mapsto f(x)$. On the same diagram sketch the graph of $x \mapsto \frac{1}{f(x)}$.



Q13. Sketch the graphs of the functions:

(1) $y = \sqrt{x^2}$ and $y = (\sqrt{x})^2$

(2) $y = \ln e^x$ and $y = e^{\ln x}$

(3) $y = \sin \arcsin x$ and $y = \arcsin \sin x$

(4) $y = \cos \arccos x$ and $y = \arccos \cos x$

(5) $y = \tan \arctan x$ and $y = \arctan \tan x$

Q14. Consider the functions $f : A \to B, g : B \to C$ defined by

$$f(x) = e^x,$$
$$g(x) = \arcsin x.$$

Find the largest possible sets A, B and C for which the composition $g \circ f$ is defined.

Q15. Consider the functions $f : A \to B$, where $A \subseteq [0, 2\pi]$ and $g : B \to C$ defined by

$$f(x) = 2\sin(3x + \pi),$$
$$g(x) = \ln x.$$

Find the largest possible sets A, B and C for which the composition $g \circ f$ is defined.

9.6 Summary of key points

transformation	**change in formula**
reflection in x-axis	$y = f(x) \to \quad y = -f(x)$
reflection in y-axis	$y = f(x) \to \quad y = f(-x)$
horizontal dilation (shrink / stretch) with a scale factor k	$y = f(x) \to \quad y = f(\frac{x}{k})$
vertical dilation (shrink / stretch) with a scale factor k	$y = f(x) \to \quad y = kf(x)$ or $\frac{y}{k} = f(x)$
reflection of negative values in x-axis	$y = f(x) \to \quad y = \lvert f(x) \rvert$
reflection of a part with positive arguments in y-axis	$y = f(x) \to \quad y = f(\lvert x \rvert)$
translation by a vector $\begin{pmatrix} a \\ b \end{pmatrix}$	$y = f(x) \to \quad y = f(x - a) + b$ or $y - b = f(x - a)$

Chapter 10

Polynomials

Definition 81 (polynomial).
*A **polynomial** is a function $P : \mathbb{R} \to \mathbb{R}$ [a] defined as follows:*

$$P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} + a_n x^n,$$

where $a_k \in \mathbb{R}$ for $k \in \{0, 1, 2, ..., n\}$ and $a_n \neq 0$.
*n is called a **degree** of polynomial P and denoted as "$\deg P$".*

[a] Later on we will extend the definition to $P : \mathbb{C} \to \mathbb{C}$.

Theorem 82 (degree of polynomials).
Let $P, Q : \mathbb{R} \to \mathbb{R}$ be polynomials and $c \in \mathbb{R}$ be a non-zero *constant. Then:*

1. $\deg(P + Q) \leq \max(\deg P, \deg Q)$
2. $\deg(P - Q) \leq \max(\deg P, \deg Q)$
3. $\deg(cP) = \deg P$
4. $\deg(P \times Q) = \deg P + \deg Q$
5. $\deg(P \circ Q) = \deg P \times \deg Q$ [a]

[a] "$P \circ Q$" denotes composition of P and Q. Compositions of functions will be discussed later in chapter 12

Task
Give examples to support all observations stated in the theorem above. When an inequality is not strict (i.e. it is "$\leq$" or "$\geq$") give two examples: one showing a strict inequality and the other showing an equality.

10.1 Algebra of polynomials

When a polynomial $P(x)$ is divided by a polynomial $D(x)$ it gives a quotient $Q(x)$ and a remainder $R(x)$. This could be expressed as follows:

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

which is equivalent to:

$$\underbrace{P(x)}_{\text{a polynomial}} = \underbrace{D(x)}_{\text{divisor}} \times \underbrace{Q(x)}_{\substack{\text{quotient} \\ \text{(divisible part)}}} \quad \underbrace{+R(x)}_{\text{remainder}}$$

where: $\deg R < \deg D$

i.e.: $\deg P = \deg D + \deg Q$

Questions 10.1.

Q1. Divide polynomial $P(x)$ by $D(x)$.

(1) $P(x) = 6x^4 + 7x^3 + 5x + 3$; $D(x) = 2x + 1$;
(2) $P(x) = 12x^4 - 17x^3 + 6x^2 + 6x - 4$; $D(x) = 3x - 2$;
(3) $P(x) = 2x^3 - 4x^2 + 5x - 1$; $D(x) = x - 2$;
(4) $P(x) = 8x^4 + 8x^3 - 8x^2 + x + 8$; $D(x) = 2x + 3$.

Q2. Divide polynomial $P(x)$ by $D(x)$.

(1) $P(x) = -8x^6 + 2x^5 + 7x^3 - 12x^2 + 7x - 6$; $D(x) = 4x^2 - x + 2$;
(2) $P(x) = 4x^6 + 12x^5 - 3x^4 + 19x^3 - 10x^2 + 5x$; $D(x) = 4x^2 + 5$;
(3) $P(x) = 2x^5 - 6x^4 + 3x^3 - x^2 + x + 1$; $D(x) = -x^2 + 2x + 1$;
(4) $P(x) = 2x^5 - 9x^4 + 11x^3 + x^2 - 5.5x - 1$; $D(x) = 2x^2 - 3x - 2$;
(5) $P(x) = 4x^4 - 6x^3 + 2x^2 - 1$; $D(x) = 2x^2 - 2x - 1$;
(6) $P(x) = 6x^4 - 9x^3 + 6x - 1$; $D(x) = 2x^2 - 3x + 1$.

Q3. Find the remainder when $P(x)$ is divided by $D(x)$.

(1) $P(x) = 2x^3 - 4x^2 + 5x - 1$; $D(x) = x - 1$;
(2) $P(x) = 2x^4 + 8x^3 - 8x^2 + x + 8$; $D(x) = x + 2$;
(3) $P(x) = 3x^4 + 9x^3 - 8x^2 + x + 15$; $D(x) = x + 3$;
(4) $P(x) = 8x^4 + 8x^3 - 8x^2 + x + 8$; $D(x) = 2x + 3$;
(5) $P(x) = 2x^4 + x^3 - 5x + 1$; $D(x) = 2x - 3$;
(6) $P(x) = -2x^5 + x^4 - 3x^2 + 5$; $D(x) = 2x - 1$.

10.2 Remainder theorem and factor theorem

Theorem 83 (Remainder theorem).
When a polynomial $P(x)$ is divided by $(x - \alpha)$ the remainder is equal to $P(\alpha)$.

Corollary 84 (to the Remainder theorem).
When a polynomial $P(x)$ is divided by $(ax + b)$ the remainder is equal to $P(-\frac{b}{a})$.

Theorem 85 (Factor theorem).
*α is a **zero of polynomial** $P(x)$ if and only if $(x - \alpha)$ is a **factor** of $P(x)$ i.e.*
$P(\alpha) = 0 \Leftrightarrow P(x)$ is divisible by $(x - \alpha)$

Theorem 86 (Fundamental theorem of algebra).
A polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0, \text{ where } a_n \neq 0$$

can be expressed in the form

$$P(x) = a_n(x - x_1)(x - x_2)...(x - x_k)(x^2 + b_1 x + c_1)(x^2 + b_2 x + c_2)...(x^2 + b_s x + c_s)$$

where

- *$x_1, x_2, ..., x_k$ are all real zeroes of the polynomial (they do not have to be unique),*
- *$b_i^2 - 4c_i < 0$ for every $i \in \{1, 2, ..., s\}$,*
- *$k + 2s = n$.*

Questions 10.2.

Q4. Knowing that $P(-1) = 4$ and $P(1) = 0$ find the remainder when $P(x)$ is divided by $x^2 - 1$.

Q5. Knowing that $P(3) = 3$ and $P(5) = 5$ find the remainder when $P(x)$ is divided by $x^2 - 8x + 15$.

Q6. Knowing that $P(-3) = 3$ and $P(2) = 2$ find the remainder when $P(x)$ is divided by $x^2 + x - 6$.

Q7. When $P(x)$ is divided by $(x + 3)$ the remainder is 1. When $P(x)$ is divided by $(x + 1)$ the remainder is 3. Find the remainder when $P(x)$ is divided by $x^2 + 4x + 3$.

Q8. Knowing that $P(-0.5) = -2.5$ and that $P(x)$ gives the remainder of 8 when divided by $(x - 3)$ find the remainder, when $P(x)$ is divided by $2x^2 - 5x - 3$.

Q9. Knowing that $P(-\frac{1}{3}) = \frac{11}{3}$ and that $P(x)$ is divisible by $(x - 1.5)$ find the remainder when $P(x)$ is divided by $6x^2 - 7x - 3$.

Q10. A polynomial $P(x) = x^3 + ax^2 - ax + 3$ is divisible by $(x + 3)$. Find the value of a.

Q11. When $P(x) = x^3 + ax^2 + bx - 2$ is divided by $(x - 1)$ the remainder is 1. When $P(x)$ is divided by $(x + 1)$ the remainder is -7. Find the values of a and b.

Q12. When $P(x) = x^3 + ax^2 - x + b$ is divided by $(2x - 1)$ the remainder is $\frac{19}{8}$. When $P(x)$ is divided by $(x + 3)$ the remainder is 5. Find the values of a and b.

Q13. When $P(x) = x^3 + 4x^2 + ax + b$ is divided by $(x + 1)$ the remainder is the same as when $P(x)$ is divided by $(x - 1)$ and three times smaller than when $P(x)$ is divided by $(x - 2)$. Find the values of a and b.

Q14. When $P(x) = ax^3 - x^2 - x + b$ is divided by $(x - 1)$ or by $(2x + 1)$ the remainder is 1. Find the values of a and b.

Q15. When $P(x) = ax^3 + bx^2 - 2x - 1$ is divided by $(x - 1)$ the remainder is 4. $P(x)$ has a zero at $x = -1$. Find the values of a and b.

Q16. When $P(x) = ax^3 + bx - 3$ is divided by $(x + 2)$ the remainder is 11 times the remainder when $P(x)$ is divided by $(x - 1)$ or by $(x + 1)$. Find the values of a and b.

Q17. When $P(x) = ax^4 - 5x^3 + bx^2 - 3x + a$ is divided by $(x - 2)$ the remainder is -5. Find the values of a and b Knowing that 1 is a zero of a polynomial.

Q18. When $P(x) = 4x^4 - 4x^3 + ax^2 + 2x + b$ is divided by $(x + 1)$ the remainder is twice the remainder when $P(x)$ is divided by $(x - 1)$. When $P(x)$ is divided by $(2x + 1)$ the remainder is 2.5. Find the values of a and b.

Q19. When $P(x) = 3x^3 + ax^2 - 9x + b$ is divided by $(x - 1)$ the remainder is -3. $P(x)$ is divisible by $(x + 2)$. Find the values of a and b.

Q20. When $P(x) = 2x^3 - 4x^2 + ax + b$ is divided by $(x - 2)$ the remainder is four times greater than when $P(x)$ is divided by $(2x - 3)$. Knowing that $P(x)$ is divisible by $(x - 1)$ find the values of a and b.

10.3 Vieta's formulas

Vieta's formulas are about the sum and products of roots of polynomials.
They are named after *François Viète* (his name is commonly used in latin form as Franciscus Vieta).

Theorem 87 (Vieta's formulae for polynomials).
For the polynomial equation:

$$\sum_{r=0}^{n} a_r x^r = 0$$

1. *the **SUM** of roots is* $\frac{-a_{n-1}}{a_n}$
2. *the **PRODUCT** of roots is* $\frac{(-1)^n a_0}{a_n}$

Questions 10.3.

Q21. For the following polynomial equations write down the sum of their roots and the product of their roots.

(1) $2x^4 + 4x^3 + 6x^2 + 8x + 10 = 0$
(2) $2x^6 - x^5 + 3x^4 + 4x^2 + 5x + 3 = 0$
(3) $3x^5 - 3x^3 + 5x + 6 = 0$
(4) $4x^6 + 3x^5 + 2x^4 + x^3 + x^2 + 1 = 0$
(5) $2x^6 - 3x^5 + 4x^4 - 5x^3 + 2x^2 + x = 0$
(6) $x^6 - 2x^4 + 3x^2 - 4 = 0$
(7) $x^5 + 2x^3 + 3x = 0$
(8) $3x^5 + 6x^4 + x^3 + 3 = 0$

Q22. The sum of the roots of the polynomial $P(x) = 3x^5 + ax^4 - 5x^2 - 2x - 1$ is 2. Find the value of the coefficient a and the product of the roots.

Q23. The sum of the roots of the polynomial $P(x) = ax^5 - 8x^4 + 3x^2 - x - 1$ is 4. Find the value of the coefficient a and the product of the roots.

Q24. The product of the roots of the polynomial $P(x) = 2x^4 - 3x^3 + 4x^2 - x + b$ is 6. Find the value of the coefficient b and the sum of the roots.

Q25. The product of the roots of the polynomial $P(x) = bx^3 - 12x^2 + cx + 8$ is 2. Find the value of the coefficient b and the sum of the roots.

Q26. The product of the roots of the polynomial $P(x) = ax^4 - 6x^3 + cx^2 + dx + e$ is 3 while the sum of them is -3. What are the coordinates of the y-intercept of the graph of P?

Q27. The product of the roots of the polynomial $P(x) = 2x^n - 3x^{n-1} + ... - 2x + 5$ is positive. Is the degree of the polynomial even or odd?

Q28. The product of the roots of the polynomial $P(x) = ax^4 - 4x^3 + cx^2 + dx + e$ is equal to the sum of the roots. Find the value of of e.

Q29. Knowing that -1, 1 and 2 are the solutions of the equation $2x^5 - 9x^4 + 5x^3 + Ax^2 - 7x - 6 = 0$, find the other two solutions.

Q30. Knowing that -4 and 1 are the solutions of the equation $4x^4 + 8x^3 + Ax^2 - 29x + 60 = 0$, find the other two solutions.

Q31. The roots of the polynomial $P(x) = ax^6 + (3b - 6)x^5 + 7x^4 + 55x^3 - \frac{81}{2}x^2 - 5x - b$, where $a, b \in \mathbb{R}$ are:

$$-\frac{3}{2}, \frac{1}{2}, \frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{1}{2} + \frac{\sqrt{3}}{2}, 3 - \sqrt{5}, 3 + \sqrt{5}$$

Find the values of a and b.

Q32. The numbers $a, a + 2, a + 5$ where $a \neq 0$ are the roots of the polynomial $P(x) = x^3 + bx^2 - \frac{1}{2}ax - \frac{1}{2}a^2$. Find the polynomial.

Q33. Consider the equation $x^3 + ax^2 + bx + a + 1 = 0$, where $a, b \in \mathbb{Q}$.

The three rational solutions to the equation form a geometric sequence with the common ratio 2.

Find the three solutions. Find the values of a and b.

Q34. Consider the equation $x^3 + ax^2 + bx + 3a = 0$, where $a, b \in \mathbb{R}^*$.

The three solutions to the equation form an arithmetic sequence with the common difference 4.

Find the three solutions. Find the values of a and b.

Q35. Let α and β be the roots of the equation $15x^2 - 20x - 18 = 0$. Without finding these roots find the value of:

(1) $\alpha^2 + \beta^2$
(2) $\frac{1}{\alpha} + \frac{1}{\beta}$
(3) $\frac{\beta}{\alpha} + \frac{\alpha}{\beta}$
(4) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
(5) $\alpha^3 + \beta^3$
(6) $|\alpha - \beta|$

Q36. Let α and β be the roots of the equation $15x^2 - 20x - 18 = 0$. Find a quadratic equation with integer coefficients whose roots are:

(1) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ (2) α^2 and β^2 (3) $\frac{\beta}{\alpha}$ and $\frac{\alpha}{\beta}$

10.4 Polynomial inequalities

To solve a polynomial inequality the following simple steps should be followed.

Step 1. transform an inequality into the form $\left(\text{a polynomial} \begin{Bmatrix} >, \geq \\ <, \leq \end{Bmatrix} 0\right)$

Step 2. factorise the polynomial, i.e. find its all linear factors,

Step 3. sketch the graph, considering carefully whether values for $x \to +\infty$ are positive or negative,

Step 4. read the solution from the sketch.

Questions 10.4.

Q37. (1) Show that $\frac{3}{2}$ is a root of $P(x) = 4x^3 + 12x^2 - 19x - 12$.

(2) Hence solve the inequality $4x^3 + 12x^2 - 19x - 12 \leq 0$.

(3) Another polynomial $T(x)$ is divided by $P(x)$. Find the remainder in this division knowing that $T(-\frac{1}{2}) = \frac{3}{4}$, $T(-4) = 13$ and $T(\frac{3}{2}) = \frac{19}{4}$

Q38. Solve the following inequalities.

(1) $(x-1)(x-2)(x-3) > 0$
(2) $(x+1)(2-x)(x+4) > 0$
(3) $(x-1)^2(x+5) \leq 0$
(4) $(3-2x)(4x+2)^2 < 0$
(5) $(3-2x)^2(4x+2) < 0$
(6) $(2+3x)(x-1)^3 \geq 0$
(7) $(x-3)(2x+1)(x-1)(x+2) < 0$
(8) $(x-3)(2x+1)^2(x+2) \leq 0$
(9) $(3-x)(2x+1)^2(x+2) \leq 0$
(10) $(2-3x)^3(x-3)^2(x+2) > 0$
(11) $(2x+1)^3(x-2)(2-3x)^3 \leq 0$
(12) $(3x-1)(x-3)^2(3x-3)^3 \geq 0$

Q39. Solve the following inequalities.

(1) $x^3 - 4x^2 + x + 6 > 0$
(2) $-x^3 + 12x - 16 \geq 0$
(3) $x^3 + x^2 - 17x + 15 < 0$
(4) $-x^3 + x^2 + 10x + 8 \leq 0$
(5) $x^3 + 2x^2 - 9x - 18 < 0$
(6) $2x^3 - 12x^2 - 2x + 60 \leq 0$
(7) $2x^3 - 14x^2 + 14x + 30 > 0$
(8) $2x^3 + 4x^2 - 8x - 16 \geq 0$
(9) $3x^3 - 6x^2 - 3x + 6 > 0$
(10) $3x^3 - 3x^2 - 42x + 72 \geq 0$
(11) $3x^3 + 9x^2 - 39x - 45 < 0$
(12) $-18x^3 - 57x^2 + 28x + 147 \leq 0$
(13) $18x^3 - 15x^2 - 100x + 125 > 0$
(14) $x^4 - 2x^3 - 7x^2 + 8x + 12 \geq 0$
(15) $-x^4 - 2x^3 - 7x^2 + 8x + 12 \geq 0$
(16) $x^4 - 2x^3 - 20x^2 + 66x - 45 \leq 0$
(17) $-x^4 - x^3 + 12x^2 + 28x + 16 > 0$
(18) $x^4 - 13x^2 + 36 \geq 0$
(19) $4x^4 - 8x^3 - 27x^2 + 27x + 54 < 0$
(20) $-4x^4 + 9x^3 + 14x^2 - 36x + 8 \leq 0$
(21) $2x^4 - 18x^3 + 34x^2 + 66x - 180 > 0$
(22) $-2x^4 + 10x^3 + 14x^2 - 58x - 60 \geq 0$
(23) $6x^4 + 7x^3 - 36x^2 - 7x + 6 < 0$
(24) $-4x^4 + 31x^2 - 21x - 18 \leq 0$
(25) $2x^4 - 2x^3 - 20x^2 + 8x + 48 > 0$
(26) $4x^4 - 28x^3 + 57x^2 - 36x \geq 0$
(27) $4x^4 - 24x^3 + 35x^2 + 6x - 9 < 0$
(28) $-2x^4 + 7x^3 + 15x^2 - 63x + 27 \leq 0$
(29) $-9x^4 + 42x^3 - 13x^2 - 84x - 36 > 0$
(30) $24x^4 - 28x^3 - 118x^2 + 219x - 90 \geq 0$
(31) $-36x^4 + 96x^3 + 161x^2 - 600x + 400 > 0$

Chapter 11

Exponential and logarithmic functions

11.1 Exponential function

Definition 88 (exponential function).
*An **exponential function** is a function defined by the equation:*

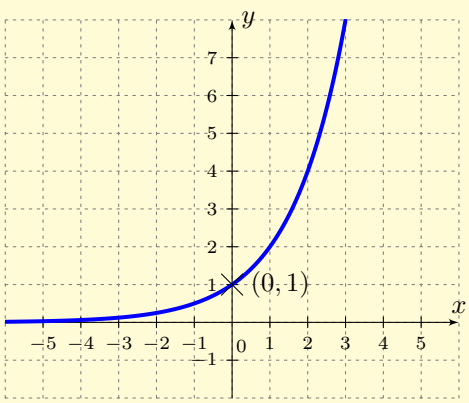
$$y = a^x,$$

where $a \in \mathbb{R}, a > 0, a \neq 1$.

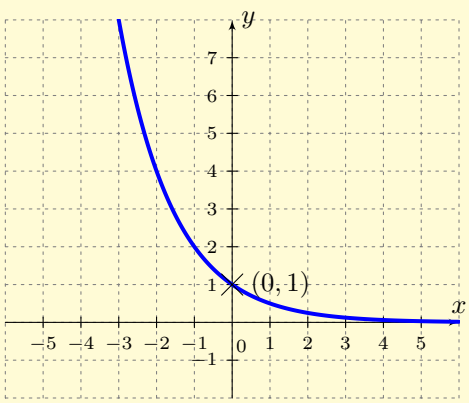
When the base of an exponential function a is equal e, the Euler's number, we call it **the exponential function**.

Properties of exponential function.
For an exponential function defined by $x \longmapsto a^x$*:*

- *the **domain** of an exponential function is the set of all real numbers* ($\mathbb{R}$)
- *the **range** of an exponential function is the set of positive real numbers* ($\mathbb{R}^+$)
- *when* $a > 1$*, an exponential function is **increasing**, as shown in figure below*
- *when* $0 < a < 1$*, an exponential function is **decreasing**, as shown in figure below*



exponential function $y = a^x$ *for* $a > 1$.



exponential function $y = a^x$ *for* $0 < a < 1$.

The fact that an exponential function decreases when its base a is between 0 and 1 ($0 < a < 1$) is essential when solving inequalities. Consider an inequality

$$a^{f(x)} > a^{g(x)},$$

where $f(x)$ and $g(x)$ are any algebraic expressions in terms of x.
While for $a > 1$ the following equivalence holds:

$$a^{f(x)} > a^{g(x)} \Leftrightarrow f(x) > g(x),$$

for $0 < a < 1$ it looks very different:

$$a^{f(x)} > a^{g(x)} \Leftrightarrow f(x) < g(x).$$

11.2 Logarithmic function

Definition 89 (logarithmic function).
*A **logarithmic function** is a function defined by the equation:*

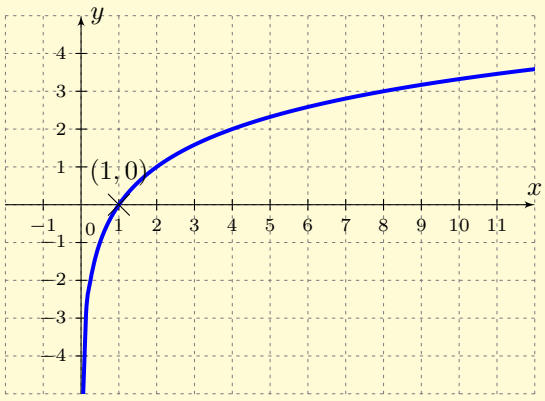
$$y = \log_a x,$$

where $a \in \mathbb{R}, a > 0, a \neq 1$.

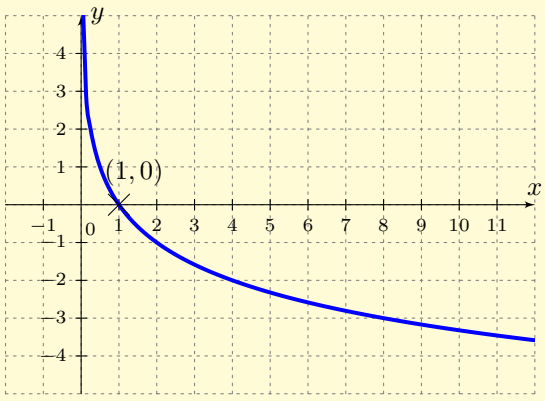
When the base of a logarithmic function a is equal to e, the Euler's number, we deal with **natural logarithm**.

Properties of logarithmic function.
For a logarithmic function defined by $x \longmapsto \log_a x$:

- *the **domain** of a logarithmic function is the set of positive real numbers* ($\mathbb{R}^+$)
- *the **range** of a logarithmic function is the set of all real numbers* ($\mathbb{R}$)
- *when* $a > 1$, *a logarithmic function is **increasing**, as shown in figure below*
- *when* $0 < a < 1$, *a logarithmic function is **decreasing**, as shown in figure below*



logarithmic function $y = \log_a x$ *for* $a > 1$



logarithmic function $y = \log_a x$ *for* $0 < a < 1$

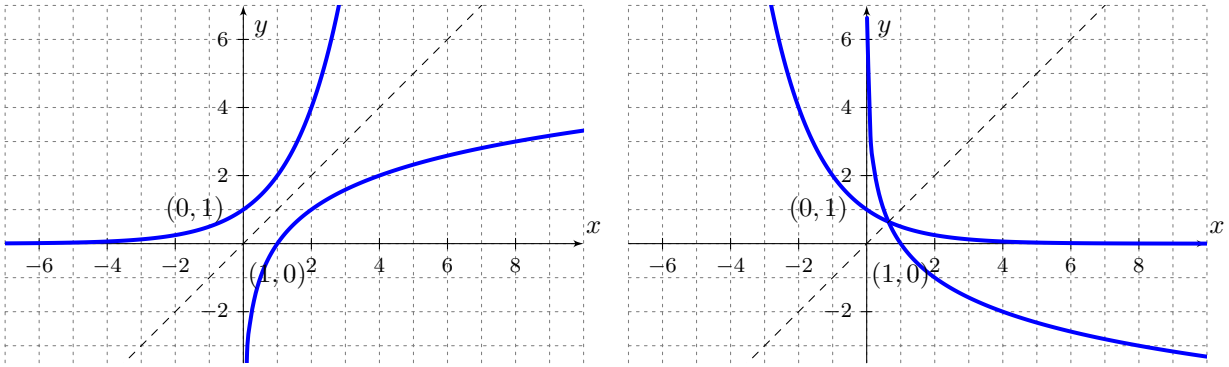
Similarly as with an exponential function the fact whether base a of a logarithmic function is *small* ($0 < a < 1$) or *large* ($a > 1$) is essential when solving inequalities.
While for $a > 1$ the following equivalence holds:

$$a^{f(x)} > a^{g(x)} \Leftrightarrow f(x) > g(x),$$

for $0 < a < 1$ a change of sign of inequality must be applied:

$$a^{f(x)} > a^{g(x)} \Leftrightarrow f(x) < g(x).$$

Placing graphs of a logarithmic and of an exponential function with the same base in the same grid shows important property of the functions (see 11.1). The property will be discussed more thoroughly in chapter 12.4.



(a) $a > 1$ (b) $0 < a < 1$

Figure 11.1: Logarithmic and exponential functions with the same base.

Chapter 12

Compositions and inverses

12.1 Compositions of functions

Definition 90 (composition of functions).
Let $f : A \to f(A)$ and $g : B \to C$ be the real functions such that $f(A) \subseteq B$.
*The **composition** of g with f is the function $(g \circ f) : A \to C$ defined as follows:*

$$(g \circ f)(x) = g(f(x))$$

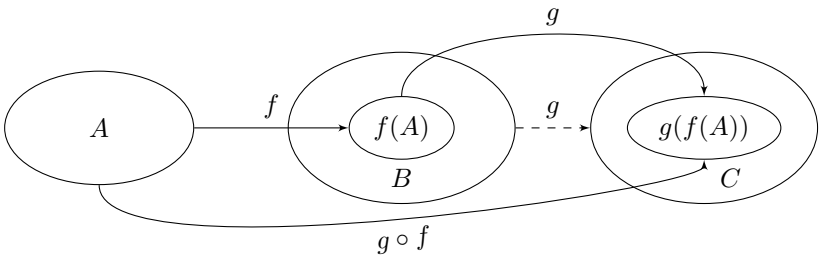


Figure 12.1: Composition of functions.

The identity function

Definition 91 (identity function).
*The **identity** function is a function $e : A \to A$ defined on any set A that transforms A* bijectively *onto itself in a way such that:*

$$e(x) = x \text{ for every } x \in A.$$

Notation.
The identity function on set A is usually denoted as e_A.

COROLLARY 92 (identity function).
For any function $f : A \to B$:

$$f \circ e_A = e_B \circ f = f$$

For simplification it is usually written as $f \circ e = e \circ f = f$

12.2 INVERSE OF A FUNCTION

DEFINITION 93 (inverse function).
Let $f : A \to B$ be a bijective *function.*
*The **inverse function** of f is the function $f^{-1} : B \to A$ such that:*

1. $f^{-1} \circ f = e_A$
2. $f \circ f^{-1} = e_B$

COROLLARY 94 (an inverse of an inverse).

$$(f^{-1})^{-1} = f$$

Figure 12.2 shows composition $(f^{-1} \circ f) : A \to A$.

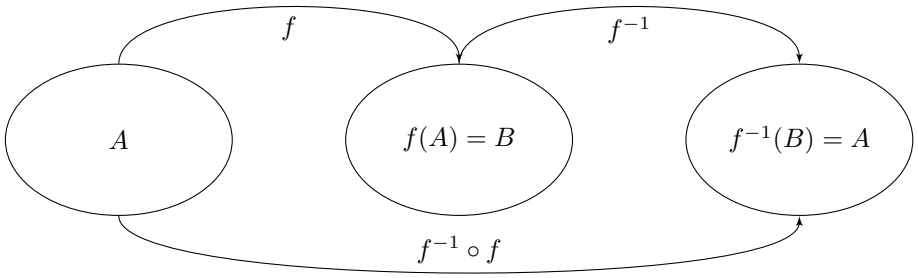


Figure 12.2: The inverse function.

As in figure 12.2 set A is shown in two positions, the figure can be redrawn as follows (see fig.12.3). To find a

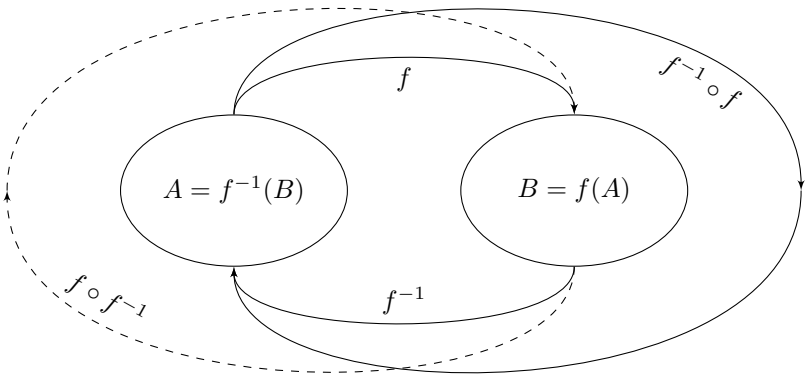


Figure 12.3: The inverse function.

method of obtaining an equation of f^{-1} when the equation of f is known let us have a look at the simplified diagram shown in figure 12.4.

EXAMPLE 95 (simple inverse function).
Find the inverse of a function $y = f(x)$ defined by $f(x) = 2x + 1$.

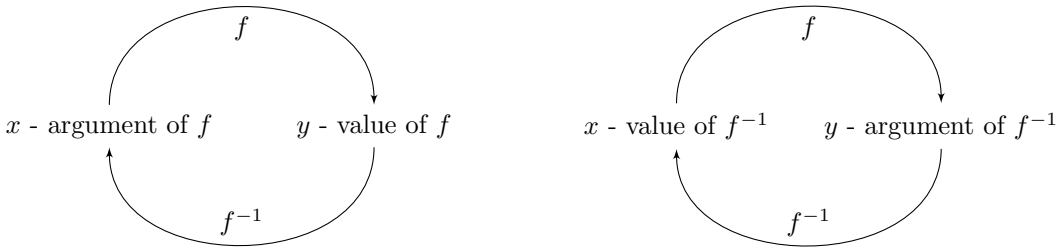


Figure 12.4: The argument and value of a function and its inverse.

As shown in figure 12.4 a variable y becomes an argument of f^{-1} while a variable x becomes its value. Thus we rearrange the equation $y = 2x + 1$ to express x in terms of y:

$$x = \frac{1}{2}(y - 1)$$

We can write the above as:

$$f^{-1}(y) = \frac{1}{2}(y - 1)$$

or, to keep a standard way of naming arguments with x's and values with y's, we will swap the two variables:

$$y = \frac{1}{2}(x - 1) \text{ i.e.}$$

$$f^{-1}(x) = \frac{1}{2}(x - 1)$$

Summing up, to obtain an equation of f^{-1} when the equation of f is known we should follow one of the two simple methods that are obviously equivalent:

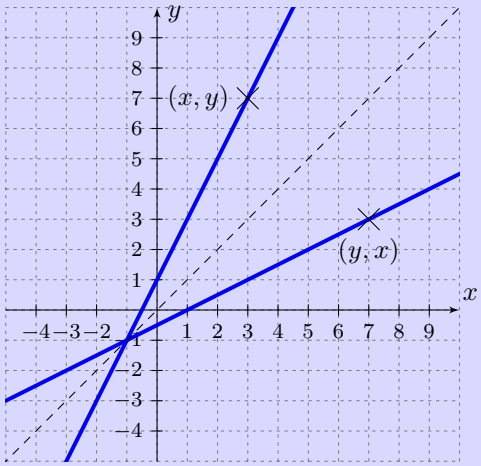
Method 1. **Step 1.** swap x with y

Step 2. rearrange the equation to express y in terms of x

Method 2. **Step 1.** rearrange the equation to express x in terms of y

Step 2. swap x with y

Let us have a look at the graphs of f and f^{-1} from example 95 placed in the same diagram.



$f(x) = 2x + 1$ and $f^{-1}(x) = \frac{1}{2}(x - 1)$

The diagram shows that point (x, y) on the graph of $x \longmapsto f(x)$ corresponds to point (y, x) on the graph of $x \longmapsto f^{-1}(x)$. It is because change of point (x, y) into point (y, x) is made by reflection in line $y = x$.

PROPERTIES OF GRAPHS OF INVERSE FUNCTIONS.
If $f^{-1} : B \to A$ is the inverse of $f : A \to B$, where $A, B \subset \mathbb{R}$ then the graphs of the two functions are symmetrical in line $y = x$.

12.3 Conditions for existence of inverse

Theorem 96 (existence of an inverse function).
A function $f : A \to B$ is invertible (i.e. the inverse of f exists) if and only if *f is a* bijection, *i.e. it is both $1-1$ and "onto".*

Proof. Actually, justification shown below is not fully formal, so it should be considered a draft of a proof.
It is easy to realise that the condition is sufficient. To verify that it is necessary let us recall an informal definition of a function.

> An assignment $f : A \to B$ is a **function** whenever
> **for every** $a \in A$ there exists **exactly one** $b \in B$ assigned to a.
> *Similarly,* $f^{-1} : B \to A$ is a function whenever
> **for every** $b. \in B$ there exists **exactly one** $a \in A$ assigned to b.

Suppose f **is not** $1-1$.
Then there are $a_1, a_2 \in A$, $a_1 \neq a_2$ and $b \in B$ such that $f(a_1) = f(a_2) = b$.
Suppose that f^{-1}, the inverse of f, exists.
Then

$$\begin{aligned} & (f^{-1} \circ f)(a_1) = a_1 \text{ and } (f^{-1} \circ f)(a_2) = a_2 \\ \Leftrightarrow \quad & f^{-1}(f(a_1)) = a_1 \text{ and } f^{-1}(f(a_2)) = a_2 \\ \Leftrightarrow \quad & f^{-1}(b) = a_1 \text{ and } f^{-1}(b) = a_2 \end{aligned}$$

which is a clear contradiction with the above definition,
i.e. f^{-1} cannot be a function as it assigns two values (a_1, a_2) to one argument b.

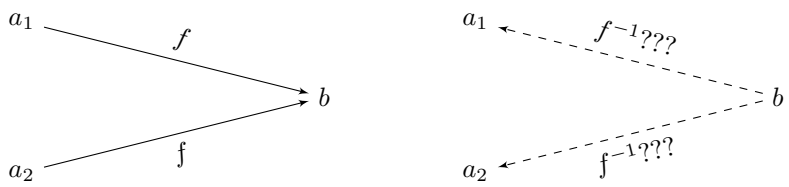


Figure 12.5: A function that is not $1-1$ has no inverse. Both a_1 and a_2 would have to be two distinct values for argument b.

Suppose f **is not "onto"**.
Then there is $b \in B$ which is not a value for any argument.
If function f^{-1} exists, then there is $a \in A$ such that

$$\begin{aligned} & f^{-1}(b) = a \\ \Leftrightarrow & f(f^{-1})(b) = f(a) \\ \Leftrightarrow \quad & b = f(a) \end{aligned}$$

which is a contradiction, as b was not to be a value of f.

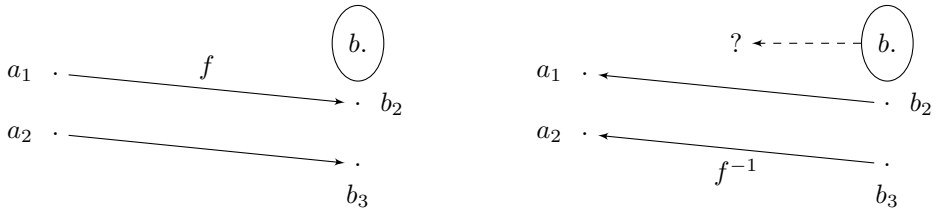


Figure 12.6: A function that is not "onto" has no inverse. b was not a value for f so it cannot be an argument for f^{-1}.

□

12.4 Logarithmic function as inverse of exponential function

Theorem 97 (logarithmic function as inverse of exponential function).
Let $a \in]0,1[\cup]1,+\infty[$. For the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by:

$$f(x) = \log_a x$$

the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}^+$ is defined by:

$$f^{-1}(x) = a^x.$$

Proof. Consider a logarithmic function with base a, where $a \in]0,1[\cup]1,+\infty[$ i.e.

$$f(x) = \log_a x, \quad x \in \mathbb{R}^+, f(x) \in \mathbb{R}$$

or:

$$y = \log_a x, \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}$$

By definition of a logarithm the latter is equivalent to:

$$x = a^y, \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}$$

Swapping x and y we get (note the change in domain and range):

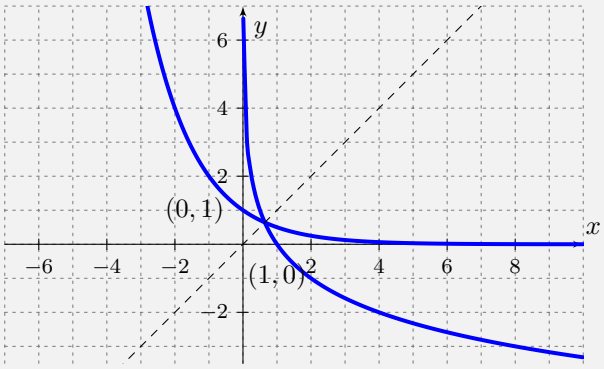
$$y = a^x, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^+$$

Finally:

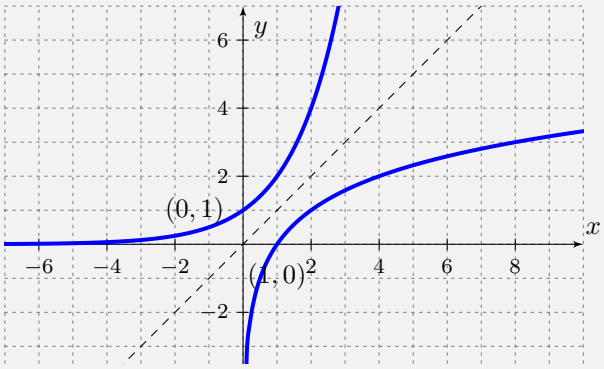
$$f^{-1}(x) = a^x, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^+$$

□

Corollary 98 (graphs of logarithmic and exponential functions).
The graphs of functions $x \longmapsto a^x, x \in \mathbb{R}$ and $x \longmapsto \log_a x, x \in \mathbb{R}^+$ are symmetrical in line $y = x$, as shown in figure below.



$0 < a < 1$



$a > 1$

Properties of logarithmic and exponential functions.

		logarithmic	***exponential***
	equation	$y = \log_a x$	$y = a^x$
	domain	$\mathbb{R}^+ =]0,+\infty[$	$\mathbb{R}$
	range	$\mathbb{R}$	$\mathbb{R}^+ =]0,+\infty[$
	x-intercept	$(1,0)$	*none*
	y-intercept	*none*	$(0,1)$
base	$0 < a < 1$	*decreasing*	
	$a > 1$	*increasing*	

Questions 12.1.

Q1. Sketch in the same set of axes graphs of the functions $f(x)$ and $f^{-1}(x)$. Write down the equation of $f^{-1}(x)$. Write down the domain and the range of both f and f^{-1}.

(1) $f(x) = \frac{2x+1}{x-2}$
(2) $f(x) = \frac{2x+4}{x-2}$
(3) $f(x) = \frac{3x-6}{x+2}$
(4) $f(x) = \frac{1-2x}{x+2}$
(5) $f(x) = \sqrt{x-2}$
(6) $f(x) = \sqrt{x+2}$
(7) $f(x) = \sqrt{2-x}$
(8) $f(x) = 2^x - 4$
(9) $f(x) = e^x - 2$
(10) $f(x) = e^{2-x}$
(11) $f(x) = \ln(x-1)$
(12) $f(x) = \ln(2-x)$

Q2. For each function $y = f(x)$ defined by the expressions shown

(i) find the expression for f^{-1}, the inverse of f,
(ii) state the domain and the range of the inverse,
(iii) sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ in the same set of axes.

(1) $f(x) = \frac{2x-1}{1-x}$
(2) $f(x) = \frac{3x+6}{x-2}$
(3) $f(x) = \frac{x-2}{2x+1}$
(4) $f(x) = \frac{1-x}{x-2}$
(5) $f(x) = e^{2x} + 1$
(6) $f(x) = 2 - e^{1-x}$
(7) $f(x) = \ln(3-x) + 1$
(8) $f(x) = 1 - \log_2(2-x)$
(9) $f(x) = \sqrt{3-x} + 2$
(10) $f(x) = 2 - \sqrt{x+1}$
(11) $f(x) = -x^2 - 2x + 3,\ x \geq -1$
(12) $f(x) = \frac{1}{2}x^2 - 2x,\ x \leq 2$

Q3. Find an expression for $(f \circ g)(x)$ and for $(g \circ f)(x)$ in each of the following cases.

(1) $f(x) = 2x + 3$; $g(x) = x^2 + 1$
(2) $f(x) = \sqrt{x-1}$; $g(x) = (x+1)^2$
(3) $f(x) = x - 2$; $g(x) = \sqrt[3]{x} + 2$

Q4. Given the functions

$$f(x) = \frac{x+2}{x-3} \text{ for } x \neq 3 \text{ and } g(x) = \frac{2x-3}{x+1} \text{ for } x \neq -1 :$$

(a) find the expression for $(f \circ g)(x)$ for all values of x where the composition exists giving your answer in its simplest terms;
(b) find the expression for f^{-1} stating clearly its domain and range;
(c) sketch in the same set of axes graphs of the functions $x \longmapsto f(x)$ and $x \longmapsto f^{-1}(x)$.

Let $h(x) = \ln(f(x))$.

(d) Find the domain and range of h.
(e) Find the inverse of h.
(f) Hence show that $(\ln \circ f)^{-1} = f^{-1} \circ \ln^{-1}$.

Let $k(x) = e^{g(x)}$. Let $\exp(x) = e^x$.

(g) Find the domain and range of k.
(h) Find the inverse of k.
(i) Hence show that $(\exp \circ g)^{-1} = g^{-1} \circ \exp^{-1}$.

Q5. Consider the function $f(x) = e^{x-2} - 1$.

(a) Find the expression for f^{-1} stating clearly its domain and range.
(b) Sketch in the same set of axes graphs of the functions $x \longmapsto f(x)$ and $x \longmapsto f^{-1}(x)$.
(c) Find the expression for $g(x)$ knowing that $f(g(x)) = x + 3$.
(d) Find the expression for $h(x)$ knowing that $h(f(x)) = 3 - x$.

Q6. Consider the functions $f : A \to B, g : B \to C$ defined by

$$f(x) = 4 - 2\sqrt{x}, \qquad g(x) = \log_2 x.$$

Find the largest possible sets A, B and C for which the composition $g \circ f$ is defined.

Q7. Consider the functions $f : A \to B, g : B \to C$ defined by

$$f(x) = \ln x, \qquad g(x) = \sqrt{3 - x}.$$

Find the largest possible sets A, B and C for which the composition $g \circ f$ is defined.

Q8. Consider the functions $g : A \to B, f : B \to C$ defined by

$$f(x) = e^x,$$
$$g(x) = \arcsin x.$$

Find the largest possible sets A, B and C for which the composition $f \circ g$ is defined.

Q9. Consider the functions $f : A \to B, g : B \to C$ defined by

$$f(x) = e^x,$$
$$g(x) = \arcsin x.$$

Find the largest possible sets A, B and C for which the composition $g \circ f$ is defined.

Q10. Consider the functions $f : A \to B, g : B \to C$ defined by

$$f(x) = 2\sin(3x + \pi),$$
$$g(x) = \ln x.$$

Find the largest possible sets A, B and C for which the composition $g \circ f$ is defined.

Q11. Consider the function $f(x) = 3 \ln |2 \log x|$.

(i) Find the domain of f.

(ii) Find the sum and the product of zeroes of f.

Q12. Consider the functions f and g defined by $f(x) = \sqrt{1 - x}$ and $g(x) = e^x$. Find the domain and range of:

(i) f (ii) g (iii) $f \circ g$ (iv) $g \circ f$

Q13. Consider the function $f(x) = x^2 - 4x + 3, x \geq a, a \in \mathbb{R}$.

(i) Find the smallest possible value of a for which f is injective.

(ii) Find the range of f.

(iii) Find the inverse of f.

Q14. For the function $y = f(x)$ defined by the expressions below find the expression for the inverse f^{-1} and state the domain and range of the inverse.

(1) $f(x) = e^{x^2} - 2, x \geq 0$ (3) $f(x) = \ln(x^2 + 1) - 1, x \leq 0$ (5) $f(x) = \sqrt{4 - \frac{x^2}{9}}, 0 \leq x \leq 6$

(2) $f(x) = e^{x^2} + 1, x \leq 0$ (4) $f(x) = \ln(1 - x^2), 0 \leq x < 1$ (6) $f(x) = \sqrt{x^2 - 1}, x \leq -1$

Q15. Find the inverse of the function $f(x, y) = (x + y, 2x - y), x, y \in \mathbb{R}$.

Chapter 13

Inequalities

Questions 13.1.

Q1. Solve the following inequalities.

(1) $x + \frac{3}{x} > 4$
(2) $\frac{4}{x} + 2x - 6 < 0$
(3) $\frac{2x+3}{x-2} \leq 0$
(4) $\frac{3x-5}{2-x} > 0$
(5) $\frac{2+x}{2-x} > 3$
(6) $\frac{3x-2}{x+1} \leq 2$
(7) $\frac{4x-2}{x-1} > 4$
(8) $\frac{x+1}{x-2} < \frac{x-2}{x+1}$
(9) $\frac{x+1}{3-2x} > \frac{x-2}{x+2}$
(10) $\frac{x+1}{2x-1} \geq \frac{2x-1}{x+1}$
(11) $\frac{x+2}{x-4} \leq \frac{2x-2}{x+4}$
(12) $|x-1| > |2x+1|$
(13) $|\frac{x+2}{2x-1}| \geq 1$
(14) $|3x-1| \leq 2|x+1|$
(15) $|x^2 - x + 2| < 4$
(16) $\frac{1}{2}|x-2| < |x+4|$
(17) $|2-x| \geq \frac{|x-1|}{2}$

Q2. Solve the inequalities. Use a calculator.

(1) $e^x \sin x < 1$ for $x < 4$
(2) $\ln x < 2\cos 2x$
(3) $\ln 2x \geq 3\sin x$
(4) $\frac{2\cos x+1}{x} > \frac{x}{4}$

Part IV

CIRCULAR FUNCTIONS AND TRIGONOMETRY

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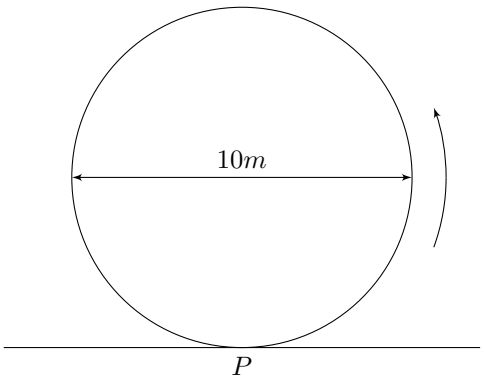
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Chapter 14

Circular functions

Task The diagram represents a Ferris wheel with a diameter of 10 metres.
Let A be a point on the wheel. The wheel starts rotating when A is at ground level. It rotates in an anticlockwise direction at a **constant rate**. One revolution takes 2 minutes.
The table below shows the height $h(t) metres$ of point P above the ground level, where t is in seconds.



t	0	5	10	15	20	25	30
$h(t)$	0	0.2	0.7	1.5			5

t	35	40	45	50	55	60
$h(t)$	6.3	7.5				

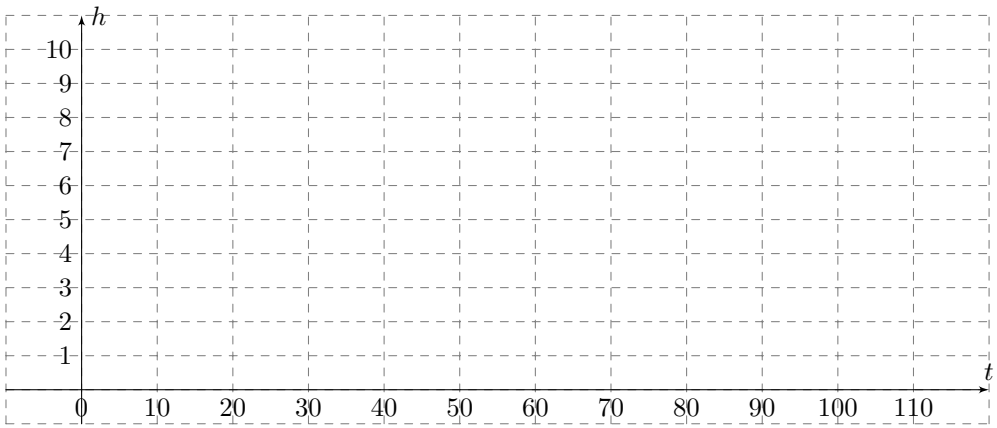
t	65	70	75	80	85	90
$h(t)$						

t	95	100	105	110	115	120
$h(t)$						

The height was measured every 5 seconds. However, some of the measurements are missing.

→ **1.** Find all missing heights of point P.

→ **2.** Sketch the graph of $h(t)$ against t for $0 \leq t \leq 120$.



14.1 Basic circular functions

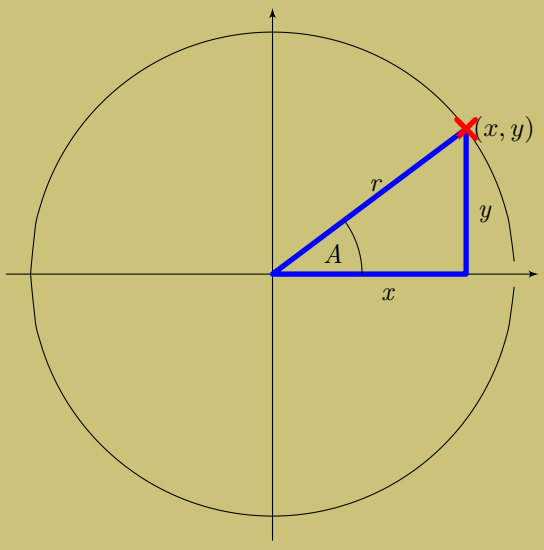
Definition 99 (basic circular functions).
The following rules show how the three basic circular functions (sine, cosine, tangent) can be defined:

$$\sin A = \frac{y}{r}$$

$$\cos A = \frac{x}{r}$$

$$\tan A = \frac{y}{x}$$

$$(r^2 = x^2 + y^2)$$



$$\textbf{\textit{Sin}} = \frac{\textbf{\textit{Opposite}}}{\textbf{\textit{Hypotenuse}}}$$

$$\textbf{\textit{Cos}} = \frac{\textbf{\textit{Adjacent}}}{\textbf{\textit{Hypotenuse}}}$$

$$\textbf{\textit{Tan}} = \frac{\textbf{\textit{Opposite}}}{\textbf{\textit{Adjacent}}}$$

SOHCAHTOA

*x - side **A**djacent to the angle*
*y - side **O**pposite to the angle*
*r - **H**ypotenuse side*

Corollary 100 (values of basic trigonometric ratios for some angles in the first quadrant).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	*no value*

14.2 Unit circle

When the hypotenuse (and the radius of the circle) $r = 1$:

$$\sin A = \frac{y}{1} = y \quad \text{and} \quad \cos A = \frac{x}{1} = x$$

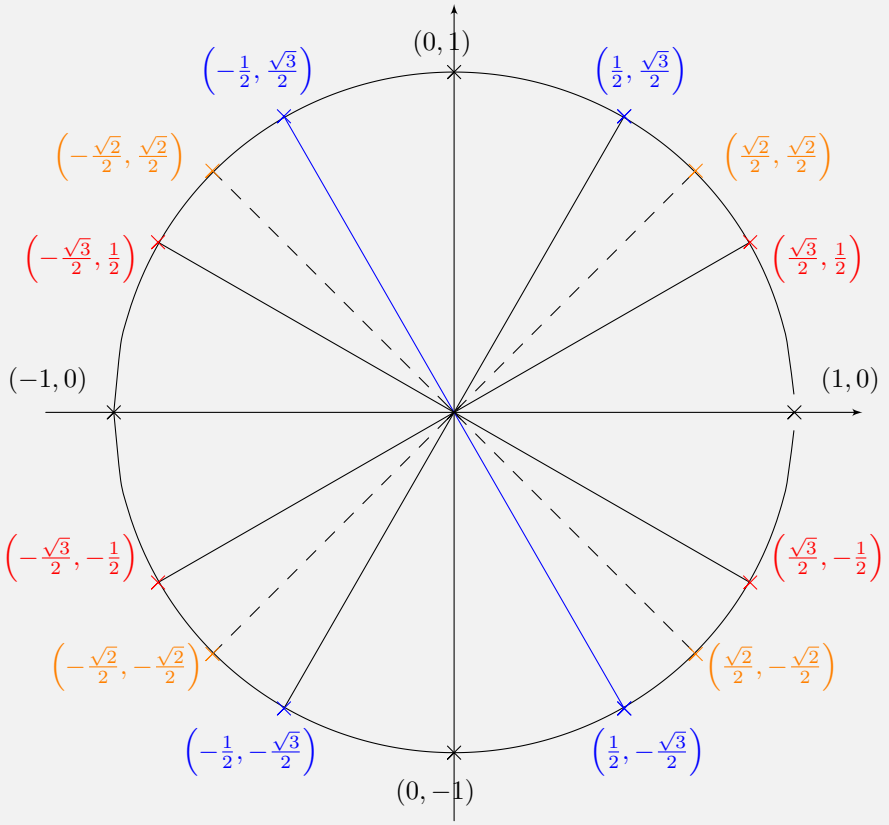
hence sine, cosine and tangent can be redefined as follows.

Definition 101 (basic circular functions).
Let $P = (x, y)$ be the point on a unit circle with centre at O. Let A be an angle measured from the positive part of the x-axis to radius OP. Then:

$$\sin A = y, \qquad \cos A = x, \qquad \tan P = \frac{y}{x} \ (x \neq 0)$$

With such a definition it is easy to get values of the trig ratios for all multiples of angles like $30°$, $45°$, $60°$.

COROLLARY 102 (values of trigonometric ratios for some angles).



QUESTIONS 14.1.

Q1. Evaluate the following.

(1) $\sin\frac{2\pi}{3}$
(2) $\cos\frac{5\pi}{6}$
(3) $\sin\frac{\pi}{2}$
(4) $\tan\left(-\frac{\pi}{2}\right)$
(5) $\tan\left(-\frac{\pi}{3}\right)$
(6) $\cos\left(-\frac{5\pi}{6}\right)$
(7) $\cos\left(-\frac{\pi}{2}\right)$
(8) $\tan\frac{2\pi}{3}$
(9) $\sin\left(-\frac{\pi}{3}\right)$
(10) $\cos\left(-\frac{2\pi}{3}\right)$
(11) $\tan\frac{\pi}{2}$
(12) $\sin\frac{5\pi}{6}$
(13) $\sin\left(-\frac{\pi}{2}\right)$
(14) $\cos\frac{\pi}{2}$
(15) $\tan\left(-\frac{2\pi}{3}\right)$
(16) $\sin\left(-\frac{5\pi}{6}\right)$
(17) $\cos\frac{2\pi}{3}$
(18) $\sin\left(-\frac{2\pi}{3}\right)$
(19) $\tan\frac{5\pi}{6}$
(20) $\cos\left(-\frac{\pi}{3}\right)$
(21) $\tan\left(-\frac{5\pi}{6}\right)$

Q2. Find all missing values of basic trigonometric ratios (sine, cosine, tangent) in the conditions specified below.

(1) $\sin A=\frac{1}{2}$, $0\le A\le\frac{\pi}{2}$
(2) $\sin A=\frac{1}{2}$, $\frac{\pi}{2}\le A\le\pi$
(3) $\sin A=-\frac{1}{2}$, $-\frac{\pi}{2}\le A\le 0$
(4) $\sin A=-\frac{1}{2}$, $-\pi\le A\le-\frac{\pi}{2}$
(5) $\cos A=\frac{4}{5}$, $0\le A\le\frac{\pi}{2}$
(6) $\cos A=-\frac{1}{3}$, $\frac{\pi}{2}\le A\le\pi$
(7) $\cos A=\frac{1}{3}$, $\frac{\pi}{2}\le A\le\pi$
(8) $\cos A=\frac{1}{3}$, $-\frac{\pi}{2}\le A\le 0$
(9) $\cos A=-\frac{3}{5}$, $-\pi\le A\le-\frac{\pi}{2}$
(10) $\tan A=\frac{3}{4}$, $0\le A\le\frac{\pi}{2}$
(11) $\tan A=\frac{4}{3}$, $-\pi\le A\le-\frac{\pi}{2}$
(12) $\tan A=-3$, $\frac{\pi}{2}\le A\le\pi$
(13) $\tan A=3$, $-\pi\le A\le-\frac{\pi}{2}$
(14) $\tan A=-1$, $-\pi\le A\le-\frac{\pi}{2}$

Q3. Express missing basic trigonometric ratios in terms of t.

(1) $\sin A=t$, $0\le A\le\frac{\pi}{2}$
(2) $\sin A=t$, $\frac{\pi}{2}\le A\le\pi$
(3) $\sin A=t$, $-\frac{\pi}{2}\le A\le 0$
(4) $\sin A=t$, $-\pi\le A\le-\frac{\pi}{2}$
(5) $\cos A=t$, $0\le A\le\frac{\pi}{2}$
(6) $\cos A=t$, $\frac{\pi}{2}\le A\le\pi$
(7) $\cos A=t$, $-\frac{\pi}{2}\le A\le 0$
(8) $\cos A=t$, $-\pi\le A\le-\frac{\pi}{2}$
(9) $\tan A=t$, $0\le A\le\frac{\pi}{2}$
(10) $\tan A=t$, $\frac{\pi}{2}\le A\le\pi$

(11) $\tan A = t$, $-\frac{\pi}{2} \leq A \leq 0$
(12) $\tan A = t$, $-\pi \leq A \leq -\frac{\pi}{2}$
(13) $\sin A = 2t$, $-\pi \leq A \leq -\frac{\pi}{2}$
(14) $\cos A = 3t$, $0 \leq A \leq \frac{\pi}{2}$
(15) $\cos A = \frac{1}{2}t$, $\frac{\pi}{2} \leq A \leq \pi$
(16) $\tan A = \frac{5t}{2}$, $-\frac{\pi}{2} \leq A \leq 0$

Q4. Find the values of A knowing that:

(1) $0 < A < \frac{\pi}{2}$, $\sin A = 0.2$
(2) $\frac{\pi}{2} < A < \pi$, $\sin A = 0.7$
(3) $-\pi < A < -\frac{\pi}{2}$, $\sin A = -0.2$
(4) $\frac{\pi}{2} < A < \pi$, $\cos A = -0.9$
(5) $-\pi < A < -\frac{\pi}{2}$, $\cos A = -0.9$
(6) $-\frac{\pi}{2} < A < 0$, $\cos A = 0.1$
(7) $\frac{\pi}{2} < A < \pi$, $\tan A = -2$
(8) $-\pi < A < -\frac{\pi}{2}$, $\tan A = 0.3$

Q5. Find the values of A, $-\pi < A \leq \pi$ knowing that:

(1) $\cos A = \frac{1}{2}$ $\sin A < 0$
(2) $\sin A = \frac{1}{2}$ $\cos A < 0$
(3) $\cos A = 0$ $\sin A > 0$
(4) $\cos A = -\frac{\sqrt{3}}{2}$ $\sin A > 0$
(5) $\sin A = -\frac{\sqrt{2}}{2}$, $\cos A < 0$
(6) $\tan A = \sqrt{3}$, $\sin A < 0$
(7) $\tan A = -\frac{1}{\sqrt{3}}$, $\cos A < 0$

Q6. Let $p = \sin(\frac{8\pi}{15})$ and $q = \cos(\frac{4\pi}{15})$. Express in terms of p and q:

(1) $\sin(\frac{4\pi}{15})$
(2) $\tan(\frac{4\pi}{15})$
(3) $\cos(\frac{8\pi}{15})$
(4) $\cos(\frac{11\pi}{15})$
(5) $\sin(\frac{11\pi}{15})$
(6) $\cos(\frac{7\pi}{15})$
(7) $\tan(\frac{7\pi}{15})$

14.3 Graphs of trigonometric functions

The graphs of sine, cosine and tangent are as follows.

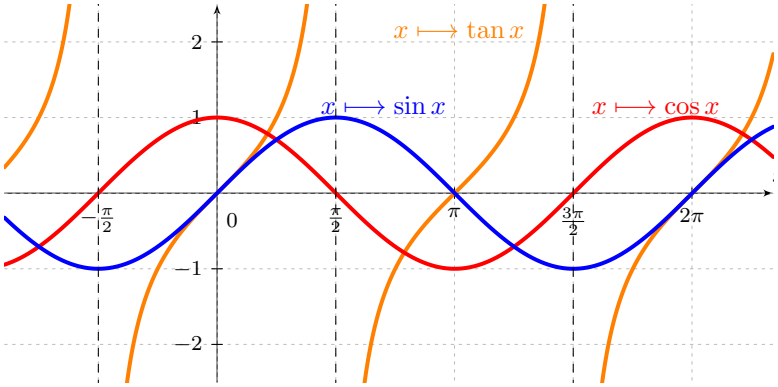
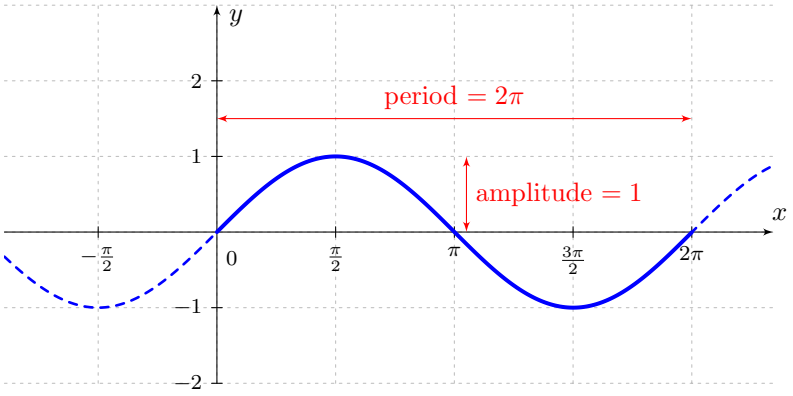


Figure 14.1: Graphs of: $x \longmapsto \sin x$, $x \longmapsto \cos x$ and $x \longmapsto \tan x$.

Notation(terminology).
used for graphs of sine or cosine



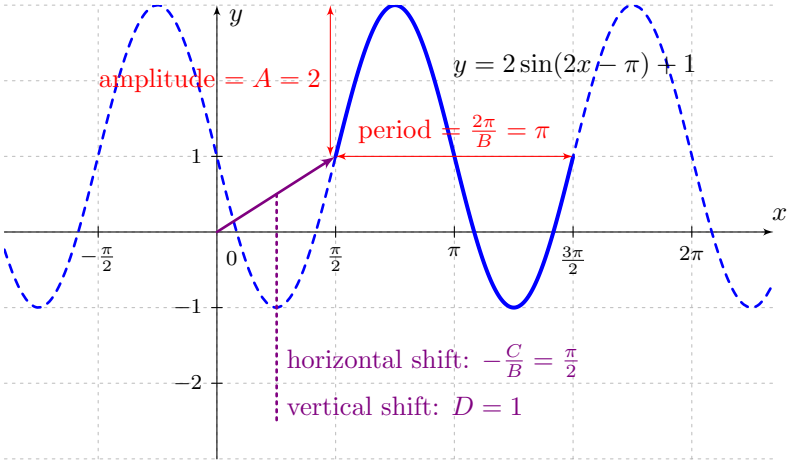
When we consider the graph defined by the equation

$$\begin{aligned} y &= A\sin(Bx+C)+D \\ &= A\sin\left(B\left(x-\left(-\frac{C}{B}\right)\right)\right)+D \end{aligned}$$

the parameters A, B, C and D are responsible for:

A	amplitude	$= \lvert A\rvert$
B	period	$= \frac{2\pi}{B}$ (or $\frac{360}{B}$ when the domain is in degrees)
C	horizontal shift	$= -\frac{C}{B}$
D	vertical shift	$= D$

Note that both horizontal and vertical shifts are applied to the graph with period and amplitude already changed, i.e. to the one defined by $y = A\sin(Bx)$.

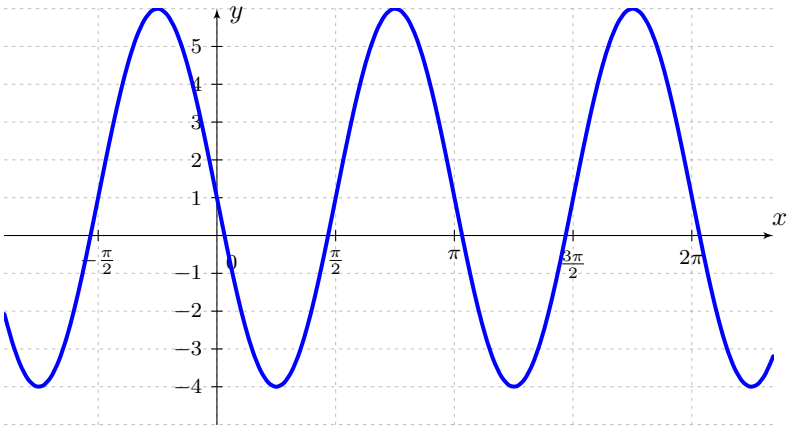


Questions 14.2.

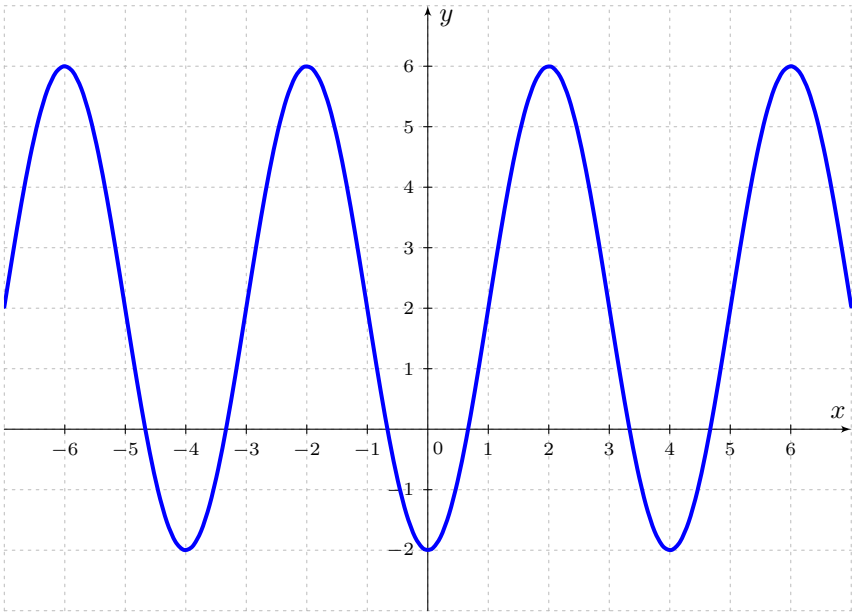
Q7. Sketch the graphs of the following.

(1)	$y_1 = 3\sin 2x$,	$y_2 = 3\sin(2x+\pi)$
(2)	$y_1 = 4\sin 2x$,	$y_2 = 4\sin(2x-\frac{\pi}{2})-1$
(3)	$y_1 = -2\sin\frac{1}{2}x+1$,	$y_2 = -2\sin(\frac{1}{2}x+\frac{\pi}{6})+1$
(4)	$y_1 = 2\sin(\frac{3}{2}x)$,	$y_2 = 2\sin(\frac{3}{2}x-\frac{\pi}{2})-\frac{1}{2}$
(5)	$y_1 = 2\cos 3x$,	$y_2 = 2\cos(3x-\pi)-1$
(6)	$y_1 = -3\cos(2x)$,	$y_2 = -3\cos(2x-\pi)$
(7)	$y_1 = -2\cos(\frac{1}{2}x)$,	$y_2 = 1-2\cos(\frac{1}{2}x+\pi)$
(8)	$y_1 = \frac{1}{2}\cos(\frac{1}{2}x)$,	$y_2 = \frac{1}{2}\cos(\frac{1}{2}x-\frac{\pi}{2})-\frac{1}{2}$
(9)	$y_1 = -\tan x$,	$y_2 = 1-\tan(x-\pi)$
(10)	$y_1 = \tan(\frac{x}{2})$,	$y_2 = \tan(\frac{x}{2}-\frac{\pi}{2})$
(11)	$y_1 = -\tan(\frac{1}{2}x)$,	$y_2 = -\tan(\frac{1}{2}x+\frac{3\pi}{2})-1$
(12)	$y_1 = -\tan(2x)$,	$y_2 = -\tan(2x-\frac{\pi}{2})$

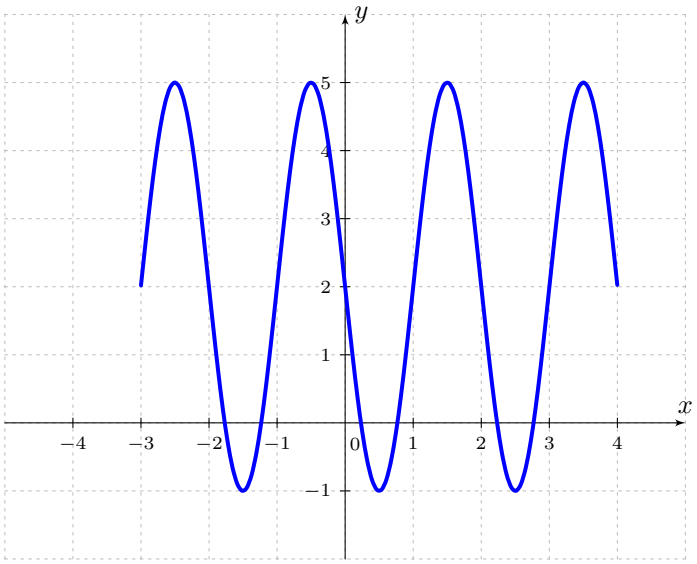
Q8. The diagram below shows a graph of $y = A\sin Bx + C$. Find the values of A, B and C.



Q9. The diagram below shows a graph of $y = p\cos qx + s$. Find the values of p, q and s.



Q10. The diagram below shows a graph of $y = a\sin(bx - c) + d$, $-3 \leq x \leq 4$, where $a, b, c, d \in \mathbb{R}^+$.

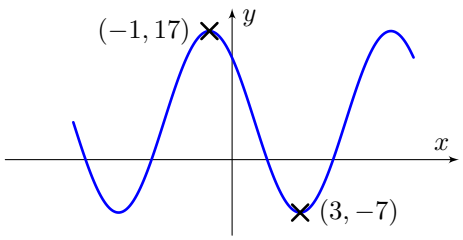


(i) Find the values of a, b, c and d.

(ii) The equation $y = m$ has exactly three solutions. Write down the value of m.

(iii) The equation $y = k$ has exactly six solutions. Write down the range of values of k.

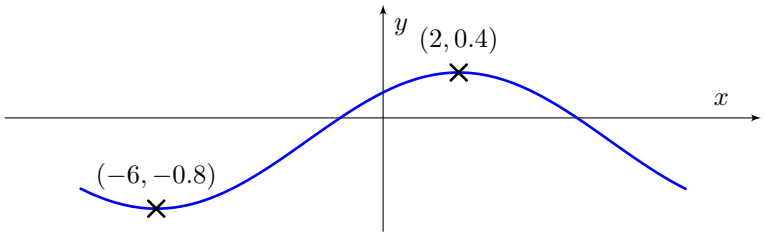
Q11. The diagram below shows a graph of a function that has a maximum point at $(-1, 17)$ and a minimum point at $(3, -7)$.



(i) Find the equation of the graph in the form $y = A\sin(Bx + C) + D$.

(ii) Find the equation of the graph in the form $y = p\cos(qx + r) + s$.

Q12. The diagram below shows a graph of a function that has a maximum point at $(2, 0.4)$ and a minimum point at $(-6, -0.8)$.



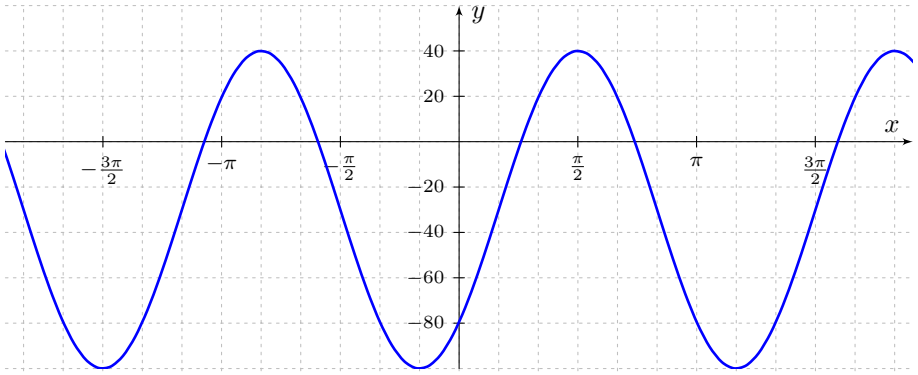
(i) Find the equation of the graph in the form $y = A\sin(Bx + C) + D$.

(ii) Find the equation of the graph in the form $y = p\cos(qx + r) + s$.

The graph is transformed by translation by $\binom{0.2}{1}$ followed by horizontal dilation by $\frac{1}{2}$.

(iii) Find the equation of the new graph.

Q13. The diagram below shows a graph of a function that has a maximum point at $(\frac{\pi}{2}, 40)$ and a minimum point at $(-\frac{\pi}{6}, -100)$.



(i) Find the equation of the graph in the form $y = A\sin(Bx + C) + D$.

(ii) Find the equation of the graph in the form $y = p\cos(qx + r) + s$.

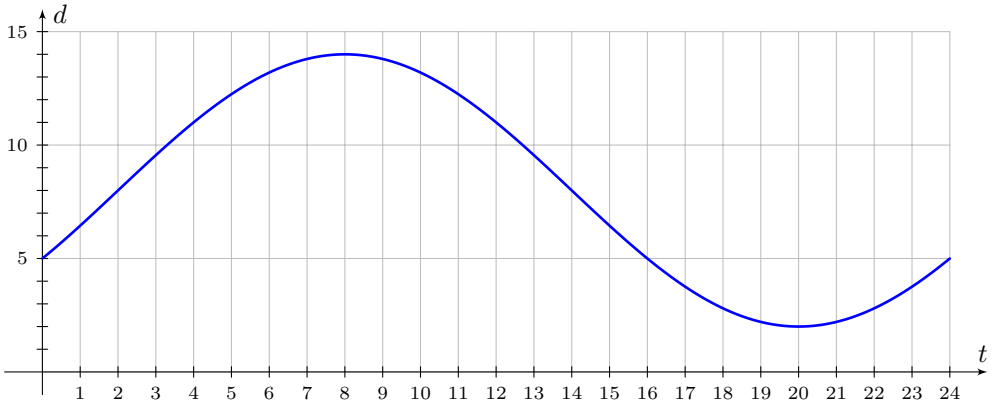
The graph is transformed by translation by $\binom{\frac{\pi}{6}}{40}$ followed by vertical dilation by $\frac{1}{2}$.

(iii) Find the equation of the new graph.

Q14. Let $f(x) = \frac{2x}{3}$ and $g(x) = \frac{1}{4}\sin(6x) + 1$. Find an expression, the period and the range of:

a) $f \circ g(x)$ b) $g \circ f(x)$

Q15. The depth of water, d metres, at point P, during a single day is modelled by the graph below. The time t is given in hours after midnight.



(i) What is the largest depth? At what time did it occur?

(ii) What is the smallest depth? At what time did it occur?

(iii) Find the equation of the graph in the form $d = A\sin(Bt + C) + D$

A boat cannot safely sail past point P if the depth is smaller than 7 metres.

(iv) Use your equation to find the time intervals when the boat can safely sail past point P on the day considered.

Q16. The depth of water, d metres, at point R, during a single day is modelled by the equation

$$d = A\sin(Bt + C) + D$$

The time t is given in hours after midnight, $0 \leq t \leq 24$. When $t = 4$ and when $t = 16$ the depth was largest and equal $18m$. The minimum depth was 7 metres.

(i) Find the values of A, B, C and D.

(ii) What percentage of time the depth is larger than 14 metres?

Q17. A metal ball is suspended on a string from the ceiling. It is pulled down and released, and then oscillates up and down. Its height, h centimetres above the floor, is modelled by the function

$$h = 210 + 4\sin((180t)^\circ),$$

where t is time in seconds after release.

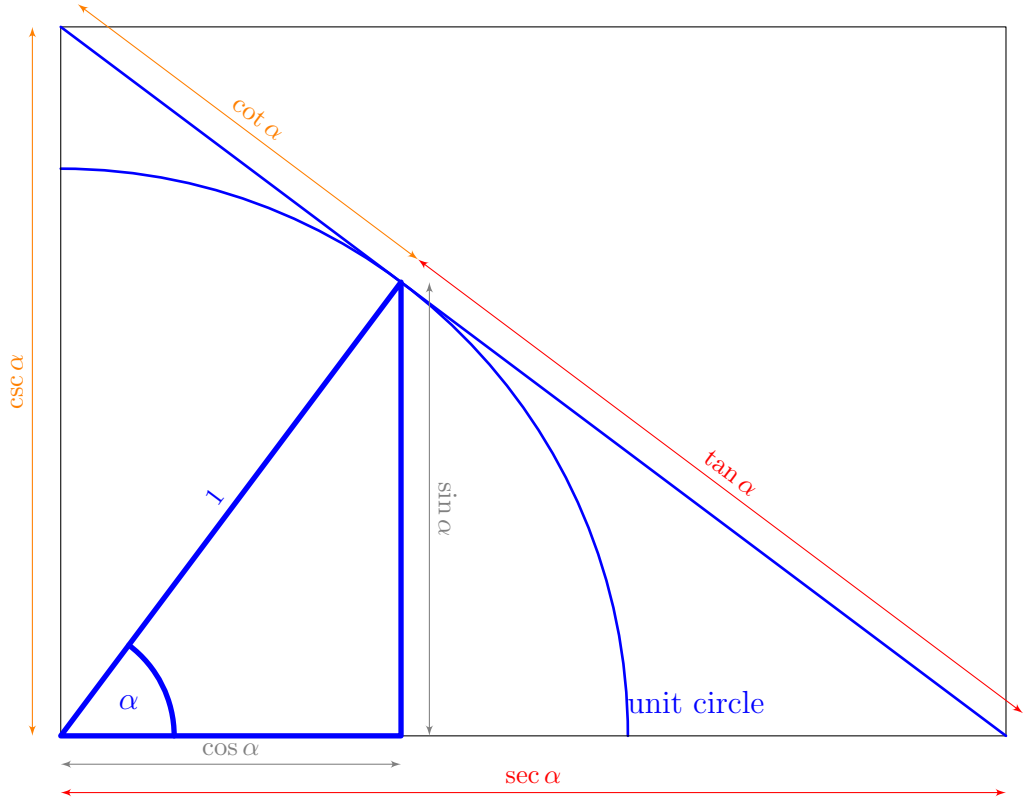
(i) Find the height of the ball after 4 seconds.

(ii) What is the period of the motion?

(iii) Find the minimum height of the ball.

(iv) Find the first time at which the height is $208cm$.

14.4 Cosecant, secant and cotangent - reciprocal circular functions



Definition 103 (reciprocal circular functions).
The reciprocal circular functions cosecant, secant *and* cotangent *are defined as follows:*

$\csc x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$	$\cot x = \frac{1}{\tan x}$
for $\sin x \neq 0$	*for* $\cos x \neq 0$	*for* $\tan x \neq 0$

Questions 14.3.

Q18. In the same set of axes sketch the graphs of:

(1) $x \longmapsto \sin x$ and $x \longmapsto \csc x$ (2) $x \longmapsto \cos x$ and $x \longmapsto \sec x$ (3) $x \longmapsto \tan x$ and $x \longmapsto \cot x$

Q19. Sketch the graphs of:

(1) $x \longmapsto \frac{1}{2}\sec 2x$ (2) $x \longmapsto \csc(x - \frac{\pi}{3})$ (3) $x \longmapsto \cot(\frac{\pi}{4} - x)$ (4) $x \longmapsto \sec(\frac{x}{2}) + 1$

Q20. Find the values of A, $-\pi < A \leq \pi$ knowing that:

(1) $\cot A = -1,\ \sin A < 0$ (2) $\sec A = -2,\ \sin A < 0$ (3) $\csc A = -2,\ \cos A > 0$

Q21. Solve the following equations for $x \in [0, 2\pi]$.

(1) $\sec x = 2$
(2) $\csc x = \frac{2}{3}\sqrt{3}$
(3) $\cot x = \sqrt{3}$
(4) $\sec(\frac{x}{2}) = -2$
(5) $\csc x = -1$
(6) $\cot 2x = 0$

Q22. Prove the following identities.

(1) $\tan x = \frac{\sin x}{\cos x}$ (2) $\sin^2 x + \cos^2 x = 1$ (3) $1 + \cot^2 x = \csc^2 x$ (4) $\tan^2 x + 1 = \sec^2 x$

Q23. Find the values of all missing reciprocal trigonometric ratios knowing that:

(1) $\csc x = 3$ and $\frac{\pi}{2} < x < \pi$
(2) $\sec x = 3$ and $-\pi < x < -\frac{\pi}{2}$
(3) $\cot x = -3$ and $\frac{\pi}{2} < x < \pi$
(4) $\csc x = -\frac{5}{3}$ and $\frac{\pi}{2} < x < \pi$
(5) $\sec x = -2.6$ and $-\pi < x < -\frac{\pi}{2}$
(6) $\csc x = -2.5$ and $-\frac{\pi}{2} < x < 0$

14.5 Inverse circular functions

In the following section we are going to consider the inverses of *sine*, *cosine* and *tangent* functions.
As stated in chapter 12, a function is invertible whenever it is bijective.
To define inverses of the trigonometric functions we have to:

- narrow their **domains** so that they become **injective** and
- specify the **ranges** to make them **surjective**.

Arcsin - inverse of sine

As shown in figure 14.2 function $x \longmapsto \sin x$ is **injective** for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, to make *sine* bijective we define it as a function:

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

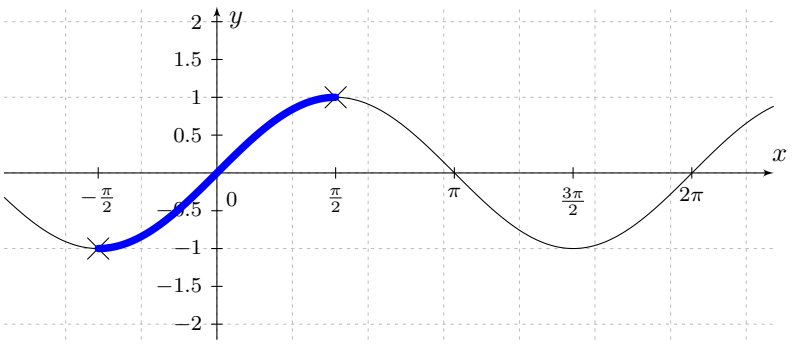


Figure 14.2: $x \longmapsto \sin x$ made bijective.

Hence, the *inverse of* $x \longmapsto \sin x$, denoted $x \longmapsto \arcsin x$, is a function:

$$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The graph of $x \longmapsto \arcsin x$ is symmetrical to the graph of $x \longmapsto \sin x$ in the line $y = x$ (see figure 14.3).

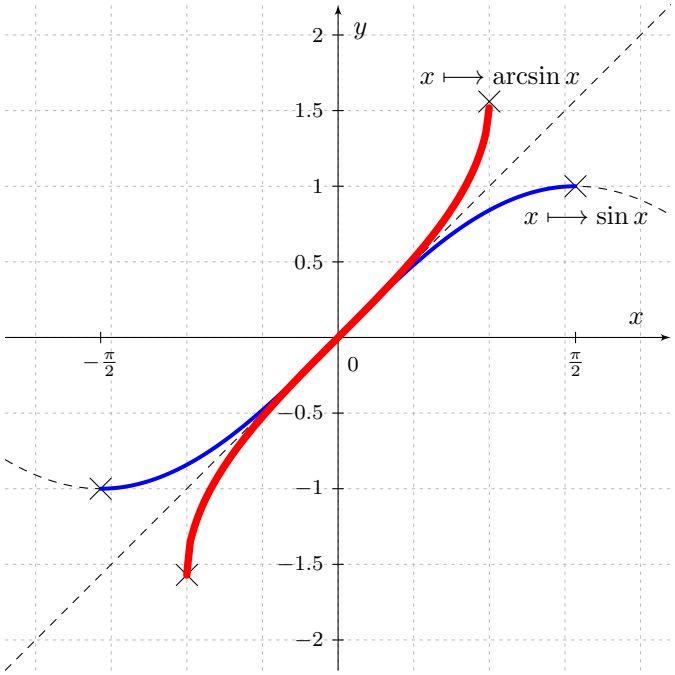


Figure 14.3: $x \longmapsto \sin x$ and $x \longmapsto \arcsin x$.

PROPERTIES OF arcsin.

1. arcsin *is an* ***odd*** *function.*

2. arcsin *is an* ***increasing*** *function.*

Arccos - inverse of cosine

As shown in figure 14.4 function $x \longmapsto \cos x$ is **injective** for $x \in [0, \pi]$. Thus, to make *cosine* bijective we define it as a function:

$$\cos : [0, \pi] \longrightarrow [-1, 1]$$

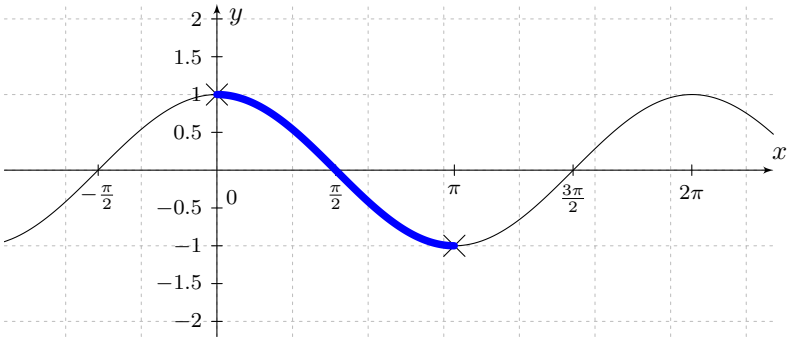


Figure 14.4: $x \longmapsto \cos x$ made bijective.

Hence, the *inverse of* $x \longmapsto \cos x$, denoted $x \longmapsto \arccos x$, is a function:

$$\arccos : [-1, 1] \longrightarrow [0, \pi]$$

The graph of $x \longmapsto \arccos x$ is symmetrical to the graph of $x \longmapsto \cos x$ in the line $y = x$ (see figure 14.5).

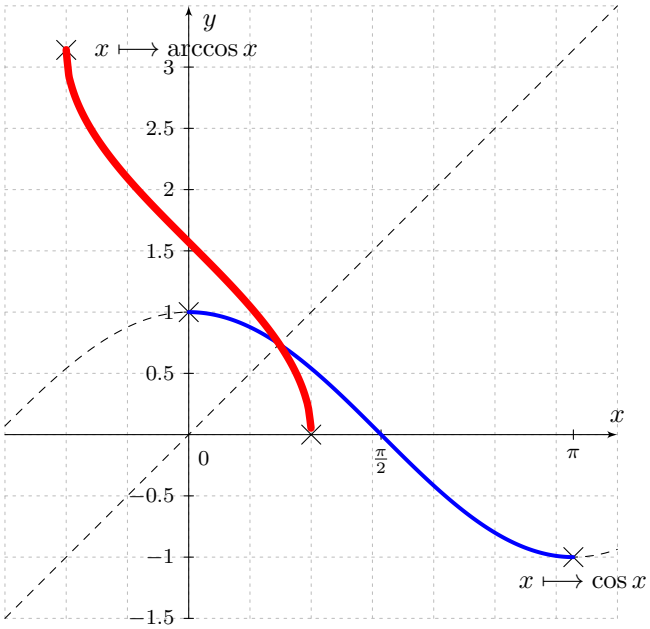


Figure 14.5: $x \longmapsto \cos x$ and $x \longmapsto \arccos x$.

PROPERTIES OF arccos.

1. arccos *is neither* ***odd*** *nor* ***even*** *function.*

2. arccos *is a* ***decreasing*** *function.*

Arctan - inverse of tangent

As shown in figure 14.6 function $x \longmapsto \tan x$ is **injective** for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, to make *tangent* bijective we define it as a function:

$$\tan : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}$$

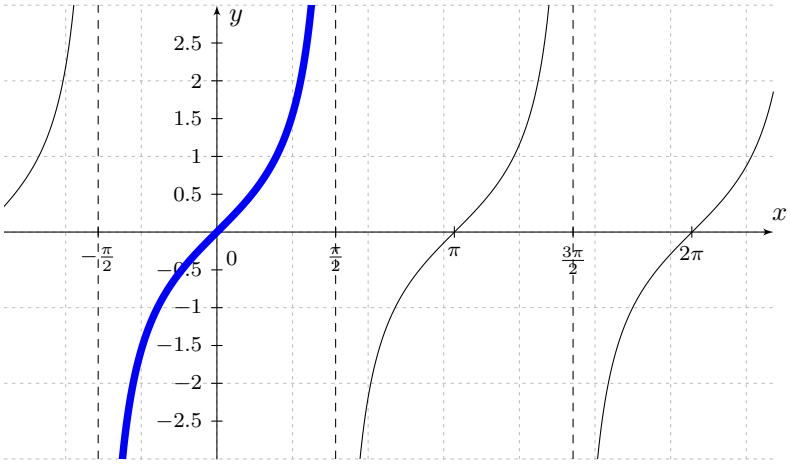


Figure 14.6: $x \longmapsto \tan x$ made bijective.

Hence, the *inverse of* $x \longmapsto \tan x$, denoted $x \longmapsto \arctan x$, is a function:

$$\arctan : \mathbb{R} \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The graph of $x \longmapsto \arctan x$ is symmetrical to the graph of $x \longmapsto \tan x$ in the line $y = x$ (see figure 14.7).

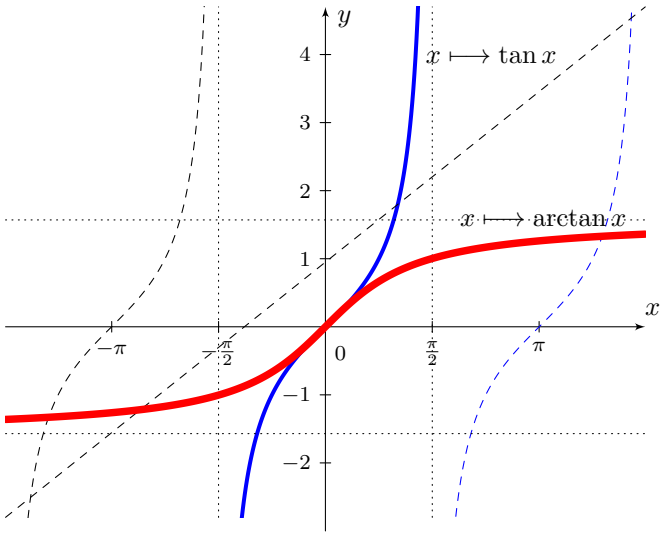


Figure 14.7: $x \longmapsto \tan x$ and $x \longmapsto \arctan x$.

PROPERTIES OF arctan.

1. arctan *is an* ***odd*** *function.*

2. arctan *is an* ***increasing*** *function.*

QUESTIONS 14.4.

Q24. Write down the domain and the range of the function.

(1) $x \longmapsto \arcsin x$ (2) $x \longmapsto \arccos x$ (3) $x \longmapsto \arctan x$

Q25. Find the exact value of $\sin A$, $\cos A$ and $\tan A$ knowing that:

(1) $A = \arccos \frac{\sqrt{2}}{3}$ (3) $A = \arcsin\left(\frac{2}{3}\right)$ (5) $A = \arctan 3$

(2) $A = \arccos\left(-\frac{\sqrt{7}}{4}\right)$ (4) $A = \arcsin\left(-\frac{3}{5}\right)$ (6) $A = \arctan\left(-\frac{2}{3}\right)$

Q26. Complete the table.

	$A \in$	$s = \sin A$	$c = \cos A$	$t = \tan A$	$\arcsin s$	$\arccos c$	$\arctan t$	A
(1)	$\left[\frac{\pi}{2}, \pi\right]$	0.768						
(2)	$\left[\frac{\pi}{2}, \pi\right]$		-0.42					
(3)	$\left[-\pi, -\frac{\pi}{2}\right]$	-0.253						
(4)	$\left[-\frac{\pi}{2}, 0\right]$		0.908					
(5)	$\left[-\pi, -\frac{\pi}{2}\right]$			3.2				
(6)	$\left[\frac{\pi}{2}, \pi\right]$			-0.2				
(7)	$\left[-\pi, -\frac{\pi}{2}\right]$	-0.029						

Q27. Simplify:

(1) $\sin(\arccos a)$
(2) $\sin(\arctan a)$
(3) $\cos(\arcsin a)$
(4) $\cos(\arctan a)$
(5) $\tan(\arcsin a)$
(6) $\tan(\arccos a)$
(7) $\cot(\arccos a)$
(8) $\cot(\arctan a)$
(9) $\cot(\arcsin a)$
(10) $\sec(\arccos a)$
(11) $\sec(\arctan a)$
(12) $\sec(\arcsin a)$
(13) $\csc(\arccos a)$
(14) $\csc(\arctan a)$
(15) $\csc(\arcsin a)$

Q28. Find the largest possible domain and the range of the following functions.

(1) $y = \arcsin(2x)$

(2) $y = -3\arccos(\frac{x}{2})$

(3) $y = \sqrt{\arctan x}$

(4) $y = \sqrt{\frac{\pi}{4} - \arctan x}$

(5) $y = \frac{1}{\arcsin x}$

(6) $y = \frac{2}{\arccos x}$

(7) $y = \sqrt{\frac{\pi}{6} - \arcsin x}$

(8) $y = \sqrt{\frac{\pi}{4} - \arccos x}$

(9) $y = \sqrt{\arccos x - \frac{\pi}{4}}$

Chapter 15

Trigonometric identities

Theorem 104 (Pythagorean identities).
For any number x the following is true:

1. $\sin^2 x + \cos^2 x = 1$

2. $\tan^2 x + 1 = \sec^2 x$

3. $\cot^2 x + 1 = \csc^2 x$

(provided that the trigonometric ratios involved in an identity exist for this x).

Proof. The $\mathbf{1}^{st}$ identity can be easily observed in a unit circle. The $\mathbf{2}^{nd}$ and $\mathbf{3}^{rd}$ can be obtained from the $\mathbf{1}^{st}$ by dividing its both sides by $\cos^2 x$ and $\sin^2 x$ respectively. □

Theorem 105 (compound angle identities).
For any numbers A and B the following is true:

1. $\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$

2. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

3. $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ $(1 \mp \tan A \tan B \neq 0)$

Corollary 106 (double angle identities).
For any number x the following is true:

1. $\sin 2x = 2 \sin x \cos x$

2. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

3. $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

Questions 15.1.

Q1. Find the exact values of $\sin(\frac{\pi}{8})$, $\cos(\frac{5\pi}{12})$ and $\tan(\frac{\pi}{12})$.

Q2. Knowing that $\cos x = \frac{1}{3}$ and $-\pi < x < 0$ find the value of $\sin(x + \frac{\pi}{3})$.

Q3. Knowing that $\sin x = -\frac{2}{3}$ and $\frac{\pi}{2} < x < \frac{3\pi}{2}$ find the value of $\cos(x - \frac{\pi}{6})$.

Q4. Knowing that $\sin x = 0.6$ and $\frac{\pi}{2} < x < \frac{3\pi}{2}$ find the value of $\tan(x - \frac{\pi}{4})$.

Q5. Prove the following.

(1) $\sin 2x \sin x = 2\cos x - 2\cos^3 x$
(2) $\frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1 - \sin 2x}{\cos 2x}$
(3) $\frac{\cos 2x}{\sin^2 x} + 2 = \csc^2 x$
(4) $\frac{\sin x}{\cot x} + \frac{\sin 2x}{2\sin x} = \cos x + \frac{1 - \cos 2x}{2\cos x}$
(5) $\frac{\sin 2x}{1 - \cos 2x - \sin^2 x} = 2\cot x$
(6) $\frac{\sin 3x}{\sin 2x \sin x} = \cot x + \cot 2x$
(7) $(\cot x - \csc x)^2 = \frac{\sec x - 1}{\sec x + 1}$
(8) $\frac{1}{2}(1 + \cot 2x)^2 = \frac{1 + \sin 4x}{1 - \cos 4x}$
(9) $\frac{2\cos^2 x}{1 - \cos 2x} = \csc^2 x - 1$
(10) $\tan 3x - \frac{\sin 2x \cos x}{\cos 3x} = \frac{1}{\cot x - \tan 2x}$
(11) $\left(\frac{\cos 2x}{\sin x}\right)^2 + 4 = 4\sin^2 x + \csc^2 x$
(12) $(\cos^3 x - \sin x + \sin^3 x)(1 - \tan x)^2 = \\ = \cos^3 x - \sin^3 x + 1.5 \sin 2x (\sin x - \cos x)$
(13) $\tan x + \cot x = 2\csc 2x$
(14) $\sec 2x + \tan 2x = \frac{1 + \tan x}{1 - \tan x}$

Q6. Show that $\arcsin(\frac{3}{5}) + \arcsin(\frac{4}{5}) = \frac{\pi}{2}$.

Q7. Show that $\arcsin(\frac{5}{13}) + \arcsin(\frac{12}{13}) = \frac{\pi}{2}$.

Q8. Show that $\arccos(\frac{40}{41}) + \arccos(\frac{9}{41}) = \frac{\pi}{2}$.

Q9. Find the exact value of p.

(1) $\arctan p = \arctan(\frac{1}{2}) + \arctan(\frac{1}{3})$
(2) $\arctan p = \arctan(\frac{1}{5}) + \arctan(\frac{2}{3})$
(3) $\arctan p = \arctan(\frac{1}{3}) - \arctan(\frac{1}{8})$
(4) $\arctan p = \arctan(\frac{1}{7}) + \arctan(\frac{1}{3})$
(5) $\arctan p = \arctan(\frac{1}{9}) - \arctan(\frac{1}{11})$
(6) $\arctan p = \arctan(\frac{2}{3}) + \arctan(\frac{3}{5})$
(7) $\arctan p = \arctan(\frac{3}{4}) - \arctan(\frac{1}{3})$
(8) $\arctan p = \arctan(\frac{5}{7}) + \arctan(\frac{1}{6})$
(9) $\arctan p = \arctan(\frac{4}{9}) - \arctan(\frac{1}{4})$
(10) $\arctan p = \arctan(\frac{7}{9}) - \arctan(\frac{1}{5})$

Q10. Consider the function $f(x) = 2\sin x + \cos x$, $x \in \mathbb{R}$.

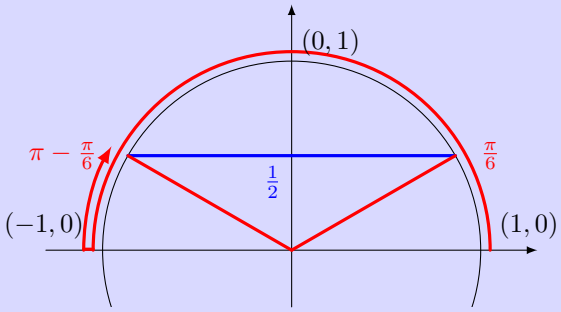
(i) Find the exact value of $R > 0$ and the value of α $(0 \leq \alpha \leq \frac{\pi}{2})$ such that $f(x) \equiv R\sin(x + \alpha)$.
(ii) State the exact largest and smallest values of the function $y = g(x)$ defined by $g(x) = 6\sin x + 3\cos x + \sqrt{5}$.
(iii) State the exact largest and smallest values of the function $y = h(x)$ defined by $h(x) = \frac{2}{2\sin x + \cos x + 3}$.

Chapter 16

Trigonometric equations and inequalities

To solve simple trigonometric equations we can use the unit circle.

Example 107 (trigonometric equation - sin 1).
Solve $\sin x = \frac{1}{2}$ *for* $0 \leq x \leq 2\pi$.



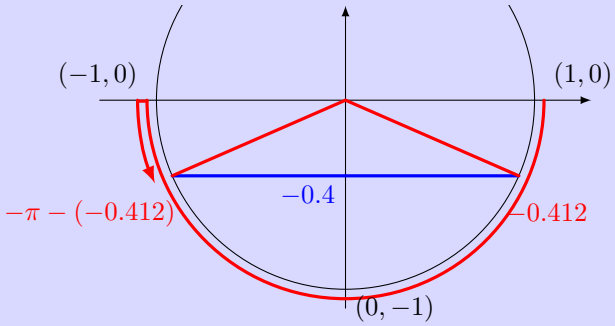
the solutions are:

$$x = \frac{\pi}{6} \text{ or } x = \pi - \frac{\pi}{6}$$

i.e.

$$x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6}$$

Example 108 (trigonometric equation - sin 2).
Solve $\sin x = -0.4$ *for* $0 \leq x \leq 2\pi$.



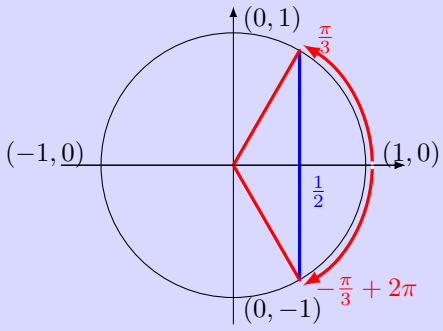
the solutions are:

$$x \approx -0.412 \text{ or } x \approx -\pi - (-0.412)$$

i.e.

$$x \approx -0.412 \text{ or } x \approx -2.73$$

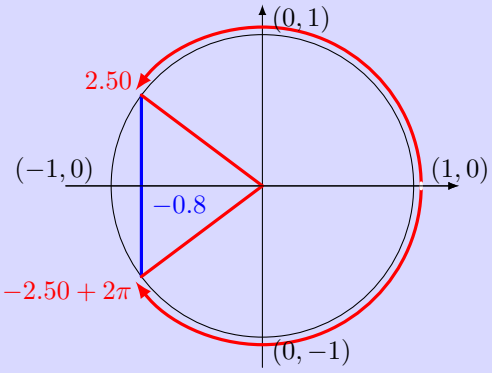
Example 109 (trigonometric equation - cos 1).
Solve $\cos x = \frac{1}{2}$ *for* $0 \leq x \leq 2\pi$.



the solutions are:

$$x = \frac{\pi}{3} \text{ or } x = \frac{5\pi}{3}$$

Example 110 (trigonometric equation - cos 2).
Solve $\cos x = -0.8$ *for* $0 \le x \le 2\pi$.



the solutions are:

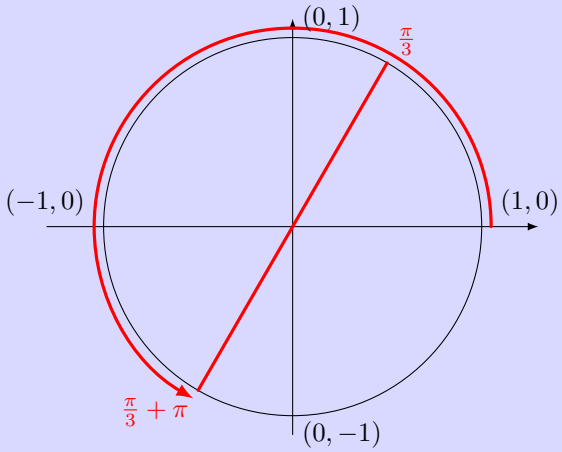
$$x \approx 2.50 \text{ or } x \approx -2.50 + 2\pi$$

i.e.

$$x \approx 2.50 \text{ or } x \approx 3.79$$

The equations that involve tangent are even easier, as two consecutive solutions differ by π:

Example 111 (trigonometric equation - tan 1).
Solve $\tan x = \sqrt{3}$ *for* $0 \le x \le 2\pi$.



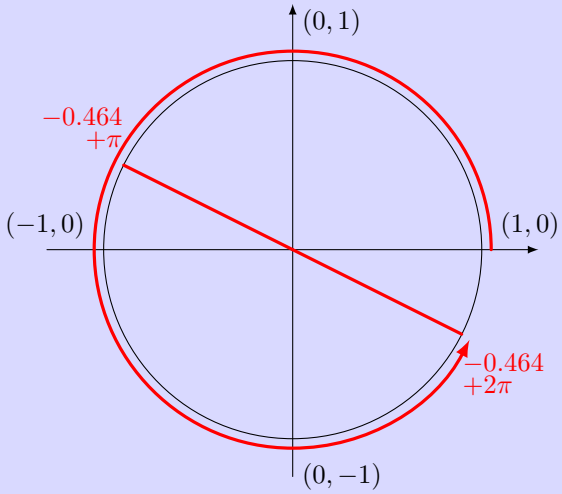
the solutions are:

$$x = \frac{\pi}{3} \text{ or } x = \frac{\pi}{3} + \pi$$

i.e.

$$x = \frac{\pi}{3} \text{ or } x = \frac{4\pi}{3}$$

Example 112 (trigonometric equation - tan 2).
Solve $\tan x = -0.5$ *for* $0 \le x \le 2\pi$.



the solutions are:

$$x \approx -0.464 + \pi \text{ or } x \approx -0.464 + 2\pi$$

i.e.

$$x \approx 2.68 \text{ or } x \approx 5.82$$

Example 113 (trigonometric equation - use of identities).
Solve $3\cos^3 x - \sin^2 x + \cos 2x - 4\cos x = 2$ *for* $0 \le x \le 2\pi$.

Let us firstly change all trigonometric ratios to $\cos x$:

$$3\cos^3 x - (1 - \cos^2 x) + 2\cos^2 x - 1 - 4\cos x = 2$$

and simplify the expression:

$$3\cos^3 x + 3\cos^2 x - 4\cos x - 4 = 0$$

then factorize:

$$3\cos^2 x(\cos x + 1) - 4(\cos x + 1) = 0$$
$$(3\cos^2 x - 4)(\cos x + 1) = 0$$

$\cos^2 x = \frac{4}{3}$	or	$\cos x = -1$
no solution		$x = \pi$

Example 114 (trigonometric equation - identities, reciprocal functions).
Solve $2\cot^2 x + \csc x - 4 = 0$ *for* $0 \le x \le 2\pi$.

Firstly we will change $\cot x$ int $\csc x$:

$$2(\csc^2 x - 1) + \csc x - 4 = 0$$

then simplify:

$$2\csc^2 x + \csc x - 6 = 0$$

and factorize:

$$(2\csc x - 3)(\csc x + 2) = 0$$

$$\csc x = \tfrac{3}{2} \quad \text{or} \quad \csc x = -2$$

to find the solutions we will change $\csc x$ into $\sin x$:

$$\sin x = \tfrac{2}{3} \qquad \sin x = -\tfrac{1}{2}$$

$$x \in \{0.730, 2.41\} \qquad x \in \left\{\tfrac{7\pi}{6}, \tfrac{11\pi}{6}\right\}$$

so finally: $x \in \left\{0.730, 2.41, \tfrac{7\pi}{6}, \tfrac{11\pi}{6}\right\}$

Example 115 (trigonometric equation - change of domain).
Solve $2\cos\left(2x - \frac{\pi}{2}\right) - 1 = 0$ *for* $0 \le x \le 2\pi$.

Let us substitute $A = 2x - \frac{\pi}{2}$ where $0 \le x \le 2\pi$

Then the equation takes the form:

$2\cos A = 1$

The domain for A is $0 \le 2x \le 4\pi$

$$-\tfrac{\pi}{2} \le 2x - \tfrac{\pi}{2} \le 4\pi - \tfrac{\pi}{2}$$

$$\boxed{-\tfrac{\pi}{2} \le A \le 3.5\pi}$$

solving the equation $\cos A = \frac{1}{2}$

we get $A = \frac{\pi}{3}$ or $A = -\frac{\pi}{3}$ (between 0 and 2π)

However, there are more solutions in the domain.

We can always subtract or add 2π to each of the solutions above.

(-2π): ~~$A = -\frac{5\pi}{3}$~~ or ~~$A = -\frac{7\pi}{3}$~~ (not in the domain)

$(+2\pi)$: $A = \frac{7\pi}{3}$ or $A = \frac{5\pi}{3}$ (both in the domain)

$(+4\pi)$: ~~$A = \frac{13\pi}{3}$~~ or ~~$A = \frac{11\pi}{3}$~~ (not in the domain)

Hence: $A \in \{-\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}\}$

i.e.: $2x - \frac{\pi}{2} \in \{-\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}\}$

$(+\frac{\pi}{2})$: $2x \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}\}$

$(\div 2)$: $x \in \{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\}$

Questions 16.1.

Q1. Solve the following equations.

(1) $\sin x = \frac{1}{2}$ for $x \in [0, 2\pi]$
(2) $\sin x = \frac{1}{2}$ for $x \in [-2\pi, 0]$
(3) $\sin x = -\frac{1}{2}$ for $x \in [-2\pi, 0]$
(4) $\sin x = -\frac{1}{2}$ for $x \in [0, 2\pi]$
(5) $\sin x = -1$ for $x \in [-3\pi, 2\pi]$
(6) $\sin x = 0.4$ for $x \in [0, 2\pi]$
(7) $\sin x = -0.4$ for $x \in [0, 2\pi]$
(8) $\sin x = 0.4$ for $x \in [-2\pi, 0]$
(9) $\sin x = -0.4$ for $x \in [-2\pi, 0]$
(10) $\sin x = -1$ for $x \in [-2\pi, 2\pi]$

Q2. Solve the following equations.

(1) $\cos x = \frac{1}{2}$ for $x \in [0, 2\pi]$
(2) $\cos x = \frac{1}{2}$ for $x \in [-2\pi, 0]$
(3) $\cos x = -\frac{1}{2}$ for $x \in [-2\pi, 0]$
(4) $\cos x = -\frac{1}{2}$ for $x \in [0, 2\pi]$
(5) $\cos x = -1$ for $x \in [-3\pi, 2\pi]$
(6) $\cos x = 0.4$ for $x \in [0, 2\pi]$
(7) $\cos x = -0.4$ for $x \in [0, 2\pi]$
(8) $\cos x = 0.4$ for $x \in [-2\pi, 0]$
(9) $\cos x = -0.4$ for $x \in [-2\pi, 0]$
(10) $\cos x = -1$ for $x \in [-2\pi, 2\pi]$

Q3. Solve the following equations.

(1) $\sec x = -2$ for $x \in [-\pi, \pi]$
(2) $\tan x = 3$ for $x \in [-\pi, \pi]$
(3) $\cot x = 3$ for $x \in [-\pi, \pi]$

Q4. Solve the following equations for $x \in [0, 2\pi]/x \in [0, 360]$.

(1) $(\sin x + 1)(2\sin^2 x - 1) = 0$
(2) $(4\sin^2 x - 3)(\sin x + 2) = 0$
(3) $(4\cos^2 x - 3)(\tan x + 2) = 0$
(4) $\sin^2 x + 1 = 2\sin x$
(5) $2\cos^2(x^\circ) + \cos(x^\circ) - 1 = 0$
(6) $\tan^2 x - 3 = 0$
(7) $\tan^2(x^\circ) - 2 = 0$
(8) $\sin^2 x - \cos x + 1 = 0$
(9) $1 - 4\cos^2 x - 4\sin x = 0$
(10) $\cos^2 x + 3\sin x - 3 = 0$
(11) $\sin x = \cos x$
(12) $2 - 3\cot x - 5\cot^2 x = 0$
(13) $2\tan^2 x + \sec x = 4$
(14) $3\tan x = 1 + 2\cot x$
(15) $2\sin^2 x + \cos^2 x = 2$
(16) $\tan^2(x^\circ) + \sec(x^\circ) - 1 = 0$
(17) $3\sec x - 2\cos x - 3\tan x = 0$
(18) $2\sin(x^\circ) + \cot(x^\circ) = \csc(x^\circ)$
(19) $15\cos x = 1 + 2\sec x$
(20) $2\tan x = 5 + 3\cot x$
(21) $4 - 2\sec x + 10\cot x - 5\csc x = 0$
(22) $4\sin x - 3\csc x - 3\cot^2 x + 1 = 0$
(23) $2\sin x + 2\cos x = \tan x + 1$

Q5. Solve the following equations for $x \in [0, 2\pi]$.

(1) $\cos(x + \pi) = -\frac{\sqrt{2}}{2}$
(2) $\cos(0.5x - \pi) = -\frac{\sqrt{3}}{2}$
(3) $\tan(0.5x - \pi) = -\sqrt{3}$
(4) $\tan(\frac{x}{3} - \frac{\pi}{2}) = -1$
(5) $2\sin(3x - \pi) = -1$
(6) $\tan(\frac{x}{2} - \frac{\pi}{4})\sqrt{3} = 1$
(7) $2\cos(2x - \frac{\pi}{3}) = 1$
(8) $4\cos^2(\frac{x}{2} + \frac{\pi}{3}) = 3$
(9) $\sqrt{3}\cot(0.5x + \pi) = -1$

Q6. Solve the following equations for $x \in [0, 2\pi]/x \in [0, 360]$.

(1) $\sin x - \cos 2x = 0$
(2) $\sin 4x^\circ = \sin 2x^\circ$
(3) $\cos 2x = (\cos x - \sin x)^2 + \sin 2x$
(4) $\cos 2x + 2\sin^2 x - \sin 2x = 0$
(5) $\sin(0.5x) = -\sqrt{3}\cos(0.5x)$
(6) $\sin^2 2x + 2\sin^3 x = 2\sin x$
(7) $\cos 2x = (\cos x - \sin x)^2 + \sin 2x$
(8) $\cos^2 x = \cos 2x + 1$
(9) $\sin 2x = \cos x$
(10) $\cos 2x - \cos^2 x = 0$
(11) $\cos 2x + \sin^2 x = \frac{1}{2}$
(12) $2\cos 4x = 2\cos^2 2x + \sin^2 2x$
(13) $\sin(6x^\circ) = \sin(3x^\circ)$
(14) $\frac{\sin^3 2x}{\cos x} = 2\sin x$

Q7. Solve the following inequalities for $x \in [0, 2\pi]/x \in [0, 360]$.

(1) $4\sin(x^\circ)\cos(x^\circ) > \sqrt{3}$
(2) $(\sin x - \cos x)^2 + \cos 2x \geq \cos^2 x - \sin 2x$
(3) $\sin x > 0.5$
(4) $\tan x \leq 1$
(5) $\cos^2 x \geq \cos 2x + 1$
(6) $2\cos^2 x + 1 \geq 2\sin^2 x$
(7) $\sin^2 x \geq \sin^4 x$
(8) $\cos x < -\frac{1}{2}$
(9) $\cos x \geq 0.6$
(10) $2\sin(x^\circ) + \sqrt{3} \leq 0$

Q8. (i) Express A in terms of B when $\sin A = \sin B$.

(ii) Solve the equation $\sin x = \sin(3x - \frac{\pi}{3})$ for $0 \leq x \leq 2\pi$.

(iii) Solve the equation $\cos 3x + \sin 3x = \cos x + \sin x$ for $0 \leq x \leq 2\pi$.

Q9. (i) Sketch the graphs of $y = \sin x$ and $y = \sin^2 x$, $0 \leq x \leq 2\pi$.

(ii) On a separate diagram sketch the graphs of $y = \sin^2 2x$ and $y = \csc^2 2x$, $0 \leq x \leq \pi$, clearly stating the coordinates of any local maximum or minimum points and the equations of any asymptotes.

(iii) Show that $\sec^2 x + \csc^2 x = 4 \csc^2 2x$.

(iv) Solve the equation $\sec^2 x + \csc^2 x = 4 - 3 \cos^2 2x$, $0 \leq x \leq \frac{\pi}{2}$.

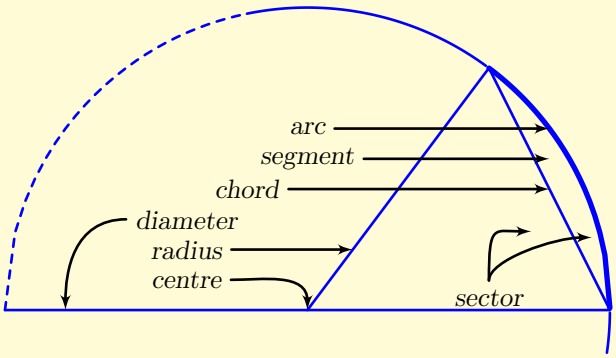
(v) Find the coordinates of the local minimum points on the graph of $y = \sec^2 2x + \csc^2 2x$, $0 \leq x \leq \frac{\pi}{2}$.

Chapter 17

Arcs, sectors, sine rule, cosine rule

17.1 Preliminary knowledge: arcs, sectors, segments,...

Notation (terminology connected to circles).

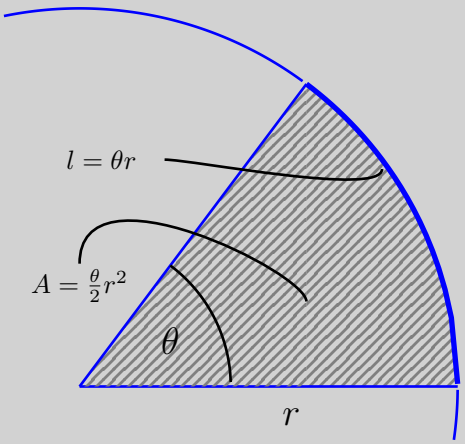


Theorem 116 (area of a sector, length of an arc).
Let:

- r *be the radius of a circle*
- θ *- the angle of a sector in* **radians**
- A *- area of the sector*
- l *- length of an arc*

Then

$$A = \frac{1}{2}\theta r^2$$
$$l = \theta r$$



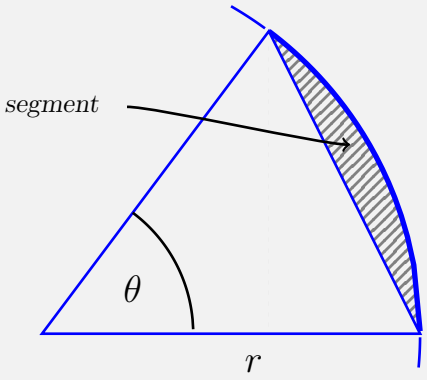
Theorem 117 (area of a triangle).
Let a and b be the sides of the triangle. Let θ be the angle between them.
Then A, the area of the triangle is

$$A = \frac{1}{2}ab\sin\theta$$

Proof. The proof can be easily conducted from the observation that the height of the triangle perpendicular to side a is equal $b\sin\theta$. □

Corollary 118 (area of a segment).
The area of a segment is a difference of the area of a sector and the area of a triangle:

$$\text{area of a segment} = \frac{1}{2}r^2(\theta - \sin\theta)$$



Example 119 (sector, segment).
In a circle with radius 8 the area of a sector with angle θ is 50. Find the area of the segment inside the sector.

The angle θ can be obtained from

$$\frac{1}{2}\theta r^2 = 50 \Rightarrow \frac{1}{2}\theta(8^2) = 50$$
$$\theta = \frac{25}{16}$$

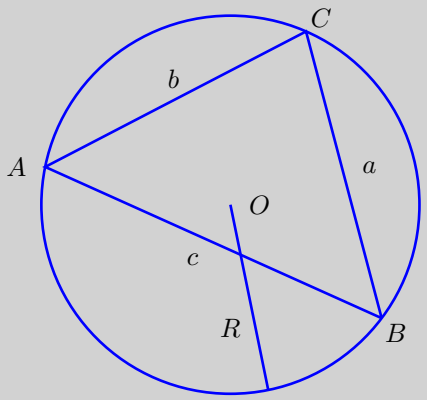
Hence the area of the segment is

$$\frac{1}{2}(8^2)\left(\frac{25}{16} - \sin\frac{25}{16}\right) \approx 18.0$$

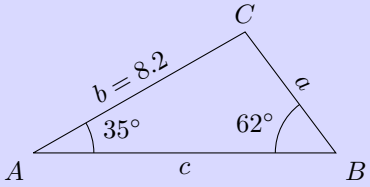
17.2 Sine rule, cosine rule

Theorem 120 (sine rule).
The lengths of sides of triangle ABC opposite to vertices A, B and C are a, b and c respectively.
Let R be the radius of the triangle's circumcircle.
Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$



Example 121 (sine rule - example 1).
Find the length of side BC in triangle ABC where $AC = 8.2$, $B\hat{A}C = 35°$ and $A\hat{B}C = 62°$.



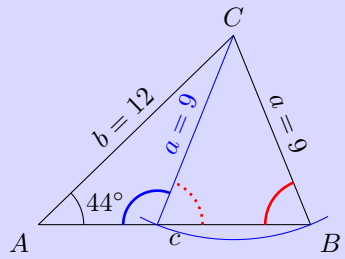
(diagram not to scale)

$\frac{a}{\sin A} = \frac{b}{\sin B}$ or $\frac{a}{\sin 35°} = \frac{8.2}{\sin 62°}$

$a = \frac{8.2 \sin 35°}{\sin 62°}$

$a \approx 5.33$

Example 122 (sine rule - example 2).
In triangle ABC the sides AC and BC are 12 and 9 respectively. The angle $B\hat{A}C = 44°$. Find angle B.



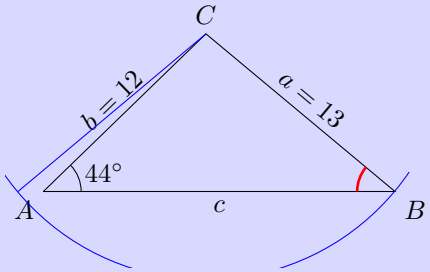
$\frac{9}{\sin 44°} = \frac{12}{\sin B}$

$\sin B = \frac{12}{9} \sin 44° \approx 0.926$

$B \approx 67.9°$ or $B \approx 180° - 67.9° \approx 112°$

this is called the ambiguous case

Example 123 (sine rule - example 3).
In triangle ABC the sides AC and BC are 12 and 13 respectively. The angle $B\hat{A}C = 44°$. Find angle B.



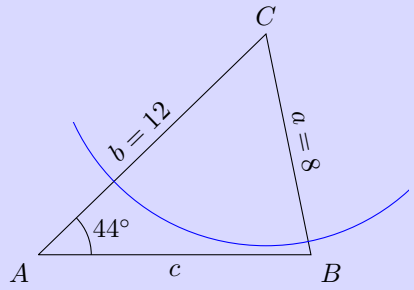
$\frac{13}{\sin 44°} = \frac{12}{\sin B}$

$\sin B = \frac{12}{13} \sin 44° \approx 0.641$

$B \approx 39.9°$ or $B \approx 180° - 39.9° \approx$ ~~$140°$~~

too large for a triangle $(44° + 140° > 180°)$

Example 124 (sine rule - example 4).
In triangle ABC the sides AC and BC are 12 and 13 respectively. The angle $B\hat{A}C = 44°$. Find angle B.



$\frac{8}{\sin 44°} = \frac{12}{\sin B}$

$\sin B = \frac{12}{8} \sin 44° \approx 1.04$

no such triangle (diagram not to scale)

Theorem 125 (cosine rule).
The lengths of sides of triangle ABC opposite to vertices A, B and C are a, b and c respectively. Then

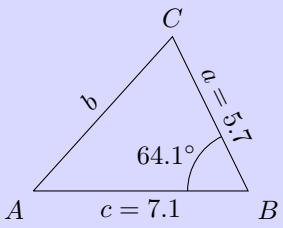
$$a^2 = b^2 + c^2 - 2bc\cos A \quad \textit{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Corollary 126 (cosine rule).
The cosine rule can be obviously applied to any angle in the triangle, so:

$$b^2 = c^2 + a^2 - 2ac\cos B \quad \textit{or} \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \textit{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 127 (cosine rule - example 1).
In triangle ABC the sides AB and BC are 7.1 and 5.7 respectively. The angle $A\hat{B}C = 64.1°$. Find the length of side AC.

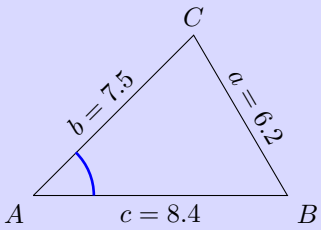


$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 5.7^2 + 7.1^2 - 2(5.7)(7.1)\cos 64.1°$

$b \approx 6.90$

EXAMPLE 128 (cosine rule - example 2).
In triangle ABC the sides AB, BC and AC are 8.4, 6.2 and 7.5 respectively. Find the angle $B\hat{A}C$.



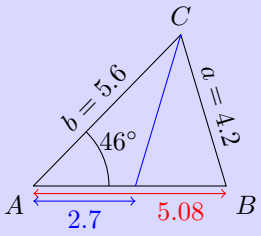
$a^2 = b^2 + c^2 - 2bc\cos A$

$6.2^2 = 7.5^2 + 8.4^2 - 2(7.5)(8.4)\cos A$

$\cos A \approx 0.701$

$A \approx 45.5°$

EXAMPLE 129 (cosine rule - example 3).
In triangle ABC the sides BC and AC are 4.2 and 5.6 respectively. Tha angle $B\hat{A}C = 46°$. Find the length AB.



$a^2 = b^2 + c^2 - 2bc\cos A$

$4.2^2 = 5.6^2 + c^2 - 2(5.6)c\cos 46°$

$c^2 + c(-11.2\cos 46°) + 13.72 = 0$

$c \approx 2.70$ or $c \approx 5.08$

QUESTIONS 17.1.

Q1. Find the size of the largest angle in a triangle with sides 6, 9 and 13.

Q2. Find the angle between two diagonals of faces of a cube.

Q3. Find the angle between two sides of a regular tetrahedron.

Q4. In a triangle $a = 8.9$, $b = 7.2$ and $C = 68°$. Find the missing angles.

Q5. In a triangle $a = 8.9$, $b = 7.2$ and $B = 68°$. Find the missing angles.

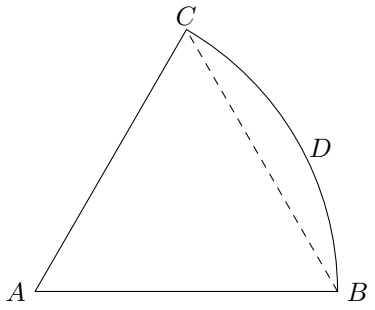
Q6. Find all missing angles and sides in the following triangles.

(1) $A = 48°, b = 6.4cm, c = 9.5cm$
(2) $A = 48°, B = 65°, c = 9.5cm$
(3) $a = 4.8cm, b = 5.2cm, c = 6.7cm$
(4) $a = 6.5cm, b = 8.2cm, B = 56°$
(5) $a = 3.4cm, b = 4.6cm, c = 5.2cm$
(6) $a = 20.4cm, b = 13.5cm, B = 31°$
(7) $A = 54°, b = 6.8cm, c = 9cm$
(8) $A = 54°, B = 61°, c = 8.9cm$
(9) $a = 5.8cm, b = 6.3cm, c = 8.1cm$
(10) $a = 7.5cm, b = 9.4cm, B = 58°$

Q7. Find the area of the triangle ABC when $a = 8.2cm, b = 7.1, C = 32°$.

Q8. Find the area of the triangle ABC when $a = 6.2cm, b = 7.3, C = 28°$.

Q9. Consider the figure $ABDC$ below:



The arc BDC has the centre at A and the radius of $3cm$. The angle $A\hat{B}C$ is $55°$. Find:

(a) the area of the figure BCD.

(b) the length of the arc BDC.

(c) the perimeter of the triangle ABC.

Q10. A yacht sails $240km$ on a bearing of $320°$, then it changes course and sails on a bearing $060°$.

a) How far must it sail to get to the point where its starting position is on bearing $180°$?

b) How far is this point from the starting position?

c) Sketch the diagram to show the positions of the yacht.

Q11. Town B is placed 4.2km on a bearing of $035°$ from town A. Town C is 5.3km on a bearing of $160°$ from town B.

(i) Find the distance from A to C.

(ii) Find a bearing of A from C.

(iii) Find a bearing of C from A.

Q12. Town A is 4.3km on a bearing of $243°$ from town B. Town B is 3.9km on a bearing of $340°$ from town C.

(i) Find the distance from A to C.

(ii) Find a bearing of A from C.

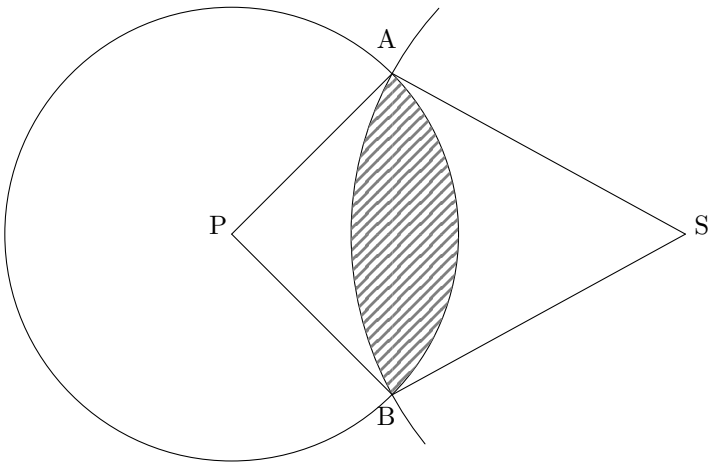
(iii) Find a bearing of C from A.

Q13. The diagram shows two circles: one with the centre at P and the radius $PA = PB = 4cm$ and the other with the centre at S and the radius $SA = SB = 6.1cm$. The angle $A\hat{P}B$ is a right angle.

a) Find the length AB.

b) Find the angle $A\hat{S}B$.

c) Find the area of the shaded region.



Q14. The diagram shows two circles: one with the centre at P and the radius $PA = PB$ and the other with the centre at S and the radius $SA = SB = 5cm$. Knowing that $AB = 3.8cm$ and $PS = 7.1cm$ find the area of the shaded region.

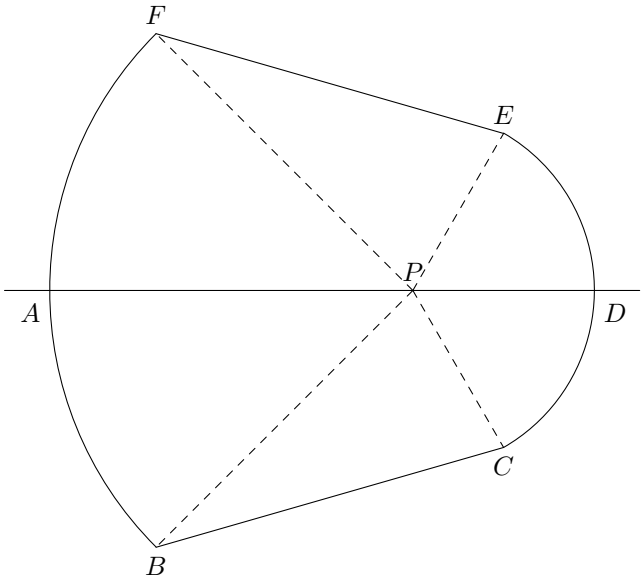
Q15. In a triangle $a = 8.9$, $b = 8.3$ and $A = 68°$. Find the missing angles.

Q16. A parallelogram has the lengths of the sides of $20cm$ and $18cm$ and the acute angle of $48°$. Find the area of the parallelogram.

Q17. In the triangle ABC angle A is $25°$, $AC = 25cm$, $BC = 12cm$. Find the angle ABC.

Q18. A student starts from a school and walks for a distance of $2km$ in a straight line towards a better place. (Or a place he thinks is better.) Then he meets his friend and they make a turn to a cinema (left) of $41°$ and walk for $1100m$ until the student's father meets them. How far from the school is the fortunate meeting?

Q19. On the diagram below line APD is a line of symmetry of the figure.



The arc BAF has the centre at P and the radius of $8cm$. The angle BPF is $90°$. The arc CDE has the centre at P and the radius of $4cm$. The angle EPC is $120°$. Find:

(a) the angle FPE;

(b) the area of the figure $ABCDEF$;

(c) the length EF;

(d) the perimeter of the figure $ABCDEF$.

Q20. A right pyramid of height $20m$ stands on a square base of side lengths $10m$. Find

(a) the length of the slant edge,

(b) the height of the sloping face,

(c) the angle between a sloping face and the base,

(d) the angle between two sloping faces that have a common edge.

17.3 Miscellaneous problems

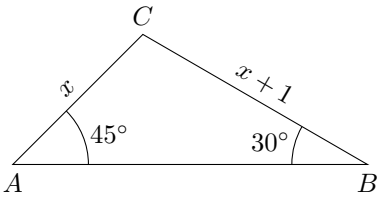
Q21. Find the acute angle between the lines $y = 2x + 5$ and $y = -\frac{1}{3}x - 2$.

Q22. Find the acute angle between the lines $7x - 2y + 5 = 0$ and $5x + 3y + 2 = 0$.

Q23. In triangle ABC angles $B\hat{A}C$ and $A\hat{B}C$ are 45° and 30° respectively. The lengths of sides AC and BC are x and $x+1$ respectively.

(i) Find the exact value of x.

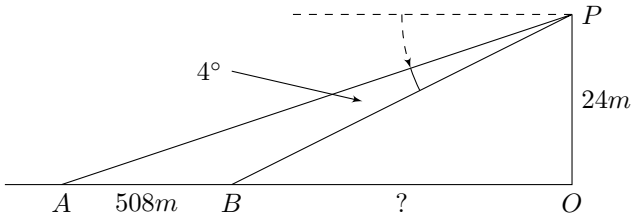
(ii) Find the exact length of side AB.



Q24. In triangle ABC point D is a point on side BC such that AD is the bisector of angle $B\hat{A}C$. Show that $\frac{BD}{CD} = \frac{AB}{AC}$.

Q25. In triangle ABC point D is the midpoint of side BC. Let $\alpha = C\hat{A}D$. Knowing that $B\hat{A}D = 2\alpha$ and that $\frac{AB}{AC} = \frac{2}{3}$ show that $\cos 3\alpha + \cos^2 \alpha = 0$.

Q26. A man stands on a tower OP that is 24 metres high. His eyes are at point P.
He watches two objects, A and B that approach the tower in a way that the points A and B are colinear with O.
The man's line of sight makes an angle of depression 4° larger when he looks down to point B than when he looks down to point A.
Find the distance from B to O.



(diagram not to scale)

Q27. In triangle ABC angle $A\hat{B}C$ is twice the angle $B\hat{A}C$. Knowing that $\frac{AC}{BC} = \frac{9}{5}$ find the value of $\cos B\hat{A}C$.

Part V

DIFFERENTIAL CALCULUS

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Chapter 18

Evaluating derivatives

Task 1

→ 1. Find the equation of the line tangent to the graph of $y = x^2$ at the point where $x = 1$.

step 1. If the line $y = mx + c$ passes through point $(1, 1)$ express c in terms of m.

step 2. Knowing that the line and the curve $y = x^2$ are to have exactly one point in common, use the discriminant to find the value of m - gradient of the tangent line.

step 3. Hence find the equation of the tangent.

→ 2. Can you follow the steps above for the function $y = x^3$?

Task 2

A distance s metres covered by a particle from a certain point in time t seconds is given by the formula

$$s = \frac{1}{2}(t + 1)^3,$$

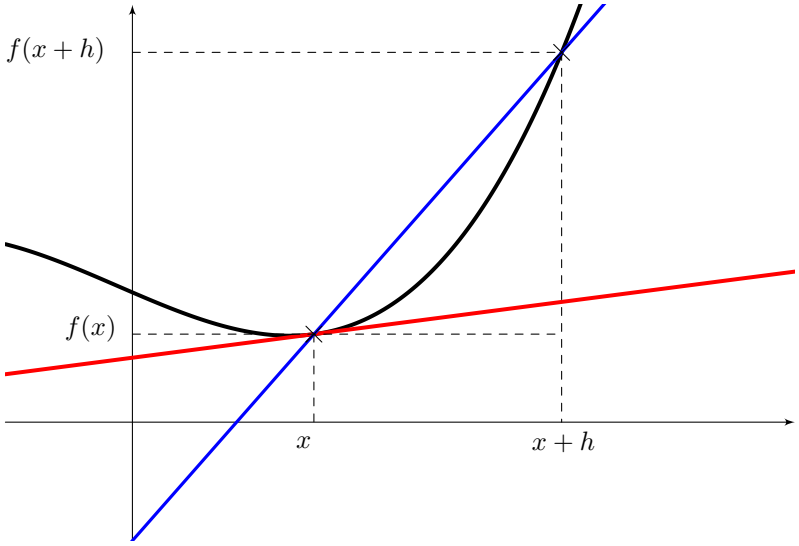
where $t \geq 0$.

→ 1. Find the average speed of the particle:

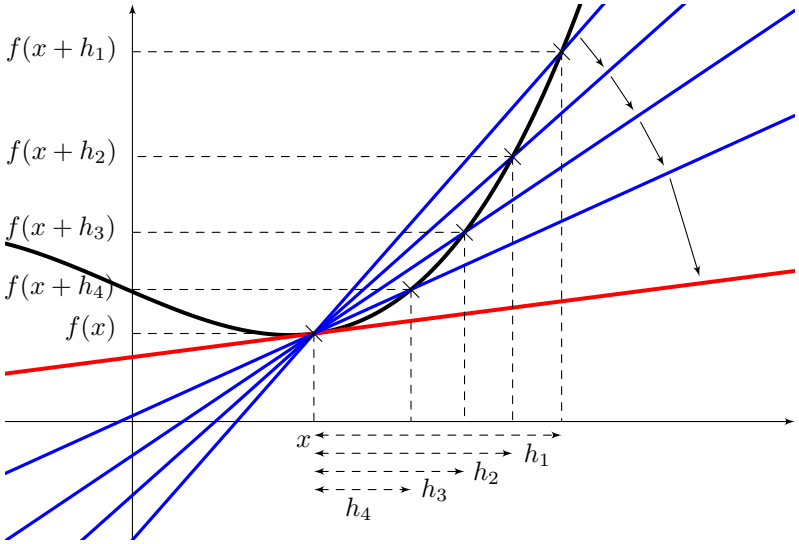
- in the first 10 seconds of the motion,
- in the first second of the motion,
- in the first 0.1 second of the motion,
- in the first 0.01 second of the motion,
- in the first 0.001 second of the motion.

→ 2. What is the speed of the particle when it starts the motion (at $t = 0$)?

Derivative - gradient of the curve.
To find the **gradient** of the **tangent** to the curve $y = f(x)$ at point $(x, f(x))$ (gradient of the curve) we will investigate gradients of **secants** passing through point $(x, f(x))$ and points $(x + h, f(x + h))$, where h is *close* to 0.



The gradient of a secant is $\frac{f(x+h)-f(x)}{h}$.
When h is getting smaller and smaller (or: when h tends to zero) secants approach the tangent.



This leads us to the definition of the derivative.

18.1 Definition of a derivative

Definition 130 (derivative at a point).
Let $f : D \longrightarrow \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}$.
*Let $x_0 \in D$. A **derivative** of the function f at the point $x = x_0$ is defined as*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Definition 131 (derivative as a function).
Let $f : D \longrightarrow \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}$.
If a derivative of f exists for all $x \in D$ then the function f' defined

$$x \longmapsto f'(x)$$

*for all $x \in D$ is called the **derivative function** or the **derivative** of f.*

NOTATION.
$f'(x)$ $\quad y'$
$\frac{d}{dx}f(x)$ $\quad \frac{dy}{dx}$

EXAMPLE 132 (tangent to curve).
We will find the equation of the tangent to the curve $f(x) = x^3$ at the point $(1, 1)$.

The tangent is given by $y = mx + b$, where m is the gradient i.e. the derivative of f at $x = 1$:

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \\ &= \lim_{h\to 0}\frac{(1+h)^3-1}{h} = \\ &= \lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h} = \\ &= \lim_{h\to 0}\frac{3h+3h^2+h^3}{h} = \\ &= \lim_{h\to 0}(3+3h+h^2) = \\ &= 3 \text{ as all terms with } h \text{ tend to } 0. \end{aligned}$$

Thus the gradient of the tangent $m = 3$. Thus the equation of the tangent is

$$\begin{aligned} & y - 1 = 3(x-1) \\ \Leftrightarrow \quad & y = 3x - 2 \end{aligned}$$

Check the above answer using GDC or appropriate graphing software.

QUESTIONS 18.1.

Q1. Use the definition of derivative to show that $\frac{d}{dx}(x^5) = 5x^4$.

Q2. Find the derivative from the first principles.

(1) $y = x^2$ (4) $y = 3x^2 + 2x$ (7) $y = \sqrt{x}$ (10) $y = \frac{1}{x^2}$
(2) $y = x^3$ (5) $y = 2x^3$ (8) $y = x\sqrt{x}$ (11) $y = x$
(3) $y = x^4$ (6) $y = \frac{1}{x}$ (9) $y = \sqrt{2x}$ (12) $y = 5$

Q3. Use the definition of derivative to show that $\frac{d}{dx}(x^n) = nx^{n-1}$, where $n \in N$.

18.2 POWER RULE

THEOREM 133 (Power rule).
If $y = x^n$, $n \in \mathbb{R}$ then $\frac{dy}{dx} = nx^{n-1}$.

EXAMPLE 134 (differentiation by the power rule).
Find $\frac{dy}{dx}$ for $y = \frac{2x^3-3}{x\sqrt{x}}$.

Expression for y can be simplified:

$$y = \frac{2x^3}{x^{1.5}} - \frac{3}{x^{1.5}} = 2x^{1.5} - 3x^{-1.5}$$

so

$$\begin{aligned} \frac{dy}{dx} &= 2(1.5)x^{1.5-1} - 3(-1.5)x^{-1.5-1} = \\ &= 3x^{0.5} + 4.5x^{-2.5} = \\ &= 3\sqrt{x} + \frac{9}{2x^2\sqrt{x}} \end{aligned}$$

Questions 18.2.

Q4. Differentiate the following with respect to x.

(1) $y = 4x^7 - 3x^4 + 5x + 9$
(2) $y = 3x^5 - 2x^3 + 6x - 7$
(3) $y = 2x^5 - 5x^3 + 4x^2 + 3x - 11$
(4) $y = \frac{2}{x}$
(5) $y = -\frac{3}{2x^2}$
(6) $y = -\frac{4}{3x^3}$
(7) $y = (2x^2 - 1)^2$
(8) $y = (2x^3 - x + 2)(x - 3)$
(9) $y = 3x^{-3} + 1.5x^4 - 1.5x^{-4}$
(10) $y = \frac{\pi\sqrt{x}}{x}$
(11) $y = \frac{2}{\sqrt{x}}$
(12) $y = -\frac{3}{x} + 3x$
(13) $y = \frac{x^2+1}{2x^3}$
(14) $y = \frac{\sqrt{2x}}{2x^2} - \frac{1}{x}$
(15) $y = \frac{\sqrt{3x^3}}{3x^2} + \frac{2}{x}$
(16) $y = \sqrt[3]{\frac{27\sqrt{x}}{8x^3}}$
(17) $y = \frac{3\sqrt[3]{x} - x^2}{2\sqrt{3x}}$
(18) $y = \frac{3x\sqrt{x^3}}{\sqrt{2x}\sqrt{x^5}} - \frac{3}{x^3\sqrt{x}} + \frac{2}{\sqrt[3]{2x}}$

18.3 Derivatives of elementary functions

Task

Investigate the graphs of $h \mapsto h$, $h \mapsto \sin h$ and $h \mapsto (\cos h - 1)$ in the neighbourhood of $h = 0$ to justify the following results.

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \qquad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$

Use the above results to find the derivative of $\sin x$ and $\cos x$ from the first principles.

Theorem 135 (Derivatives of elementary functions).
The functions with equations shown below have the following derivatives.

$y = \sin x$	$\frac{dy}{dx} = \cos x$	$y = \sec x$	$\frac{dy}{dx} = \sec x \tan x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$	$y = \csc x$	$\frac{dy}{dx} = -\csc x \cot x$
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x$	$y = \cot x$	$\frac{dy}{dx} = -\csc^2 x$
$y = e^x$	$\frac{dy}{dx} = e^x$	$y = a^x$	$\frac{dy}{dx} = a^x (\ln a)$
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$	$y = \log_a x$	$\frac{dy}{dx} = \frac{1}{x \ln a}$

Theorem 136 (Product rule).
To differentiate the product of two functions the following rule may be used:

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

which can be written in simple way as:

$$(uv)' = u'v + uv'$$

Example 137 (product rule).
Differentiate $x^2 \cos x$.

$$\frac{d}{dx}\left(x^2 \cos x\right) = \left(\frac{d}{dx}x^2\right)\cos x + x^2\left(\frac{d}{dx}\cos x\right) =$$
$$= 2x\cos x - x^2 \sin x$$

Theorem 138 (Quotient rule).
To differentiate the quotient of two functions the following rule may be used:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

which can be written in simple way as:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example 139 (product rule).
Differentiate $\frac{x^2}{\cos x}$.

$$\frac{d}{dx}\left(\frac{x^2}{\cos x}\right) = \frac{\left(\frac{d}{dx}x^2\right)\cos x - x^2\left(\frac{d}{dx}\cos x\right)}{\cos^2 x} =$$
$$= \frac{2x\cos x + x^2\sin x}{\cos^2 x} = x\sec x(2 + x\tan x)$$

Questions 18.3.

Q5. Use derivatives of $\sin x$ and $\cos x$ to find the derivatives of the following.

(1) $y = \tan x$ (2) $y = \cot x$ (3) $y = \sec x$ (4) $y = \csc x$

Q6. Differentiate the following with respect to x. Other letters denote constants.

(1) $y = (3x^2 - 5)\cos x$
(2) $y = \frac{2\sin x}{\tan x}$
(3) $y = \frac{x^2+2x+3}{(x+1)^2}$
(4) $y = \frac{x}{\tan x}$
(5) $y = \frac{(x-1)(x+2)}{(x-2)(x+1)}$
(6) $y = 2x^3t^4 - t^3x^2$
(7) $y = (2x^5 - 10x^4)\cot x$
(8) $y = \frac{2\cos x}{\tan x}$
(9) $y = \frac{x^3+1.5x^2+3}{(x+1)^2}$

Q7. Differentiate the following with respect to x.

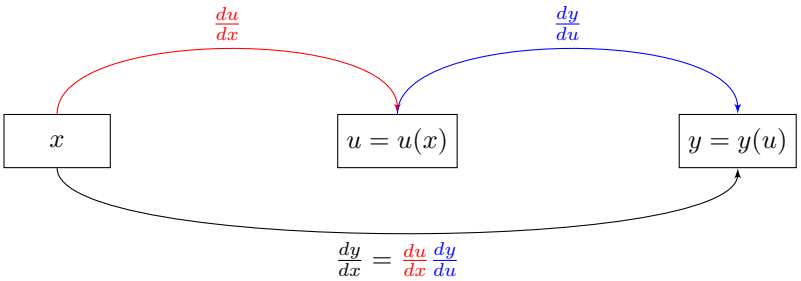
(1) $y = x^2e^x$
(2) $y = x^3\sin x$
(3) $y = \sin x\cos x$
(4) $y = (1 - 3x^2)\cos x$
(5) $y = e^x\sin x$
(6) $y = x\ln x$
(7) $y = e^x\ln x$
(8) $y = \sin 2x$
(9) $y = \cos 2x$
(10) $y = \sin 3x$

Q8. Let functions f and g be such that $f(5) = 2$, $f'(5) = -1$, $g(5) = -3$ and $g'(5) = 4$.
Let $p(x) = f(x)g(x)$ and $q(x) = \frac{f(x)}{g(x)}$. Find $p'(5)$ and $q'(5)$.

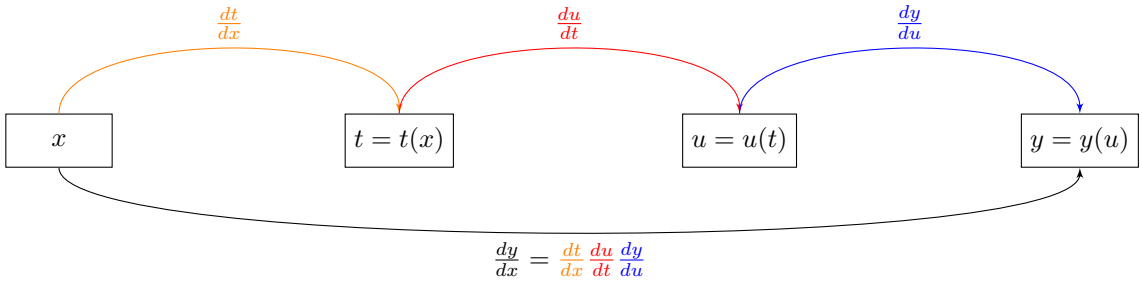
18.4 Chain rule

Theorem 140 (Chain rule).
Let $u(x)$ be a function with derivative $\frac{du}{dx}$.
Let $y(u)$ be a function with derivative $\frac{dy}{du}$.
Then the derivative of the composition $y = y(u(x))$ with respect to x is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$



Corollary 141 (Chain rule).
If $y = y(u(t(x)))$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dt}\frac{dt}{dx}$ etc.



Example 142 (chain rule).
Suppose that an airplane's height at time t is given by the function $h(t)$ and that the oxygen concentration at height x is given by the function $c(x)$. Then $(c \circ h)(t)$ describes the oxygen concentration around the plane at time t. Then

$\frac{dh}{dt}$	*denotes the rate of change of the height t of the plane in time t*
$\frac{dc}{dh}$	*denotes the rate of change of the concentration of oxygen c at the height h*
$\frac{dc}{dt} = \frac{dc}{dh}\frac{dh}{dt}$	*denotes the rate of change of the concentration of oxygen c in time t*

Example 143 ($\frac{d}{dx}(\sin^4 x)$).
Evaluate the derivative of $y = \sin^4 x$.

If $y = \sin^4 x$ then $y = y(u(x))$ where:

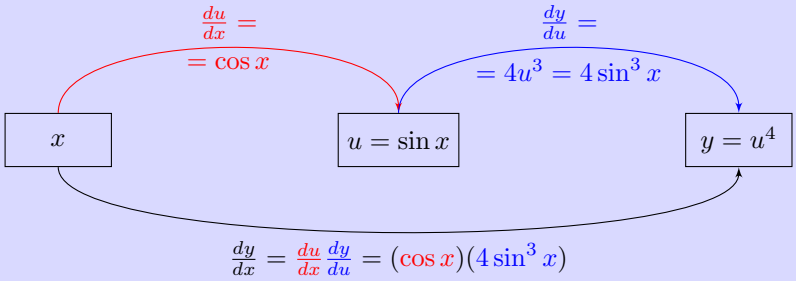
$$\begin{aligned} y &= u^4 \\ u &= \sin x \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{dy}{du} &= 4u^3 \\ \frac{du}{dx} &= \cos x \end{aligned}$$

$\frac{dy}{du}$ can be expressed in terms of x:

$$\frac{dy}{du} = 4u^3 = 4\sin^3 x$$

By the chain rule:

$$\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du} = (\cos x)(4\sin^3 x) = 4\sin^3 x \ \cos x$$



Example 144 ($\frac{d}{dx}(\ln(e^{3x^2-x}+2))$).
Evaluate the derivative of $y = \ln(e^{3x^2-x}+2)$.

If $y = \ln(e^{3x^2-x}+2)$ then $y = y(u(t(x)))$ where:

$$\begin{aligned} y &= \ln u \\ u &= e^t + 2 \\ t &= 3x^2 - x \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{dy}{du} &= \frac{1}{u} \\ \frac{du}{dt} &= e^t \\ \frac{dt}{dx} &= 6x - 1 \end{aligned}$$

$\frac{du}{dt}$ can be expressed in terms of x:

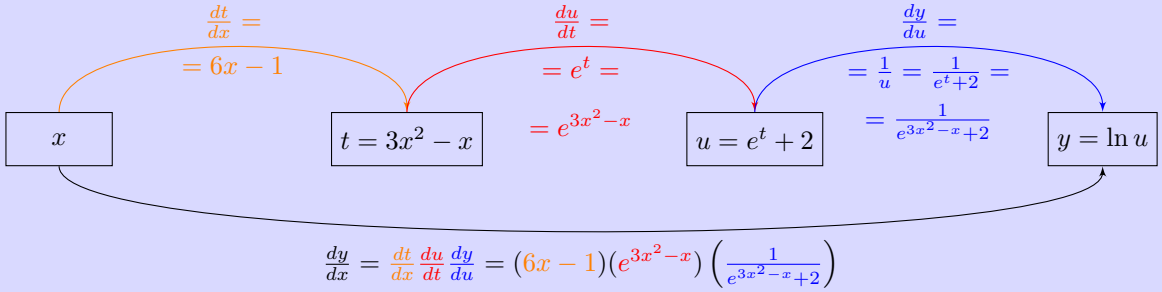
$$\frac{du}{dt} = e^t = e^{3x^2-x}$$

and so can $\frac{dy}{du}$:

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{e^t+2} = \frac{1}{e^{3x^2-x}+2}$$

By the chain rule:

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{du}{dt}\frac{dy}{du} = (6x-1)(e^{3x^2-x})\frac{1}{e^{3x^2-x}+2} = \frac{(6x-1)e^{3x^2-x}}{e^{3x^2-x}+2}$$



Questions 18.4.

Q9. Differentiate the following with respect to x. Other letters denote constants.

(1) $y = \sin^3 2x$
(2) $y = e^{2\sin 3x} + 1$
(3) $y = 3^{2x^3-x+1}$
(4) $y = \log_2(\sin^2 x)$
(5) $y = \cos\ln(2x^3 + x - 3)$
(6) $y = (3x^2 + 2x - 1)^{10}$
(7) $y = (\sin^4 x)(\cos^4 x)$
(8) $y = \frac{x}{\tan^2 x}$
(9) $y = (2\sin^3 \pi x - 2)^4$
(10) $y = \frac{2}{\sqrt[5]{1+4x-3x^4}}$
(11) $y = \sin^2 3x$
(12) $y = e^{\cos 2x}$
(13) $y = 3^{x^2+1}$
(14) $y = \log_2(x^2 + 2x + 3)$
(15) $y = \sin\ln(2x^3)$
(16) $y = (2x + 1)^{15}$
(17) $y = (\sin^3 x)(\cos^5 x)$
(18) $y = (2\sin^2 3x + 1)^3$
(19) $y = \frac{1}{\sqrt[3]{1-2x^5}}$
(20) $y = \frac{e^{3x^2+1}}{\sqrt{x}}$
(21) $y = \left(\frac{2x^2+x-1}{x+2}\right)^4$
(22) $y = -\frac{3}{e^{2x}}$
(23) $y = (e^{2x})\sin x$
(24) $y = \sec^2 x$
(25) $y = (x + 1)^2(2x - 3)^3$
(26) $y = \sin^2(3x)^2$
(27) $y = \sin(2e^x)$
(28) $y = \pi^{\cos^2 x+1}$
(29) $y = \frac{\ln(3x^2-x)}{3x^2-x}$
(30) $y = \log_2(\sin x)$
(31) $y = \sin\ln(2e^3)$
(32) $y = (\sin 2x)^{15}$
(33) $y = (\tan^3 x)(\cos^5 x)$
(34) $y = x\csc x$
(35) $y = (2e^{6x} + e^x - 1)^3$
(36) $y = \frac{1}{\sqrt[3]{(1-2\sqrt{x})^5}}$
(37) $y = x^2e^{2x}$
(38) $y = \left(\frac{e^x}{\sqrt{x}+1}\right)^4$
(39) $y = \frac{1}{4}e^{2\ln(4x-2)}$
(40) $y = sixteen$
(41) $y = (3x^4 - 2x - 7)\ln(3x^4 - 2x - 7)$
(42) $y = (2x^3 + x + 1)\ln(2x^3 + x + 1)$

Q10. Let functions f and g be such that $f(1) = 2$, $f(2) = 1$, $f'(1) = 3$, $f'(2) = -1$, $g(1) = -1$, $g(2) = -3$, $g'(1) = 2$ and $g'(2) = -2$.
Let $h(x) = (g \circ f)(x)$. Find $h'(1)$ and $h'(2)$.

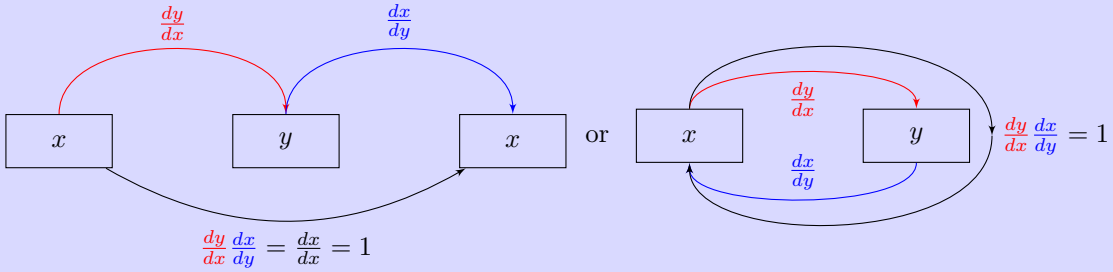
Q11. The line $y = 4x + b$ is tangent to the curve $y = \tan x + \sqrt{3}$, $0 \le x \le \pi$. Find the value of b.
(You may like to follow the steps: *firstly*, find the point where $\frac{dy}{dx} = 4$ and *secondly*, find b.)

Q12. Find the equation of the normal to the curve $y = (2x^2 - 7)^3$ at a point where $y = 1$.

18.5 Derivatives of inverse functions

Corollary 145 (to the chain rule).
If y is an invertible and differentiable function of x then

$$\frac{dy}{dx}\frac{dx}{dy} = 1.$$



Example 146 (derivative of $\arctan x$).
Find the derivative of $y = \arctan x$.

If $y = \arctan x$ then $x = \tan y$.
Then $\frac{dx}{dy} = \sec^2 y$ and hence:

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Theorem 147 (derivatives of inverse trigonometric functions).
For $y = \arcsin x$, $-1 \le x \le 1$ the derivative is $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, where $-1 < x < 1$.
For $y = \arccos x$, $-1 \le x \le 1$ the derivative is $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$, where $-1 < x < 1$.
For $y = \arctan x$, $x \in \mathbb{R}$ the derivative is $\frac{dy}{dx} = \frac{1}{1+x^2}$, where $x \in \mathbb{R}$.

Questions 18.5.

Q13. Knowing that $\frac{d}{dx}e^x = e^x$ show that $\frac{d}{dx}\ln x = \frac{1}{x}$.

Q14. Let $y = \arctan x$. Show that $\frac{dx}{dy} = \sec^2 y$ and hence find $\frac{dy}{dx}$.

Q15. Let $y = \arcsin x$. Find $\frac{dy}{dx}$.

Q16. Let $y = \arccos x$. Find $\frac{dy}{dx}$.

Q17. A function $f : \mathbb{R} \to \mathbb{R}$ is invertible and $f(e) = 1$. The inverse of f is given by

$$f^{-1}(x) = xe^{2x^2+4x-5}$$

(i) Find the exact value of the gradient of $y = f(x)$ at the point where $x = e$.
(ii) Find the coordinates of a point on the graph of f where its derivative is not defined.

Q18. A function $f : [-2\pi, 2\pi] \to [-2\pi, 2\pi]$ is invertible. The inverse of f is given by

$$f^{-1}(x) = \sin x - x$$

(i) Show that $f(\pi) = -\pi$.
(ii) Find the exact value of the gradient of $y = f(x)$ at the point where $x = \pi$.
(iii) Find the coordinates of all points on the graph of f where its derivative is not defined.

Chapter 19

Piecewise functions

Questions 19.1.

Q1. Consider a function:

$$f(x) = \begin{cases} x^2 + 2x - 1 & \text{for } x \in [-2, 1] \\ -2x + 6 & \text{for } x \in]1, 3] \end{cases}$$

(i) Sketch the graph of $x \longmapsto f(x)$.

(ii) Explain why f is not continuous.

(iii) Translate the linear part of the graph to make it continuous. Write down the new equation of f.

(iv) Is the derivative of f continuous over the whole domain? Why?

(v) Change the linear "piece" of f to make its derivative continuous.

Q2. Consider function g such that:

- $g(x) = \begin{cases} x^2 + 2x - 1 & \text{for } x \in [-2, 1] \\ ax^2 + bx + c & \text{for } x \in]1, 3] \end{cases}$ and
- both g and its derivative are continuous.

(i) If $a = -1$ find the values of b and c.

(ii) If $b = 8$ find the values of a and c.

(iii) If $c = 0$ find the values of a and b.

Q3. Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 2x - 3 & \text{for } x < 2 \\ ax^2 + bx + c & \text{for } x \geq 2 \end{cases}$$

(i) Show that for $a = 1$, $b = -8$ and $c = 13$ the function is continuous while its derivative is not.

(ii) Let $a = -1$ and $c = -11$. Find the value of b for which the function is continuous. Is the derivative of f continuous for the value?

(iii) Let $a = -1$. Find the values of b and c for which both the function and its derivative are continuous.

Q4. Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} mx - 3 & \text{for } x < 1 \\ \frac{3x^2 + bx - 9}{4} & \text{for } x \geq 1 \end{cases}$$

Find the values of m and b for which the function is continuous and so is its derivative.

Q5. Consider 3 intervals:

$$A = [-3, -1[,$$
$$B = [-1, 1[,$$
$$C = [1, 3].$$

Create a *piecewise* function y such that y:

(a) is quadratic and non-negative on each of the three intervals;

(b) is continuous;

(c) is concave up ($\cup$-shaped) on A and C, concave down ($\cap$-shaped) on B;

(d) is symmetrical in y-axis

(e) has a continuous derivative for each $x \in A \cup B \cup C$.

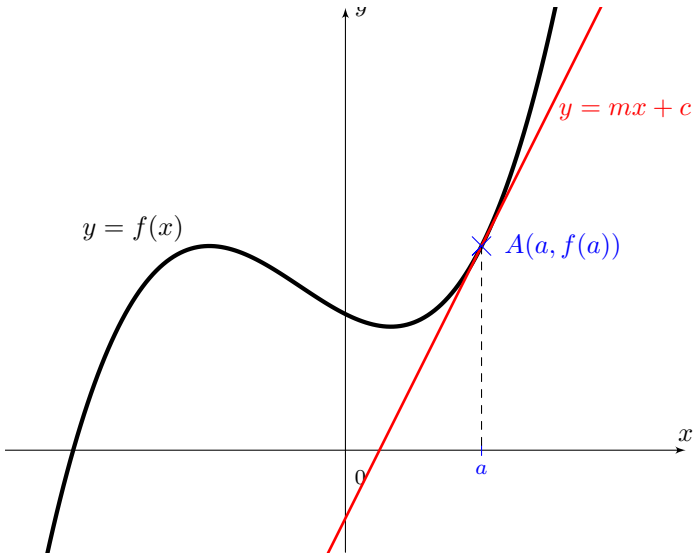
Chapter 20

Lines tangent and normal to curves

To find the equation

$$y = mx + c$$

of the tangent to a curve defined by the equation $y = f(x)$ at point $A(a, f(a))$ we have to firstly find the slope m of the tangent. For that we differentiate the equation of the curve and then we evaluate the value of the derivative at $x = a$.



$$m = f'(a)$$

Then we plug in the coordinates of A to the equation of the tangent to find the value of the coefficient c:

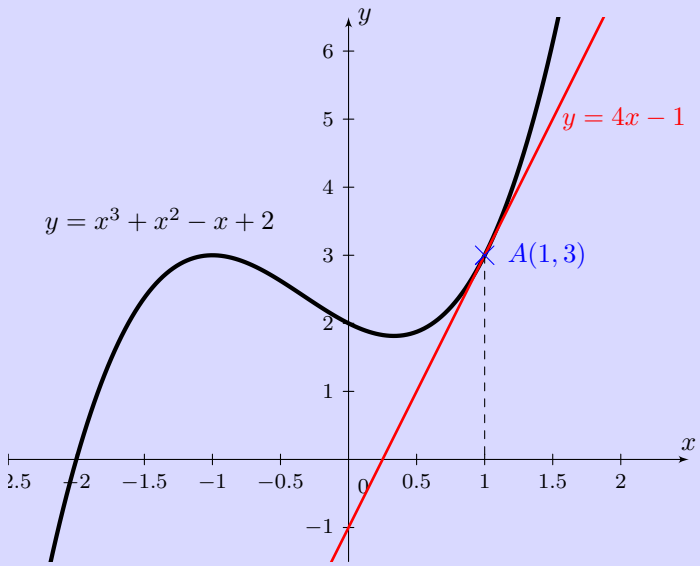
$$y = mx + c$$
$$y = f'(a)x + c$$

and for $(x, y) = (a, f(a))$:

$$f(a) = f'(a)a + c$$
$$\text{so finally } c = f(a) - af'(a)$$

and the tangent is $y= (f'(a))x + (f(a) - af'(a))$.

Example 148 (tangent to a curve).
Find the equation of the tangent to the curve given by $f(x) = x^3 + x^2 - x + 2$ at the point A where $x = 1$.



The y-coordinate of A is

$$f(1) = 1^3 + 1^2 - 1 + 2 = 3$$

The derivative of the equation of the curve is:

$$f'(x) = 3x^2 + 2x - 1$$

so the slope of tangent is:

$$m = f'(1) = 3(1)^2 + 2(1) - 1 = 4.$$

The equation of tangent is therefore given by $y = 4x + c$.
To find the value of c coordinates of A are used:

$$3 = 4 \times 1 + c$$
$$c = -1$$

so the equation of the tangent is

$$y = 4x - 1.$$

Questions 20.1.

Q1. Find the equations of the tangents of the following curves at the points indicated.

(1) $y = 2x^3 + x^2 - 2x - 1$ at $(2, 15)$,
(2) $f(x) = \frac{2x}{x-1}, x \neq 1$ at $(-1, 1)$,
(3) $y = x^2 e^x$ at $(1, e)$,
(4) $y = x \sin^2 x$ at $x = \frac{\pi}{2}$,
(5) $f(x) = \ln(ex)$ at $x = 1$,
(6) $y = \arctan(x^3)$ at $x = -1$,
(7) $f(x) = \frac{e^x}{e^{2x}+1}$ at $x = \ln 2$,
(8) $y = x\sqrt{x}$ at $(4, 8)$,
(9) $f(x) = (2 \sin x + 1) \cos x$ at $(0, 1)$

Q2. For the curves from **Q1** find the equations of the normals to the curves at the points indicated.

Q3. Consider the curve $y = x^3 + 4x^2 + x - 6$.

(a) Find the equation of the tangent to the curve at the point where $x = -1$.
(b) Find the coordinates of the point where this tangent meets the curve again.
(c) Find the equation of the normal to the curve at the point where $x = -1$.

Q4. The line $y = 2x + b$ is tangent to the curve $y = \frac{1}{(3-x)^2}$. Determine the equation of the tangent and the coordinates of the points where it meets the curve.

Q5. The line $y = -\frac{1}{2}x + c$ is normal to the curve $y = 2 \sin 2x$ at the point (x, y), where $0 \leq x \leq \frac{\pi}{2}$. Determine the equation of the normal and the coordinates of the points where it meets the curve.

Q6. For the curve defined by $x \mapsto \frac{-x}{x^2+1}$ find the equation of the normal at the origin, and the equations of the tangents that are parallel to the x-axis. Find also the points where the tangents and the normal intersect.

Q7. For the curve defined by $x \mapsto \frac{-2x}{x^2+4}$ find the equation of the normal at the origin, and the equations of the tangents that are parallel to the x-axis. Find also the points where the tangents and the normal intersect.

Q8. The tangent to the curve $y = \sqrt{x}$ at the point (x, y) intersects the x-axis at the point $(-2, 0)$. Determine the coordinates of the point where the tangent meets the curve. Find the equation of the tangent.

Q9. The tangent to the curve $y = x^3 - x$ at the point (x, y) intersects the curve at the point $(-2, -6)$. Find the coordinates of the point where the tangent meets the curve. Find the equation of the tangent.

Q10. The tangent to the curve $y = \frac{\sin(Ax)}{B}$ at the point $(1, 0)$ has the equation $y = -x + 1$. Find the smallest positive value of A and the value of B.

Q11. The tangent to the curve $y = e^{Ax}$ at the point where $x = 0$ intersects the x-axis at $(-3, 0)$. Find the value of A and the equation of the tangent.

Q12. The tangent to the curve $y = \ln(Ax+B)$ at the x-intercept of the curve is given by the equation $y = 2x+3$. Find the values of A and B.

Q13. The tangent to the curve $y = \frac{1}{x+1}$ passes through the origin. Determine the equation of the tangent and the coordinates of the point where it meets the curve.

Q14. The line $yc = x + 1$ is tangent to the curve $y = \frac{1}{3-x}$. Determine the value of c and the coordinates of the point where the tangent meets the curve.

Q15. The normal to the curve $y = x^2 - 6x + 6$ at the point (x, y) intersects the curve again at the point $\left(\frac{3}{2}, -\frac{3}{4}\right)$. Find the coordinates of the point of intersection of the normal and the curve. Find the equation of the normal.

Q16. The normal to the curve $y = e^{Ax}$ at the point where $x = 0$ intersects the x-axis at $\left(\frac{1}{2}, 0\right)$. Find the value of A and the equation of the normal.

Q17. The normal to the curve $y = \sqrt{x}$ at the point (x, y) intersects the x-axis at the point $(4.5, 0)$. Find the coordinates of the point of intersection of the normal and the curve. Find the equation of the normal.

Q18. The normal to the curve $y = \frac{\sin(Ax)}{B}$ at the point $(1, 0)$ has the equation $y = \frac{1-x}{4}$. Find the smallest positive value of A and the value of B.

Q19. The normal to the curve $y = \ln(Ax)$ at the x-intercept of the curve intersects the y-axis at point $\left(0, \frac{1}{4}\right)$. Find the value of A and the coordinates of the point where the normal meets the curve.

Q20. The normal to the curve $y = \frac{3x+5}{2x+2}$ at the point (x, y) intersects the x-axis at the point where $x = \frac{1}{2}$. Find the equation of the normal. Find the coordinates of all points of intersection of the normal and the curve.

Q21. The normal to the curve $y = \frac{x^2}{x+2}$ has the equation $y = \frac{x+4}{c}$. Find the value of c and coordinates of all points where the normal intersects the curve.

Q22. The line $y = 16x - 9$ is a tangent to the curve $y = 2x^3 + ax^2 + bx - 9$ at the point $(1, 7)$. Find the values of a and b.

Q23. A tangent to the graph of $y = e^x$ passes through the origin.

(i) Sketch the graphs of $y = e^x$ and the tangent on the same set of axes and find the equation of the tangent.

(ii) Use your sketch to justify that the area of the region completely bounded by the graph of $y = e^x$, the tangent and the y-axis is smaller than $\frac{1}{2}$.

Q24. The function $f(x) = \frac{1}{2}\sqrt{4 - 6x - 8x^2 - 2x^3}$ is defined for $-2 \leq x \leq \sqrt{2} - 1$. The tangent to the graph of f is parallel to the line $3x + 4y = 0$. Find the equation of the tangent.

Chapter 21

Further derivatives. Use of derivatives in sketching graphs

Definition 149 (second derivative).
Let y be a function of x. A derivative of y, $\frac{d}{dx}(y) = \frac{dy}{dx}$ is a function of x itself.
Its derivative: $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is denoted by $\frac{d^2y}{dx^2}$ and is called the ***second derivative*** *of y.*
Similarly, the third *or* further *derivatives of y may be obtained, i.e.*

$$\frac{d^3y}{dx^3} = \frac{d}{dy}\left(\frac{d^2y}{dx^2}\right)$$
$$\frac{d^4y}{dx^4} = \frac{d}{dy}\left(\frac{d^3y}{dx^3}\right)$$
$$\vdots$$
$$\frac{d^ny}{dx^n} = \frac{d}{dy}\left(\frac{d^{n-1}y}{dx^{n-1}}\right), n \in \mathbb{N}^+$$

Example 150 (2^{nd} and 3^{rd} derivative).
Let $y = 4x^5 - 3x^3 + 4x^2 - 5x + 7$. Then:

first *derivative of y is*	$\frac{dy}{dx} = 20x^4 - 9x^2 + 8x - 5,$
second *derivative of y is*	$\frac{d^2y}{dx^2} = 80x^3 - 18x + 8,$
third *derivative of y is*	$\frac{d^3y}{dx^3} = 240x^2 - 18.$

Note.

n^{th} derivative of y for $n \geq 6$ is	$\frac{d^ny}{dx^n} = 0$

Notation.
2^{nd} derivative of y is denoted as $\frac{d^2y}{dx^2}$ or y''.

2^{nd} derivative of $f(x)$ is denoted as $\frac{d^2}{dx^2}f(x)$ or $f''(x)$.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$
$$(y')' = y''$$
$$\frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}f(x)$$
$$(f'(x))' = f''(x)$$

n^{th} derivative of y is denoted as $\frac{d^n y}{dx^n}$.
n^{th} derivative of $f(x)$ is denoted as $\left(f^{(n)}(x)\right)$.

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^n y}{dx^n}$$

Questions 21.1.

Q1. Find the second derivative of the following.

(1) $e^{-0.5x}(\sin 2x + \cos 2x)$ (2) $(2x^3 + x^2 - 3)\ln(3x+1)$ (3) $\frac{x^2+4x+4}{x^2-3x+2}$

Q2. Find the formula for the n^{th} derivative of the following.

(1) e^{3x} (2) $\cos(ax+b)$ (3) $\frac{1}{ax^2}$

Q3. Consider the function $y = e^{-\frac{1}{4}x^2+x}$.

(i) Find $\frac{dy}{dx}$ and show that the only solution of $\frac{dy}{dx} = 0$ is $x = 2$.
(ii) By considering values of $\frac{dy}{dx}$ for $x < 2$ show that the function is **increasing** for $x < 2$.
(iii) Deduce that the function is decreasing for $x > 2$.
(iv) What is the value of $\frac{d^2y}{dx^2}$ for $x = 2$?
(v) When is $\frac{d^2y}{dx^2}$ positive, and when negative?

Q4. Find the first and second derivatives of the following:

(1) $y_1 = x^3 - 3x$ (2) $y_2 = xe^x$ (3) $y_3 = x\ln x$ (4) $y_4 = \frac{x^2}{x^2+1}$

For each of the functions above:

(i) find zeroes of the function,
(ii) find intervals where the function is positive / negative (i.e. its values are positive / negative),
(iii) find zeroes of the first derivative of the function,
(iv) find intervals where the first derivative is positive / negative,
(v) find zeroes of the second derivative of the function,
(vi) find intervals where the second derivative is positive / negative,
(vii) are the values of the 2^{nd} derivative at zeroes of the 1^{st} derivative positive or negative?
(viii) use a calculator or appropriate software to graph the functions and investigate what information about the graphs does the second derivative give us.

21.1 Stationary and inflection points

Theorem 151 (use of 1^{st} derivative).
Let $y = f(x)$ be a function whose derivative $f'(x)$ is defined at $x = x_0$. Then:

if $f'(x_0) = 0$ *then* $(x_0, f(x_0))$ *is a* ***stationary point*** *(minimum or maximum or stationary inflection point)*
if $f'(x_0) > 0$ *then* f *is* ***increasing*** *in the* neighbourhood *of* $(x_0, f(x_0))$
if $f'(x_0) < 0$ *then* f *is* ***decreasing*** *in the* neighbourhood *of* $(x_0, f(x_0))$

THEOREM 152 (use of 2^{nd} derivative).
Let $y = f(x)$ be a function whose first derivative $f'(x)$ and second derivative $f''(x)$ are defined at $x = x_0$. Then:

*if $f''(x_0) > 0$ then f is concave **UP** ($\cup$) in the* neighbourhood *of $(x_0, f(x_0))$*
*if $f''(x_0) < 0$ then f is concave **DOWN** ($\cap$) in the* neighbourhood *of $(x_0, f(x_0))$*
*if $f''(x)$ **changes sign** at $x = x_0$*
*then f has an **inflection point** at $(x_0, f(x_0))$*

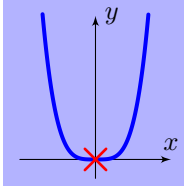
THEOREM 153 (nature of stationary points - 2^{nd} derivative test).
Let $y = f(x)$ be a function whose first derivative $f'(x)$ and second derivative $f''(x)$ are defined at $x = x_0$. Then:

*if $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a (local) **minimum** at $(x_0, f(x_0))$*
*if $f'(x_0) = 0$ and $f''(x_0) < 0$ then f has a (local) **maximum** at $(x_0, f(x_0))$*
if $f'(x_0) = 0$ and $f''(x_0) = 0$ then further investigation is required

Note that the final case of theorem 153 (both first and second derivative equal zero) is not decisive. The three examples below show that when

$$f'(x_0) = 0 \text{ and } f''(x) = 0$$

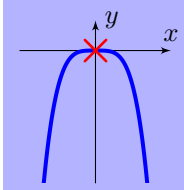
then point $(x_0, f(x_0))$ can be minimum, maximum as well as stationary inflection point.

EXAMPLE 154 ($y' = 0, y'' = 0$ - minimum).


Consider $y = x^4$. Then $y' = 4x^3$ and $y'' = 12x^2$.
At $x = 0$ both $y' = 0$ and $y'' = 0$.
Let us investigate how y', y'' and thus y behave for x's below zero and above zero.

x	< 0	0	> 0	*conclusion*
y'	$-ive$	0	$+ive$	*y' changes from $-$ to $+$, so f from decreasing to increasing*
y''	$+ive$	0	$+ive$	*y'' is positive, so f concave up*

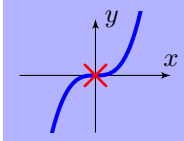
*The conclusions above show that point $(0, 0)$ is a **minimum**.*

EXAMPLE 155 ($y' = 0, y'' = 0$ - maximum).


Consider $y = -x^4$. Then $y' = -4x^3$ and $y'' = -12x^2$.
At $x = 0$ both $y' = 0$ and $y'' = 0$.
Let us investigate how y', y'' and thus y behave for x's below zero and above zero.

x	< 0	0	> 0	*conclusion*
y'	$+ive$	0	$-ive$	*y' changes from $+$ to $-$, so f from increasing to decreasing*
y''	$-ive$	0	$-ive$	*y'' is negative, so f concave down*

*The conclusions above show that point $(0, 0)$ is a **maximum**.*

EXAMPLE 156 ($y' = 0, y'' = 0$ - inflection point).


Consider $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$.
At $x = 0$ both $y' = 0$ and $y'' = 0$.
Let us investigate how y', y'' and thus y behave for x's below zero and above zero.

x	< 0	0	> 0	*conclusion*
y'	$+ive$	0	$+ive$	*y' is positive, so f increases*
y''	$-ive$	0	$+ive$	*y'' changes from $-$ to $+$, so f from concave DOWN to concave UP*

*The conclusions above show that point $(0, 0)$ is an **inflection point**.*

21.2 KINDS OF INFLECTION POINTS

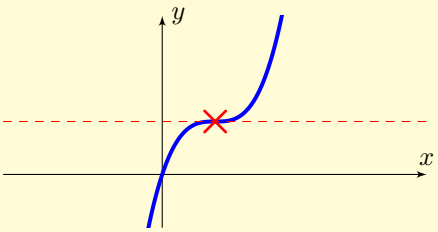
Inflection points can be either **stationary** or **non-stationary**. In either case the sign of second derivative *after* the point must differ from the sign *before* the point.

PROPERTIES OF INFLECTION POINTS.

1. *stationary* *inflection point*

It appears when:

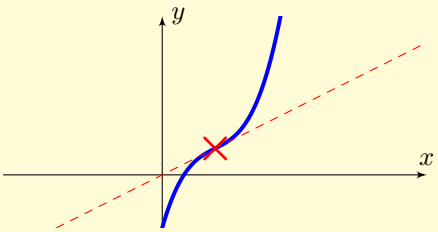
- $f'(x) = 0$ *(stationary)*
- $f''(x) = 0$ *(possibility of inflection)*
- $f''(x)$ *changes sign* *(inflection point)*



2. *non-stationary* *inflection point*

It appears when:

- $f'(x) \neq 0$ *(non-stationary)*
- $f''(x) = 0$ *(possibility of inflection)*
- $f''(x)$ *changes sign* *(inflection point)*

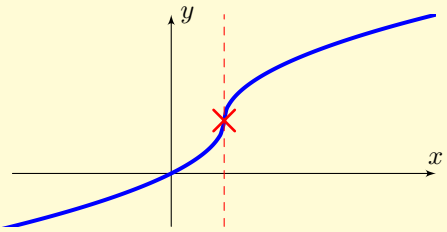


3. *"vertical"* *inflection point*

It is a special case of non-stationary point. In this case, however, the tangent to the graph at the point is vertical which means that there is no slope as it would be infinite. This in turn means that $f'(x)$ has no value at the point. In consequence $f''(x)$ cannot exist at the point.

Finally, a "vertical" inflection point appears when:

- $f'(x)$ *is not defined* *(vertical)*
- $f''(x)$ *is not defined* *(as it cannot be)*
- $f''(x)$ *changes sign* *(inflection point)*



QUESTIONS 21.2.

Q5. For each of the following functions find its first derivative and second derivative. Investigate signs of both derivatives at $x = 0$ and in the neighbourhood of $x = 0$. State, whether point $(0, f(0))$ is:

- a minimum
- a maximum
- a stationary inflection point
- a non-stationary inflection point
- none of the above

(1) $y = x^3$ (2) $y = x^3 + x^2$ (3) $y = x^3 + x$ (4) $y = x^4$ (5) $y = x^4 - x^2$ (6) $y = x^4 + x$ (7) $y = x^{\frac{1}{3}}$

Q6. For the function $f(x) = x^3 + x$:

(i) find the zeroes
(ii) find the first and second derivatives and their zeroes
(iii) hence sketch the graph of the function.

Q7. For the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$:

(i) find the zeroes
(ii) find the first and second derivatives and their zeroes
(iii) hence sketch the graph of the function.

Q8. For the function $f(x) = x^2 + \frac{1}{x^2}$:

(i) find the zeroes

(ii) find the first and second derivatives and their zeroes

(iii) hence sketch the graph of the function.

Q9. The function f is defined by $f(x) = e^{2x-x^2}$. Find the coordinates of the point(-s) of inflection of the graph of $y = f(x)$.

Q10. The function f is defined for $x \in]0, 1]$ by $f(x) = x^3 \ln x$. Find the minimum of the function and justify it is a minimum.

Q11. For each of the functions below:

(i) find coordinates of stationary points, state and justify their nature,

(ii) find the coordinates of inflection points,

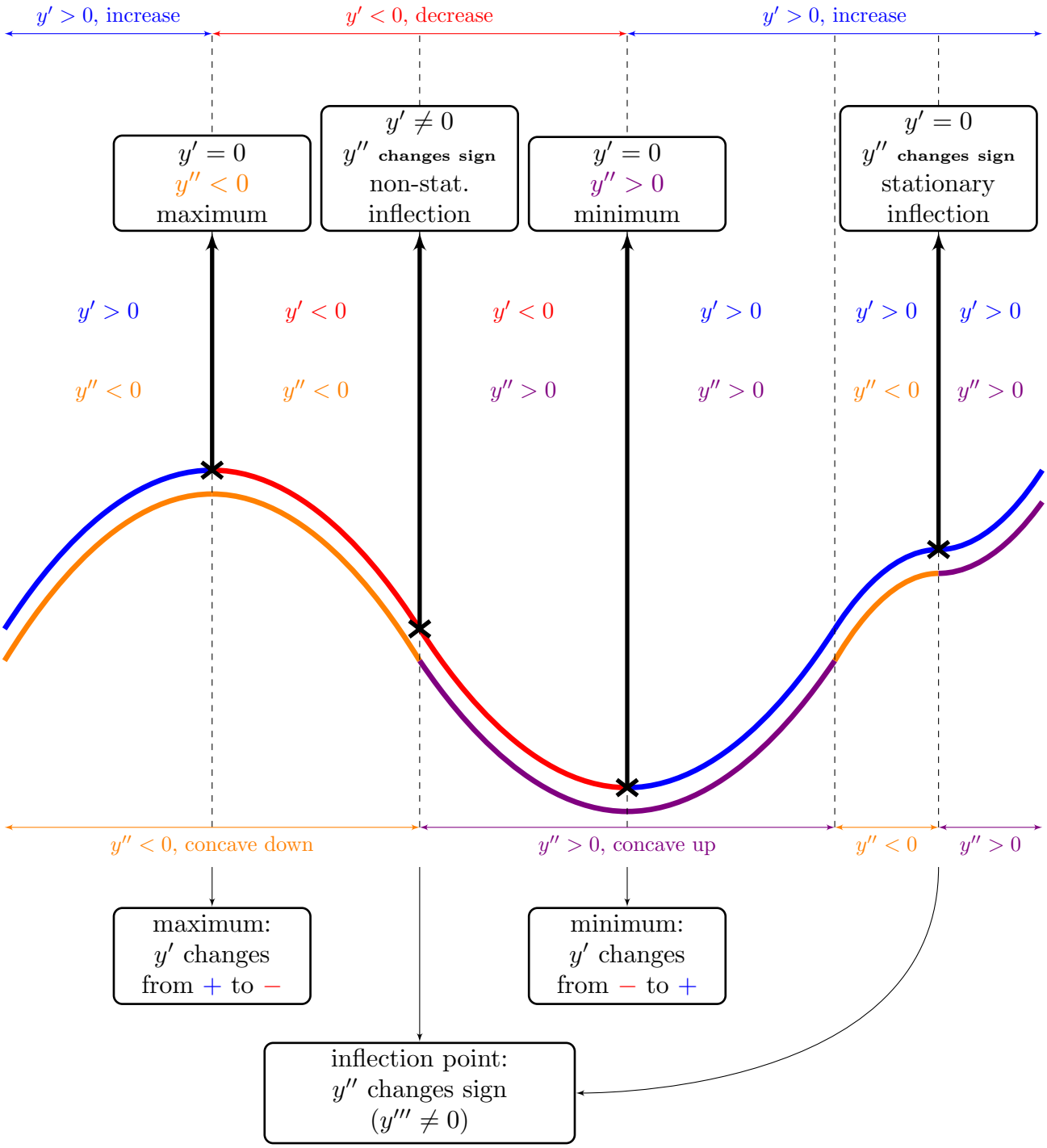
(iii) sketch the graph of the function.

(1) $f(x) = xe^{-x}$
(2) $f(x) = x^2e^{-x}$
(3) $f(x) = (x-1)^2e^{-x}$
(4) $f(x) = \frac{\ln 2x}{x}$
(5) $f(x) = \frac{\ln x}{x^2}$
(6) $f(x) = \frac{e^x}{x}$
(7) $f(x) = \frac{e^x}{\sqrt{x}}$
(8) $f(x) = 9x^3 - 3x^2 - 8x + 4$
(9) $f(x) = 3x^4 - 4x^3 - 24x^2 + 48x - 1$
(10) $f(x) = 3x^4 - 4x^3 - 96x^2 + 192x + 256$
(11) $f(x) = x^4 - 6x^2 - 8x + 24$

Q12. For the curve defined by $x \mapsto \frac{-2x+2}{x^2+1}$ find:

(i) the coordinates of the inflection points,

(ii) the equation of the normal at one chosen inflection point.


$y' > 0$, increase
$y' < 0$, decrease
$y' > 0$, increase
$y' = 0$
$y'' < 0$
maximum
$y' \neq 0$
y'' changes sign
non-stat.
inflection
$y' = 0$
$y'' > 0$
minimum
$y' = 0$
y'' changes sign
stationary
inflection
$y' > 0$
$y'' < 0$
$y' < 0$
$y'' < 0$
$y' < 0$
$y'' > 0$
$y' > 0$
$y'' > 0$
$y' > 0$
$y'' < 0$
$y' > 0$
$y'' > 0$
$y'' < 0$, concave down
$y'' > 0$, concave up
$y'' < 0$
$y'' > 0$
maximum:
y' changes
from + to −
minimum:
y' changes
from − to +
inflection point:
y'' changes sign
$(y''' \neq 0)$

Chapter 22

Applications of differential calculus

22.1 Implicit differentiation

Questions 22.1.

Q1. Differentiate relations implicitly. You do not need to rearrange the equations to get $\frac{dy}{dx}$.

(1) $x^3 - y^2 = 2x^2y$
(2) $x^2 + 4xy - y^2 = 8$
(3) $\tan(x+y) - \tan(xy^2) = 0$
(4) $\arctan y^2 = x(1+y)$
(5) $\arctan(2\sqrt{xy}) = \frac{x}{y}$
(6) $\arctan xy = x + y$
(7) $\sin(x^2y) + \cos^2(xy) = 0$
(8) $\ln(\frac{x^2}{y}) = x$
(9) $3e^{xy} = 6$
(10) $e^{x\cos y} = \ln 2$
(11) $\ln(x^2y) = \cos(\pi)$
(12) $\sin(x^3y) + \cos(xy^3) = 0$
(13) $e^{2xy} = \frac{e^x}{e^y+1}$

Q2. Find the slope of the curve $\sqrt{x^2+y^2} - 2xy = 29$ at $(3,-4)$.

Q3. Find the slope of the curve $x^2 + y^4 + 2xy^2 = 4$ at the point where $x = 1$.

Q4. Find the slope of the curve $x^3 + y^3 + 2x^2y^2 = 1$ at $(1,-2)$.

Q5. Find the first derivative of the following relations.

(1) $x - \frac{x^3}{y} = \frac{1}{x}$
(2) $e^{xy} = y$

Q6. Find the first and the second derivative of the following relations.

(1) $xy - y = 6$
(2) $\sqrt{xy} + \sqrt{x} = 3$

Q7. Consider the relation $xy = y^2 + 1$. Show that $\frac{d^2y}{dx^2} = \frac{2y(y-x)}{(2y-x)^3}$.

Q8. Consider the relation $x^2 + \ln(x+y) = 1$.

(i) Prove that the point $(1,0)$ lies in the graph of the relation.
(ii) Find the gradient of the graph at $(1,0)$.
(iii) Find the value of the second derivative at $(1,0)$.

(iv) Sketch the graph of the relation in the region of the point $(1, 0)$.

Q9. A curve has equation $4x^2 + 9y^2 = 36$.

(i) Substituting smallest possible value for x^2 and then for y^2 find the domain and range of the relation.

(ii) Use implicit differentiation to find $\frac{dy}{dx}$.

(iii) Eliminate y from your expression above.

(iv) Sketch the curve.

Q10. A curve has equation $x = y^2 - 1$.

(i) Find the domain and range of the relation.

(ii) Use implicit differentiation to find $\frac{dy}{dx}$.

(iii) Eliminate y from your expression above.

(iv) Find points on axis and sketch the curve.

Q11. A curve has equation $\frac{e^{xy^2}}{x^2-2xy} = 3y$. Differentiate the equation implicitly.

Q12. Find the equations of the tangents of the following curves at the points indicated.

(1) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point(s), where $x = -4$,

(2) $2xy - 19 = \sqrt{x^2 + y^2}$ at $(3, 4)$.

22.2 Curves defined parametrically

Questions 22.2.

Q13. A curve is defined parametrically by $\begin{cases} x = 5\cos t \\ y = 5\sin t \end{cases}$, where $0 \leq t \leq 2\pi$.

(i) State the range of possible values of x and y.

(ii) Evaluate $\frac{dy}{dt}$, $\frac{dx}{dt}$ and hence $\frac{dy}{dx}$.

(iii) Show that $\frac{dy}{dx} = 0$ at points $(0, 5)$ and $(0, -5)$.

(iv) Show that $\frac{dy}{dx}$ does not exist at points $(5, 0)$ and $(-5, 0)$.

(v) Find the values of t for which $\frac{dy}{dx}$ takes values of 1 and -1 and hence find the coordinates of the points on the curve for these values of t.

(vi) Sketch the curve.

(vii) Find the cartesian equation of the curve.

Q14. A curve is defined with parametric equations $x = \cos t + 2, y = \sin t - 3$, $0 \leq t \leq 2\pi$.

(i) Find $\frac{dy}{dx}$.

(ii) Sketch the curve. Find its cartesian equation.

Q15. A curve is given with parametric equations $x = 2\cos 2t, y = \csc t$, $0 \leq t \leq \pi$. The point P on the curve has x-coordinate 1.

(i) Find the value of the parameter t at P and the y-coordinate of P.

(ii) Find $\frac{dy}{dx}$ and the equation of the tangent to the curve at P.

(iii) Sketch the curve. Find its cartesian equation.

Q16. A curve is given with parametric equations $x = 3\cos t, y = 4\sin t - 3$, $0 \leq t \leq 2\pi$. The point P on the curve has y-coordinate -1.

(i) Find the value of the parameter t at P and the x-coordinate of P.

(ii) Find $\frac{dy}{dx}$ and the equation of the normal to the curve at P.

(iii) Sketch the curve. Find its cartesian equation.

Q17. A curve is defined with parametric equations $x = 8\cos t - 2, y = 4\sqrt{2}\sin t$, where $0 \leq t \leq 2\pi$. The point P on the curve has x-coordinate 2.

(i) Find the y-coordinate of P.

(ii) Evaluate $\frac{dy}{dx}$ and show that the normal to the curve at P passes through the origin.

(iii) Sketch the curve. Find its cartesian equation.

Q18. A curve is defined with parametric equations $x = \sec t - 4, y = \tan t$ where $-\frac{\pi}{2} < t < \frac{3\pi}{2}, x \neq \frac{\pi}{2}$. Points P and Q on the curve have y-coordinate 1.

(i) Find the x-coordinates of P and Q.

(ii) Evaluate $\frac{dy}{dx}$ and the point where the normals to the curve at P and Q intersect.

(iii) Sketch the curve. Find its cartesian equation.

Q19. A curve is defined with parametric equations $y = 4t + 1, x = 2t^2 - 3$, where $t \in \mathbb{R}$. Find the equation of a tangent to the curve at the point where $x = -2.5$.

Q20. For a curve defined parametrically by $x = a\cos t, y = a\sin t$, where a is a positive constant and $0 \leq t \leq 2\pi$ show that a normal to the curve at any point passes through the origin.

Q21. A curve is defined with parametric equations

$$x = \frac{t+1}{1-t}, y = \frac{1}{t+2}, -2 < t < 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{1}{2}\left(\frac{1-t}{t+2}\right)^2$.

(ii) Find an equation for the normal to the curve at the point where $t = 0$.

(iii) Find the cartesian equation of the curve.

Q22. Consider a curve defined parametrically by $x = 8\cos t, y = 6\sin 2t$, $0 \leq t \leq \pi$.Find the coordinates of all points on the curve where tangents are horizontal. Find the equations of the tangents.

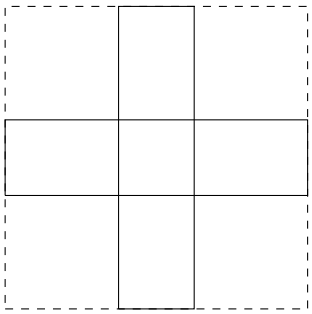
22.3 Optimisation

Questions 22.3.

Q23. A farmer wants to enclose the rectangular piece of land with a rope 60 metres long. What should be the dimensions of the plot so that it has the largest possible area? Justify your answer.

Q24. A farmer wants to enclose the rectangular plot of land with a rope 60 metres long. The plot is to run along the wall so that the wall would constitute one of its sides while the other three would be formed by the rope. What should be the dimensions of the plot so that it has the largest possible area? Justify your answer.

Q25. A cuboid (without a top) is made from a square $30cm \times 30cm$ sheet of paper (see the diagram). A cuboid has a square base. What are the dimensions of a cuboid that has maximum volume?



Q26. If $x + y = 4$, where $x, y \geq 0$, find the exact minimum value of $x^2 + y^3$.

Q27. Find the point on the curve $y = x^2 - 4x + 1$ that is closest to the point $(8, 0)$.

Q28. A rectangular sheet of metal with perimeter $60cm$ is fold in a way that it forms a side surface of a cylinder. Given that the volume of the cylinder is largest possible find the dimensions of the rectangle.

Q29. Consider the curve defined by $y = x^2 - 1$. A line tangent to the curve bounds the triangle with the smallest possible area in in the 4^{th} quadrant. Find the coordinates of the point where the tangent meets the curve.

22.4 Applying derivatives in kinematics

Questions 22.4.

Q30. A particle is moving along a straight line. Its displacement from a fixed origin is given by $s = 0.25t - 1$, where $0 \leq t \leq 8$ is time in hours and s is in kilometres.

(i) At what time is the particle at origin?

(ii) What is the initial displacement of the particle? (e.g. displacement when the observation starts)

(iii) What is the final displacement of the particle?

(iv) What is the total distance travelled by the particle?

(v) What is the average particle's speed over time $0 \leq t \leq 8$?

(vi) What is the average particle's speed over time $1 \leq t \leq 4$?

(vii) What can you say about the particle's acceleration?

Q31. The displacement of a particle from a fixed origin is given by $s = \frac{1}{2}t^2 + \frac{3}{2}t + 1$, where $0 \leq t \leq 4$ is time in seconds and s is in metres.

(i) What is the initial displacement of the particle? (e.g. displacement when the observation starts)

(ii) What is the final displacement of the particle?

(iii) What is the total distance travelled by the particle?

(iv) Use your calculator to find the average particle's speed over time:

(a) $0 \leq t \leq 4$ (c) $1 \leq t \leq 3$ (e) $1 \leq t \leq 1.5$ (g) $1 \leq t \leq 1.05$
(b) $1 \leq t \leq 4$ (d) $1 \leq t \leq 2$ (f) $1 \leq t \leq 1.1$ (h) $1 \leq t \leq 1.01$.

(v) What is the particle's speed when t=1?

(vi) What can you say about the particle's acceleration?

displacement	s	m, km
velocity *(change of displacement in time)*	$v = \frac{ds}{dt}$	$\frac{m}{s}, \frac{km}{h}$
acceleration *(change of velocity in time)*	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	$\frac{m}{s^2}, \frac{km}{h^2}$ $\frac{(\frac{m}{s})}{s}, \frac{(\frac{km}{h})}{h}$

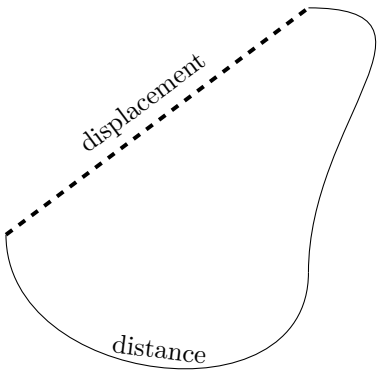


Figure 22.1: Distance and displacement.

Q32. The displacement of a particle is given by $s = 4te^{-\frac{t}{2}}$, where $0 \leq t \leq 8$ is in seconds and s is in metres.

(i) Find the expression for velocity in terms of t.

(ii) When the particle's displacement takes the maximum value?

(iii) What is the particle's maximum displacement?

(iv) What is the smallest particle's displacement from the origin after it reaches the maximum?

(v) Find the expression for acceleration in terms of t.

Q33. *(idea taken from the Internal Assessment task published in IB Teacher Support Material)*
The formula $y = 2.5t^3 - 15t^2$ represents the position of the elevator, y, measured in metres ($y = 0$ represents ground level) and t represents time measured in minutes ($t = 0is$ the starting time). We know that the trip up and down the shaft, ignoring time spent is approximately six minutes and that the depth of the shaft is no more than 100 metres.

(i) Use your GDC in parametric mode to visualize the motion of the elevator in the shaft (you may want to view the motion along the line $x = -1$).

(ii) Comment on the motion shown by the function. In what aspects does it differ from the motion you would expect to observe in reality?

(iii) Graph the displacement, velocity and acceleration functions versus time in one set of axes. Show clearly zeroes, maximum/minimum/inflection points.

(iv) Explain the meaning of the negative, positive and zero values of the velocity graph,

(v) Explain the relationship between velocity and acceleration in the intervals when the elevator speeds up, slows down, and is at rest.

(vi) Evaluate the usefulness and identify the problems of the model in the given situation.

Q34. Velocity in ms^{-1} of a particle is given by $v = t^3 - 6t^2 + 11t$, where $0 \leq t \leq 4$ is time in seconds.

(i) Find the time interval when velocity is decreasing.

(ii) Find the minimum value of acceleration.

(iii) Given that the initial displacement was -13.5 metres find the expression for the displacement.

Q35. A particle has displacement defined by $s = 4\cos t + 3\sin t$, $t \geq 0$.

(i) What is the particle's initial position?

(ii) (a) What is the particle's maximum displacement from the origin?
(b) What is the particle's maximum displacement from its initial position?

(iii) What is the particle's maximum speed?

(iv) Show that its acceleration, a, is given by $a = -s$.

(v) Describe the motion of the particle.

Q36. The displacement measured in metres of a particle from a fixed origin is defined by $s(t) = 5\sin(20\pi t - 5\pi) - 1$, where $0 \leq t \leq 0.1$ is time in hours.

(i) Find the maximum displacement of the particle from the origin.

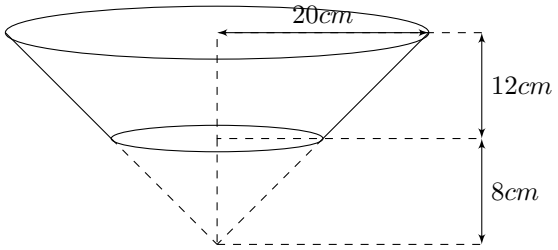
(ii) Find the total distance travelled by the particle.

(iii) Find maximum velocity.

(iv) Find the average speed of the particle.

22.5 Related rates

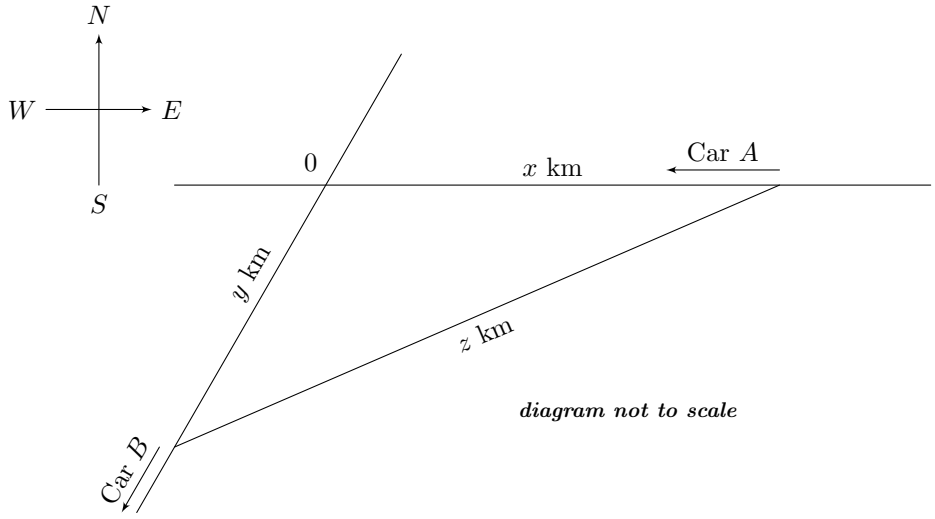
Questions 22.5.



Q37. Water is poured into a pot shown in the diagram at a rate of $3\pi cm^3 s^{-1}$. Find the rate at which the level of water is rising after 1 minute and 31 seconds.

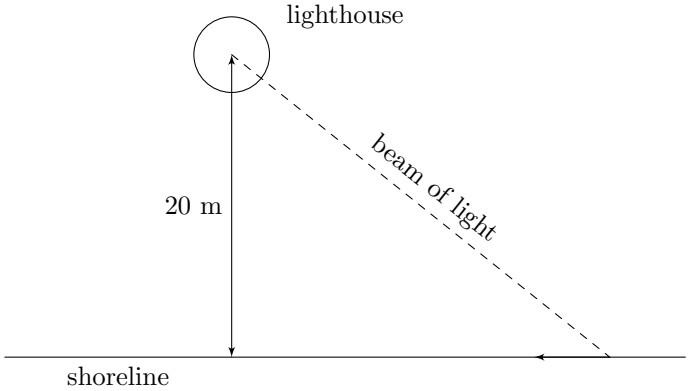
Q38. Car A is travelling on a straight east-west road in a westerly direction at 90km h^{-1}.
Car B is travelling on a straight road at 60km h^{-1} in a SW direction such that the angle AOB is 120°.

The roads intersect at the point O. When car A is x km east of O, and car B is y km south of O, the distance between the cars is z km.



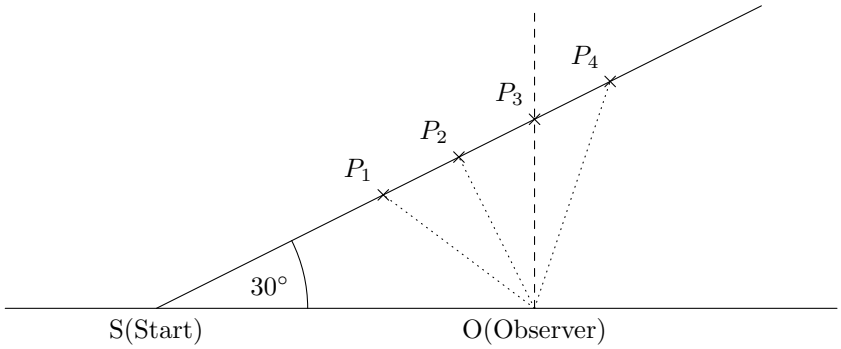
Find the rate of change of z when car A is 1.5 km east of O and car B is 0.9 km south-west of O.

Q39. A lamp at a lighthouse rotates at a rate of 12π radians per minute. A lighthouse is situated on an island 20 metres from a shoreline. A beam of light from the lighthouses lamp indicates a point that travels along the shoreline.



What is the speed of the point when its position is closest to the lighthouse? What is the speed of the point when it is 25 metres from the lighthouse? Give the answer in metres per second.

Q40. A plane pulled away from the ground at the point S. It moves along a straight line with constant velocity $200\frac{km}{h}$ making an angle of elevation of 30°. An observer standing $3km$ from S watches the plane rising.



Find the rate of change of the distance between the plane and the observer when the plane is directly over the observer.

Q41. (*idea by Marcin Kołaszewski*)

(a) Two straight roads intersect at an angle of 30°. Car A is travelling on one of the roads while car B on the other. Both cars are travelling towards the intersection and with a constant speed of $100\frac{km}{h}$. The cars will collide at the intersection. What is the rate of change of the distance between the cars at the moment of collision?

(b) Two straight roads intersect at an angle of 30°. Car A is travelling on one of the roads while car B on the other. Both cars are travelling towards the intersection and with constant speeds of $100\frac{km}{h}$ and $300\frac{km}{h}$ respectively. The cars will collide at the intersection. What is the rate of change of the distance between the cars at the moment of collision?

Part VI

INTEGRAL CALCULUS

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Chapter 23

Integrals

23.1 Indefinite integrals

Task
Find a function whose derivative is

(1) $\frac{dy}{dx} = 3x^2 + 4x - 2$

(2) $\frac{dy}{dx} = 4e^{2x} + 3x$

(3) $\frac{dy}{dx} = \sqrt{x}$

(4) $\frac{dy}{dx} = \frac{3}{x^2} + \frac{2}{x^3}$

(5) $\frac{dy}{dx} = \frac{3}{x} + \frac{1}{3x}$

Definition 157 (indefinite integral).
*If $\frac{dy}{dx} = f'(x)$ then $y = f(x) + c$, where $c \in \mathbb{R}$ is called the **antiderivative** of $f'(x)$.*
***Indefinite integral** of $h(x)$ is a family (a set) of all antiderivatives of $h(x)$, denoted as $\int h(x)dx$:*

$$\int h(x)dx = H(x) + C \;\Leftrightarrow\; h(x) = \frac{d}{dx}H(x)$$

Properties of indefinite integrals.

1. $\int \frac{d}{dx}(h(x))dx = h(x) + c$
2. $\frac{d}{dx}\int h(x)dx = h(x)$
3. $\int kh(x)dx = k\int h(x)dx$
4. $\int (g(x) \pm h(x))dx = \int g(x)dx \pm \int h(x)dx$

Theorem 158 (power rule).
For $x, n \in \mathbb{R}$, $n \neq -1$:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$$

Example 159 (integration: power rule).
Evaluate $\int \frac{3x^2+2x^4}{4x^3\sqrt{x}}dx$.

$$\int \frac{3x^2+2x^4}{4x^3\sqrt{x}}dx =$$

let's express a square root as a power:
$$= \int \frac{3x^2+2x^4}{4x^{3.5}}dx =$$

now split the fractions:
$$= \int \frac{3x^2}{4x^{3.5}} + \frac{2x^4}{4x^{3.5}}dx =$$

and simplify them:
$$= \int \frac{3}{4}x^{-1.5} + \frac{1}{2}x^{0.5}\ dx =$$

and now the power rule:
$$= \frac{3}{4} \times (-2)x^{-0.5} + \frac{1}{2} \times \left(\frac{2}{3}\right) x^{1.5} + c =$$

hence the final answer is:
$$= -\frac{3}{2\sqrt{x}} + \frac{1}{3}x\sqrt{x} + c$$

Questions 23.1.

Q1. Evaluate.

(1) $\int 2xdx$
(2) $\int x^4dx$
(3) $\int 2x^3dx$
(4) $\int 3x^5dx$
(5) $\int 3x^{4.5}dx$
(6) $\int 2x^{\frac{4}{3}}dx$
(7) $\int 5x^2(2x-1)dx$
(8) $\int \frac{3}{x^3}dx$
(9) $\int \frac{2}{x^5}dx$
(10) $\int 2x\sqrt{x}dx$
(11) $\int \frac{2}{x\sqrt{x}}dx$
(12) $\int \sqrt{2x^9}dx$
(13) $\int \frac{3x^4}{x^2\sqrt{x}}dx$
(14) $\int \frac{x^2+x-6}{x-2}dx$
(15) $\int (x+\frac{1}{x})^3dx$
(16) $\int \frac{(x+2)^3}{x^2}dx$
(17) $\int \frac{4x^3\sqrt{x}-3x^2}{2x^3}dx$
(18) $\int \frac{2x(\sqrt[3]{2x^2})}{\sqrt{2x}}dx$
(19) $\int \frac{(2x^3+3)^2}{(2x)^3}dx$
(20) $\int \sqrt{t}(\frac{1}{t^2}-\frac{3}{t})dt$
(21) $\int (y-\frac{2}{y})^2dy$
(22) $\int \frac{(u-1)^2}{\sqrt{u}}du$
(23) $\int 2\sqrt[3]{v^4} - \frac{3}{\sqrt[4]{v^3}}dv$
(24) $\int a(1+\frac{1}{a^2})(a^2-1)da$
(25) $\int 2x^3\sqrt{x}(1-x^2)dx$
(26) $\int (6x^3-6x^2+3x+4)dx$
(27) $\int \frac{(2x\sqrt{x}-x)^2}{2x^2}dx$
(28) $\int \left(\frac{1}{x\sqrt{x}} - \sqrt{2x}\right)dx$
(29) $\int \left(\sqrt{\frac{3}{x}} - \sqrt{\frac{x}{3}}\right)dx$

Theorem 160 (integral of x^{-1}).

- *For* $x > 0$ $\int \frac{1}{x}dx = \ln x + c$
- *For* $x \neq 0$ $\int \frac{1}{x}dx = \ln|x| + c$

Example 161 (integral of $\frac{1}{x}$).
Evaluate $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$.

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \int x - 2 + \frac{1}{x}\ dx =$$
$$= \frac{1}{2}x^2 - 2x + \ln x + c$$

More basic integrals

THEOREM 162 (integrals of exponential and trigonometric functions).

- $\int \sin x \, dx = -\cos x + c$
- $\int \cos x \, dx = \sin x + c$
- $\int \sec^2 x \, dx = \tan x + c$
- $\int e^x \, dx = e^x + c$
- $\int a^x \, dx = \frac{1}{\ln a} a^x + c$

THEOREM 163 ("inverse" of the chain rule for linear expressions).
If

$$\int f(x)dx = F(x) + c$$

then

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + c$$

EXAMPLE 164 (inverse chain rule for liner expressions).
Evaluate:

(1) $\int (3x-1)^9 \, dx$ ***(2)*** $\int \cos(4x-\pi) \, dx$ ***(3)*** $\int \sin(1-\frac{\pi}{2}x) \, dx$ ***(4)*** $\int e^{0.02x+1} \, dx$

The coefficient in a box refers to $\frac{1}{a}$ from the theorem above.

(1) $\int (3x-1)^9 \, dx = \boxed{\frac{1}{3}} \, \frac{1}{10}(3x-1)^{10} + c = \frac{1}{30}(3x-1)^{10} + c$

(2) $\int \cos(4x-\pi) \, dx = \boxed{\frac{1}{4}} \, \sin(4x-\pi) + c$

(3) $\int \sin(1-\frac{\pi}{2}x) \, dx = \boxed{\left(-\frac{2}{\pi}\right)} \left(-\cos(1-\frac{\pi}{2}x)\right) + c = \frac{2}{\pi}\cos(1-\frac{\pi}{2}x) + c$

(4) $\int e^{0.02x+1} \, dx = \boxed{\frac{1}{0.02}} \, e^{0.02x+1} + c = 50e^{0.02x+1} + c$

QUESTIONS 23.2.

Q2. Evaluate.

(1) $\int \frac{1}{2x} dx$
(2) $\int \sin 5x dx$
(3) $\int (3x-2)^6 dx$
(4) $\int (2x+5)^5 dx$
(5) $\int \sqrt{2x+3} dx$
(6) $\int e^{3x} dx$
(7) $\int e^{3x+1} dx$
(8) $\int e^{1-3x} dx$
(9) $\int (e^x + e^{-x})^2 dx$
(10) $\int \cos(\pi x) dx$
(11) $\int (\sin x + \cos x)^2 dx$
(12) $\int 2^x dx$
(13) $\int 2^{2x} dx$
(14) $\int 2^{2x+1} dx$
(15) $\int \sec^2 2x dx$
(16) $\int \sin(2x-\pi) dx$
(17) $\int \cos(\frac{x-\pi}{2}) dx$
(18) $\int \frac{(e^x+2)^2}{e^x} dx$
(19) $\int \cos(\frac{\pi x-\pi}{2}) dx$
(20) $\int \sin(\pi - x) dx$
(21) $\int \frac{1}{x+2} dx$
(22) $\int \frac{1}{2x+4} dx$
(23) $\int \frac{2}{3-2x} dx$
(24) $\int \sec^2(\pi - 2x) dx$
(25) $\int (e^{2x} + x) dx$
(26) $\int (\sin 2x + \sec^2 3x) dx$
(27) $\int \frac{dx}{2x+1}$
(28) $\int 3^{2x} dx$
(29) $\int \left(\sqrt{100x-1} - \frac{1}{\sqrt{100x-1}}\right) dx$
(30) $\int (\cos(4x+2) + e^{-x}) dx$
(31) $\int \frac{\ln 2 dx}{2x-1}$
(32) $\int 5^{3x+4} dx$

Integration with use of trigonometric identities

Example 165 (integral of a square of sine).
Evaluate $\int \sin^2 x \, dx$.

From the formula for the cosine of a double angle:

$$\cos 2x = 1 - 2\sin^2 x$$

$\sin^2 x$ can be expressed as $\frac{1}{2} - \frac{1}{2}\cos 2x$.
Hence:

$$\int \sin^2 x \, dx = \frac{1}{2}\int 1 - \cos 2x \, dx = \\ = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Corollary 166 (integrals of squares of sine or cosine).
To integrate a square of sine or cosine we need to firstly express it in terms of a cosine of a double angle using either $\boxed{\cos 2A = 1 - 2\sin^2 A}$ *or* $\boxed{\cos 2A = 2\cos^2 A - 1}$.

Example 167 (integral of a product of sine and cosine).
Evaluate $\int \sin x \cos 3x \, dx$.

Let us cosider two compound angle identities:

$$\begin{cases} \sin(A+B) = \sin A\cos B + \cos A\sin B \\ \sin(A-B) = \sin A\cos B - \cos A\sin B \end{cases}$$

Adding the two equations we get:

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

and hence:

$$\sin A\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$

Using the obtained identity in the integral we get:

$$\int \sin x\cos 3x \, dx = \int \frac{1}{2}\left(\sin(x+3x) + \sin(x-3x)\right) = \\ = \int \frac{1}{2}\left(\sin 4x - \sin 2x\right) = \\ = -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + c$$

Questions 23.3.

Q3. Evaluate.

(1) $\int (\cos x + 2)^2 \, dx$ (2) $\int \sin^2(2x - \pi) \, dx$ (3) $\int (2\sin x - \cos x)^2 \, dx$

Q4. Evaluate.

(1) $\int \sin 2x \cos 3x \, dx$ (2) $\int \sin 4x \cos x \, dx$ (3) $\int \sin(3x - \pi)\cos(2x + \pi) \, dx$

Q5. Use identities for $\cos(A \pm B)$ to evaluate:

(1) $\int \cos 2x \cos 3x \, dx$
(2) $\int \cos 4x \cos x \, dx$
(3) $\int \cos(5x - \pi)\cos(2x + \pi) \, dx$
(4) $\int \sin 2x \sin 3x \, dx$
(5) $\int \sin 4x \sin x \, dx$
(6) $\int \sin(5x - \pi)\sin(2x + \pi) \, dx$

Integrals of inverse trigonometric functions

THEOREM 168 (integrals of inverse trigonometric functions).

- $\int \frac{1}{1+x^2}dx = \arctan x + c$
- $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + c$
- $\int \frac{-1}{\sqrt{1-x^2}}dx = \arccos x + c$

EXAMPLE 169 (integrals of inverse trigonometric functions).
Evaluate:

(1) $\int \frac{dx}{\sqrt{1-9x^2}}$ ***(2)*** $\int \frac{dx}{4+x^2}$ ***(3)*** $\int \frac{dx}{4x^2+4x+5}$

(1) $\int \frac{dx}{\sqrt{1-9x^2}} = \int \frac{dx}{\sqrt{1-(3x)^2}} = (\text{by theorem 163}) = \frac{1}{3}\arcsin(3x) + c$

(2) $\int \frac{dx}{4+x^2} = \frac{1}{4}\int \frac{dx}{1+\left(\frac{x}{2}\right)^2} = (\text{by theorem 163}) = \frac{1}{4} \times \frac{1}{\left(\frac{1}{2}\right)}\arctan\left(\frac{x}{2}\right) + c = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c$

(3) $\int \frac{dx}{4x^2+4x+5} = \int \frac{dx}{(2x+1)^2+4} = \frac{1}{4}\int \frac{dx}{(x+\frac{1}{2})^2+1} = \frac{1}{4}\arctan(x+\frac{1}{2}) + c$

QUESTIONS 23.4.

Q6. Evaluate.

(1) $\int \frac{1}{1+4x^2}dx$
(2) $\int \frac{1}{1+2x^2}dx$
(3) $\int \frac{4}{4+x^2}dx$
(4) $\int \frac{3}{2+x^2}dx$
(5) $\int \frac{2}{1+(x-1)^2}dx$
(6) $\int \frac{1}{x^2+2x+2}dx$
(7) $\int \frac{1}{4x^2+12x+10}dx$
(8) $\int \frac{1}{\sqrt{1-2x^2}}dx$
(9) $\int \frac{1}{\sqrt{2-4x^2}}dx$
(10) $\int \frac{1}{\sqrt{1-(x+1)^2}}dx$
(11) $\int \frac{1}{\sqrt{2x-x^2}}dx$
(12) $\int \frac{1}{\sqrt{4x-4x^2}}dx$

Q7. Find the integrals.

(1) $\int (\frac{x^2\sqrt{x}+3x}{x\sqrt{x}-2x^{1.5}})^2 dx$
(2) $\int \sqrt{\frac{x-2}{4}}dx$
(3) $\int \sqrt{\frac{2}{x-2}}dx$
(4) $\int \sin 3x dx$
(5) $\int \sec^2 0.5x dx$
(6) $\int \cos^2 3x dx$
(7) $\int \frac{4}{1+x^2}dx$
(8) $\int \frac{1}{2+4x^2}dx$
(9) $\int 2^{3x}dx$
(10) $\int (\sqrt{2})^x dx$
(11) $\int \frac{2}{3x-6}dx$
(12) $\int e^{3x+2} - \sqrt{e^x}dx$

23.2 DEFINITE INTEGRALS

QUESTIONS 23.5.

Q1. Given $\frac{dy}{dx} = 2x - \sin x$ and $y = 2$ for $x = 0$, find the expression for y in terms of x.

Q2. Given $\frac{dy}{dx} = e^x - 2x$ and $y = 3$ when $x = 0$, find the expression for y in terms of x.

Q3. Given that $\frac{dy}{dx} = 2\cos(3x - \frac{\pi}{2}) + \frac{3}{\pi}$ and $y = 0$ when $x = \frac{\pi}{9}$, find an expression for y in terms of x.

Q4. Given that $\frac{dy}{dx} = 4\sin(2x - \frac{\pi}{2}) + \frac{16}{\pi}$ and $y = 0$ when $x = \frac{\pi}{8}$, find an expression for y in terms of x.

Q5. Find the equation of the curve $y = f(x)$ knowing that

- $f(0) = -1$
- $f'(0) = -2\pi$
- $f''(x) = 2\pi^2 \sin(\pi x)$

Q6. (a) Find $\frac{d}{dx}(x\sin x)$.

(b) **Hence**, find $\int x\cos x dx$.

(c) **Hence**, find $\int_0^{\pi} x\cos x dx$.

Q7. (a) Find $\frac{d}{dx}(xe^x)$.

(b) **Hence**, find $\int xe^x dx$.

(c) **Hence**, find $\int_1^2 xe^x dx$

Q8. (a) Velocity of a particle (in $\frac{m}{s}$) is given by $v = \frac{3}{4}t^2 + 1$, where t is time measured in seconds. Knowing that the initial displacement is 3 metres, find the distance covered by the particle in the first 2 seconds.

(b) Velocity of a particle (in $\frac{m}{s}$) is given by $v = \frac{3}{4}t^2 + 1$, where t is time measured in seconds. Find the distance covered by the particle in the first 2 seconds.

DEFINITION 170 (definite integral).
*The **definite integral***

$$\int_a^b f(x)dx$$

is defined informally as the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$.

We are going to use integration to evaluate areas in the next chapter. For now we are going to use the following

THEOREM 171 (fundamental theorem of calculus).
If

$$F'(x) = f(x)$$

then

$$\int_a^b f(x)dx = F(b) - F(a)$$

NOTATION.

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

EXAMPLE 172 (definite integral).
Evaluate $\int_2^8 \frac{2}{\sqrt{x}}\, dx$.

$$\int_2^8 \frac{2}{\sqrt{x}}\, dx = \int_2^8 2x^{-0.5}\, dx =$$
$$= \left[\frac{2}{0.5}x^{0.5}\right]_2^8 = \left[4x^{0.5}\right]_2^8 =$$
$$= 4\sqrt{8} - 4\sqrt{2} = 8\sqrt{2} - 4\sqrt{2} = 4\sqrt{2}$$

QUESTIONS 23.6.

Q9. Evaluate the following integrals. Give answers in exact form.

(1) $\int_0^2 (x+1)dx$

(2) $\int_{-3}^6 (x^2 + 2x - 1)dx$

(3) $\int_{-3}^1 (2 + 4x - x^2)dx$

(4) $\int_{-5}^1 (x+2)^2 dx$

(5) $\int_4^{16} \sqrt{x}dx$

(6) $\int_1^9 x\sqrt{x}dx$

(7) $\int_1^e \frac{1}{x}dx$

(8) $\int_1^2 \frac{2}{x^2}dx$

(9) $\int_1^2 \frac{dx}{x^3}$

(10) $\int_4^9 \frac{1}{\sqrt{x}}dx$

(11) $\int_0^1 (x^2 - x + 2\sqrt{x})dx$

(12) $\int_1^2 (2x + \frac{1}{x})dx$

(13) $\int_2^3 \left(\frac{x^2+1}{x^2}\right)dx$

(14) $\int_1^2 \left(\frac{x+1}{x}\right)^2 dx$

(15) $\int_1^4 \frac{x+1}{\sqrt{x}}dx$

PROPERTIES OF DEFINITE INTEGRALS.

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$

3. $\int_a^b f(x)dx \pm \int_a^b g(x)dx = \int_a^b (f(x) \pm g(x))dx$

4. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

QUESTIONS 23.7.

Q10. Given that $\int_{-1}^{5} f(x)dx = 6$ evaluate the following.

(1) $\int_{-1}^{5}(f(x)+2)dx$

(2) $\int_{5}^{-1} 2f(x)dx$

(3) $\int_{-1}^{5}(f(x)-2x)dx$

(4) $\int_{-1}^{2}(f(x)+3)dx + \int_{2}^{5}(f(x)-2)dx$

(5) $\int_{3}^{-1}(3-f(x))dx - \int_{3}^{5}(2-f(x))dx$

(6) $\int_{1}^{-1}(f(x)+x)dx + \int_{1}^{5}(x^2-f(x))dx$

Q11. Evaluate the following integrals. Give answers in exact form.

(1) $\int_2^4 (2x-4)^3 dx$

(2) $\int_2^5 \sqrt{x-1}dx$

(3) $\int_{\ln 2}^{\ln 3} e^x dx$

(4) $\int_1^e \frac{1}{2x}dx$

(5) $\int_{\frac{e}{2}}^{e} \frac{1}{2x-e}dx$

(6) $\int_{\frac{\pi}{4}}^{\pi} \sin(2x-\pi)dx$

(7) $\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \cos(\frac{\pi}{2}-x)dx$

(8) $\int_0^{\ln 2} e^{2x}dx$

(9) $\int_0^1 \frac{1}{1+x^2}dx$

(10) $\int_{-\frac{1}{2}}^{0} \frac{1}{1+4x^2}dx$

(11) $\int_{\sqrt{3}}^{3} \frac{1}{9+x^2}dx$

(12) $\int_e^{2e} \frac{dx}{e(x+e)}$

(13) $\int_{\ln 2}^{\ln 6}(e^{2x}-2)dx$

(14) $\int_1^4 (x+2)^2 dx$

Q12. Given that $\int_{a-1}^{a+1} \frac{dx}{2x+1} = 1$ find the value of a.

Q13. Give the **exact** values in simplest form.

(1) $\int_e^{2e} \frac{dx}{\pi(2x-e)}$

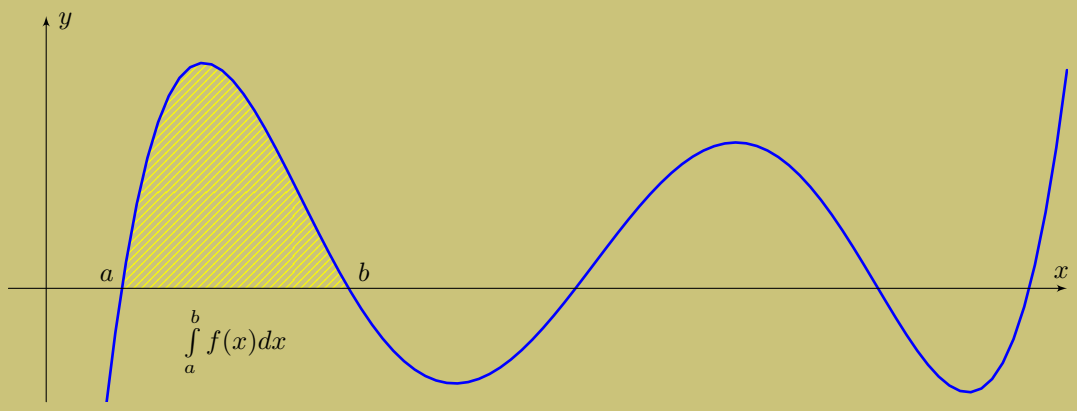
(2) $\int_{\ln 2}^{\ln 5}(e^{4x}+e^x)dx$

Chapter 24

Applications of integration

24.1 Area under a curve

Definition 173 (definite integral as a signed area).
A definite integral is equal to a "signed" (i.e. positive or negative) area between a curve and the x-axis and the limits of the integral.

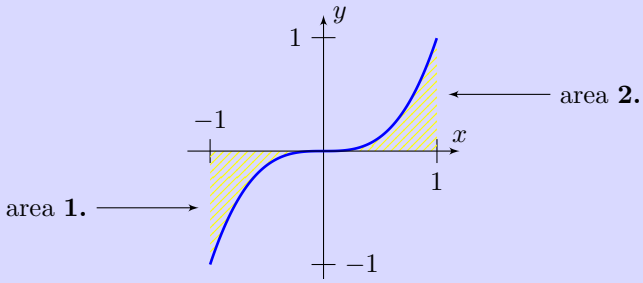


More precisely: If $\frac{d}{dx}F(x) = f(x)$ then the signed area *bounded by the graph of f, the x-axis and the lines $x = a$ and $x = b$ is given by $F(b) - F(a)$, i.e. by $\int_a^b f(x)\,dx$.*

The area given by definite integrals can be negative when the curve is below the x-axis or when the lower limit of the integral is larger than the upper limit, hence the area is called "signed" (see the properties below).

An **important note** must be made here.
We learned firstly about the definite integrals and now we are about to discover their applications in areas and then volumes.
However, the definite integral is defined as the area between the curve and the x-axis and then the Fundamental Theorem of Calculus (theorem 171) shows us that the anti-derivative is the same as the integral.

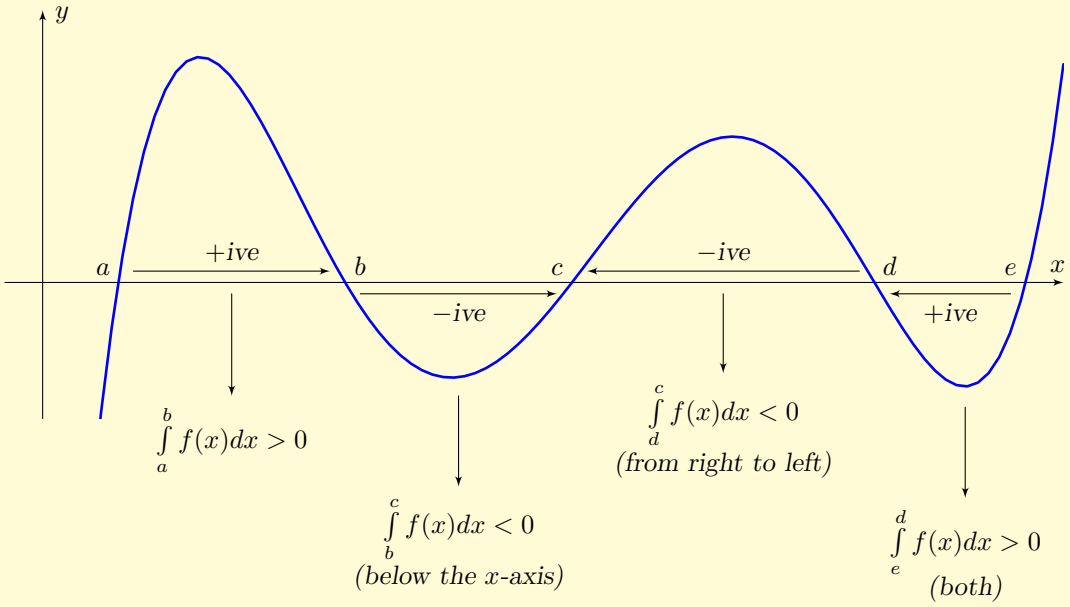
Example 174 (signed area).
*Find the areas bounded by the graph of $y = x^3$, the x-axis and the line: **1.** $x = -1$, **2.** $x = 1$*



1. $\int_{-1}^{0} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{0} = 0 - \frac{1}{4} = -\frac{1}{4}$

2. $\int_{0}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4} - 0 = \frac{1}{4}$

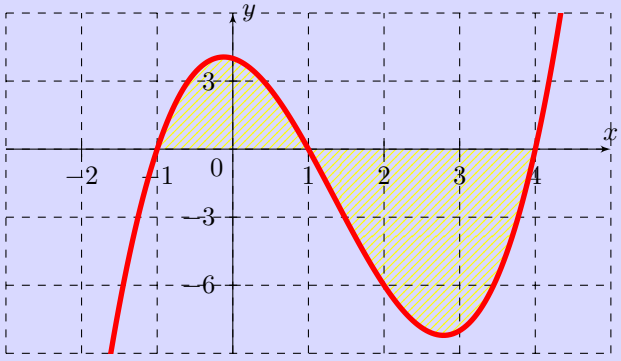
Properties of "orientated" area.



Example 175 (area bounded by the curve and the x-axis).
Find the total area of all regions bounded by the curve defined by $y = x^3 - 4x^2 - x + 4$ and the x-axis.

The curve has three x-intercepts:

$$\begin{aligned} x^3 - 4x^2 - x + 4 &= 0 \\ x^2(x-4) - (x-4) &= 0 \\ (x^2-1)(x-4) &= 0 \\ (x+1)(x-1)(x-4) &= 0 \\ x = -1, x = 1 \text{ or } x &= 4 \end{aligned}$$



As it can be observed, the region between $x = -1$ and $x = 1$ is above the x-axis, while the region between $x = 1$ and $x = 4$ is below the x-axis.
Hence the area is:

$$\int_{-1}^{1} x^3 - 4x^2 - x + 4\,dx - \int_{1}^{4} x^3 - 4x^2 - x + 4\,dx =$$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_{-1}^{1} - \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_{1}^{4} =$$

$$= \left(\left(\frac{1}{4} - \frac{4}{3} - \frac{1}{2} + 4\right) - \left(\frac{1}{4} + \frac{4}{3} - \frac{1}{2} - 4\right)\right) - \left(\left(\frac{1}{4}(4^4) - \frac{4}{3}(4^3) - \frac{1}{2}(4^2) + 4(4)\right) - \left(\frac{1}{4} - \frac{4}{3} - \frac{1}{2} + 4\right)\right) =$$

$$= \frac{16}{3} - \left(-15\frac{3}{4}\right) = 21\frac{1}{12}$$

Questions 24.1.

Q1. Show that the total area of two regions enclosed by the graph of $y = \sin x$ and the x-axis between $x = 0$ and $x = 2\pi$ is equal 4.

Q2. Find the total area of the regions bounded by the curve $y = f(x)$ and the x-axis for:

(1) $y = x^3 - 4x^2$,
(2) $y = x^3 + 3x^2 - x - 3$,
(3) $y = x^3 + 7x^2 + 4x - 12$.

Q3. Find the total area of the region bounded by the curve $y = e^{2x}$, the x-axis and the lines $x = 0$ and $x = 1$.

Q4. Find the total area of the region bounded by the curve $y = e^{-3x}$, the x-axis and the lines $x = 0$ and $x = -\ln 2$.

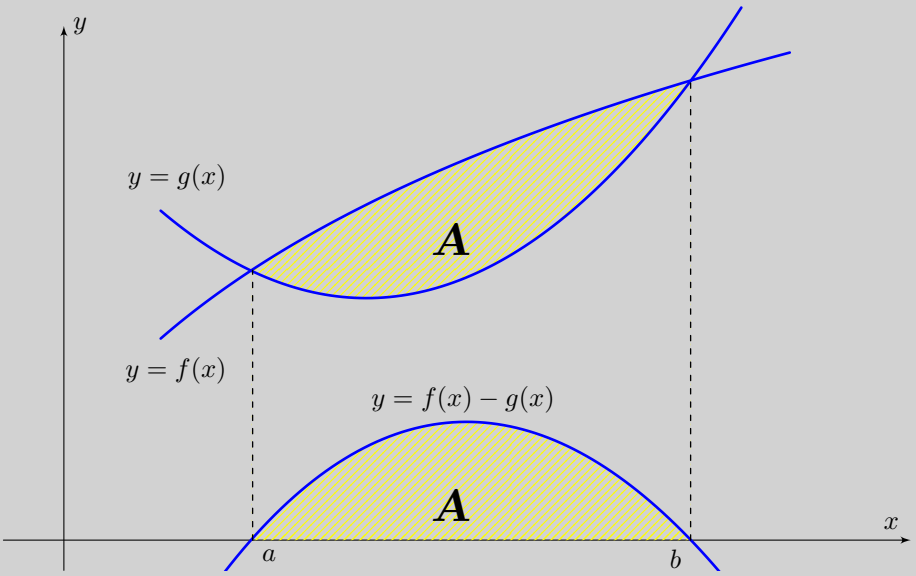
24.2 Area between curves

Theorem 176 (area between curves).
Let $y = f(x)$ and $y = g(x)$ be two functions whose graphs intersect at $x = a$ and $x = b$,where $a < b$.
If for every $x \in]a, b[$

$$f(x) \geq g(x)$$

then the area of the region bounded by the two graphs between $x = a$ and $x = b$ is given by

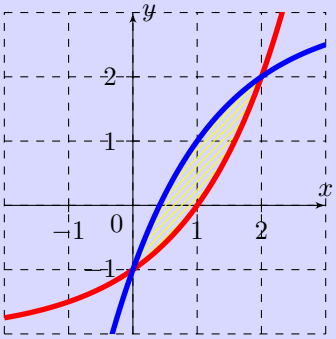
$$\int_a^b f(x) - g(x)\, dx.$$



Example 177 (area between curves).
Find the area of the region bounded by the curves defined by $y = 2^x - 2$ and $y = 3 - \left(\frac{1}{2}\right)^{x-2}$.

The diagram shows the graphs of $y = 2^x - 2$ and $y = 3 - \left(\frac{1}{2}\right)^{x-2}$.
To find the intersections of the two curves we will solve the equation:

$$\begin{aligned} 2^x - 2 &= 3 - \left(\frac{1}{2}\right)^{x-2} \\ 2^x - 2 &= 3 - 2^{-x+2} \\ 2^x - 2 &= 3 - \frac{4}{2^x} \\ (2^x)^2 - 5(2^x) + 4 &= 0 \\ (2^x - 1)(2^x - 4) &= 0 \end{aligned}$$



So the intersections are at $2^x = 1$, i.e. $x = 0$ and at $2^x = 4$, i.e. $x = 2$.
It can be observed that the curve $y = 2^x - 2$ is below the region considered and the curve $y = 3 - \left(\frac{1}{2}\right)^{x-2}$ is above it.
(E.g. the value for $x = 1$ can be checked to find out that $1 > 0$.)
Hence the area of the region is:

$$\int_0^2 \left(3 - \left(\frac{1}{2}\right)^{x-2}\right) - (2^x - 2)\ dx =$$
$$= \int_0^2 5 - \left(\frac{1}{2}\right)^{x-2} - 2^x\ dx =$$
$$= \left[5x - \frac{1}{\ln\left(\frac{1}{2}\right)}\left(\frac{1}{2}\right)^{x-2} - \frac{1}{\ln 2}(2^x)\right]_0^2 =$$
$$= \left[5x + \frac{1}{\ln 2}\left(\frac{1}{2}\right)^{x-2} - \frac{1}{\ln 2}(2^x)\right]_0^2 =$$
$$= \left(5(2) + \frac{1}{\ln 2}\left(\frac{1}{2}\right)^{2-2} - \frac{1}{\ln 2}(2^2)\right) - \left(5(0) + \frac{1}{\ln 2}\left(\frac{1}{2}\right)^{0-2} - \frac{1}{\ln 2}(2^0)\right) =$$
$$= \left(10 + \frac{1}{\ln 2} - \frac{4}{\ln 2}\right) - \left(\frac{4}{\ln 2} - \frac{1}{\ln 2}\right) = 10 - \frac{6}{\ln 2}$$

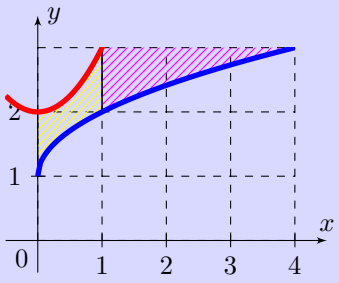
EXAMPLE 178 (use of integration to find area).
Find the area bounded by the curves $y = x^2 + 2$, $y = \sqrt{x} + 1$, the y-axis and the line $y = 3$.

We are going to evaluate the are using a few different methods.

method 1.
sum of areas of two regions:

R_1: between $y = \sqrt{x} + 1$ and $y = x^2 + 2$ for $0 \le x \le 1$,

R_2: between $y = \sqrt{x} + 1$ and the line $y = 3$ for $1 \le x \le 4$.



Area $R_1 = \int_0^1 (x^2 + 2) - (\sqrt{x} + 1)\ dx = \int_0^1 x^2 - x^{0.5} + 1\ dx =$
$= \left[\frac{1}{3}x^3 - \frac{2}{3}x^{\frac{2}{3}} + x\right]_0^1 = \frac{1}{3} - \frac{2}{3} + 1 = \frac{2}{3}$

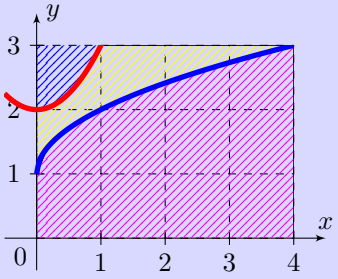
Area $R_2 = \int_1^4 3 - (\sqrt{x} + 1)\ dx = \int_1^4 2 - x^{0.5}\ dx =$
$= \left[2x - \frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \left(8 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right) = \frac{4}{3}$

So the total area is $\frac{2}{3} + \frac{4}{3} = 2$.

method 2.
difference of the area of the rectangle (4×3) and the two regions:

R_1: between $y = x^2 + 2$, y-axis and the line $y = 3$,

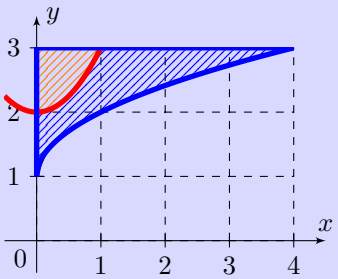
R_2: between $y = \sqrt{x} + 1$, x-axis, y-axis and the line $x = 4$.



$$\text{Area } R_1 = \int_0^1 3-(x^2+2)\ dx = \int_0^1 1-x^2\ dx =$$
$$= \left[x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}$$
$$\text{Area } R_2 = \int_0^4 \sqrt{x}+1\ dx = \int_0^4 x^{0.5}+1\ dx =$$
$$= \left[\frac{2}{3}x^{1.5} + x\right]_0^4 = \frac{16}{3} + 4 = \frac{28}{3}$$

So the area of the region considered is $12 - \frac{2}{3} - \frac{28}{3} = 2$.

method 3.
difference of the areas of two regions:



R_1: between $y = x^2 + 2$, y-axis and the line $y = 3$,

R_2: between $y = \sqrt{x} + 1$, y-axis and the line $y = 3$.

We will evaluate the area of R_2 from the perspective of y-axis.

$$\text{Area } R_1 = \int_0^1 3-(x^2+2)\ dx = \int_0^1 1-x^2\ dx =$$
$$= \left[x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}$$
$$\text{Area } R_2 = \int_1^3 x\ dy =$$
$$\boxed{y = \sqrt{x}+1 \ \Rightarrow\ x = (y-1)^2}$$
$$= \int_1^3 (y-1)^2\ dy =$$
$$= \left[\frac{1}{3}(y-1)^3\right]_1^3 = \frac{8}{3}$$

So the area of the region considered is $R_1 - R_1 = \frac{8}{3} - \frac{2}{3} = 2$.

Questions 24.2.

Q5. (non-calculator) Find the total area of the region enclosed by the curve $y = 3 - x^2$ and the line $y = x - 3$.

Q6. (non-calculator) Find the total area of the two regions enclosed by the curve $y = \sqrt{x+3} - 2$ and the lines $x = -3$ and $x = 6$ and the x-axis.

Q7. (non-calculator) Find the total area of the two regions enclosed by the curve $y = \sqrt{x+3} - 2$ and the lines $y = -2$ and $y = 1$ and the y-axis.

Q8. (calculator) Calculate the area enclosed by the curves $y = \ln x$ and $y = e^x - e$, $x > 0$.

Q9. (non-calculator) The curve is defined with the equation $f : x \mapsto -0.5x^2 + 2x + 1.5$.

(i) Write down $f'(x)$.
(ii) Write down $f'(0)$.

N is a normal to the graph of f at its y-intercept.

(iii) Write down the gradient of the normal.
(iv) Find the equation of the normal.
(v) Find the coordinates of the other point where the normal meets the curve.

R is the region bounded by the graph of f and the normal N.

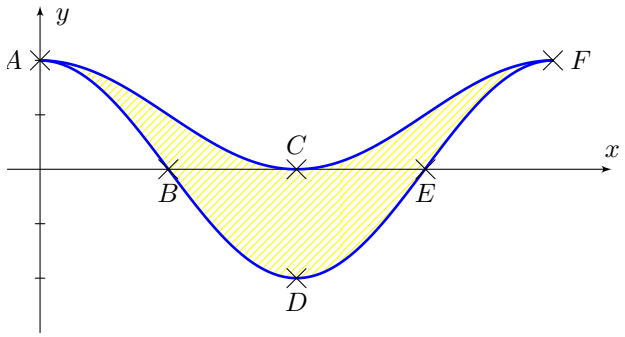
(vi) Give the expression describing the area of R. Simplify the expression.

(vii) Find the area of R.

Q10. (calculator) Find the area bounded by the graphs of $y = e^{2x}$ and $y = -x^2 + 2$.

Q11. (non-calculator) Find the area bounded by the graphs of $y = \frac{1}{1+x^2} - 1$ and $y = 0.5x^2 - 1$.

Q12. (non-calculator) The diagram below shows the region bounded by the graphs of the functions $y = \cos 2x$ and $y = \cos^2 x$.



(i) Identify, which curve is the graph of $y = \cos 2x$ and which of $y = \cos^2 x$.

(ii) Write down the coordinates of the points A, B, C, D, E and F.

(iii) Write down the expression representing the area of the shaded region.

(iv) Find the area.

Q13. (calculator) Let $f(x) = \sec^2 x - 2$, $-k < x < k$, where $k > 0$.

(i) State the highest possible value of k for which $f(x)$ is defined.

(ii) Sketch the graph of f showing clearly x- and y-intercepts.

A normal to the graph of f at one of its x-intercepts has a positive gradient.

(iii) Find the equation of the normal.

The normal intersects the graph of f once more.

(iv) Find the coordinates of the point of intersection.

(v) Write down the expression representing the area bounded by the graph of f and the normal.

(vi) Find the area giving your answer correct to 4 d.p.

Q14. (calculator) Find the area bounded by the graphs of $y = 1 - e^{0.5x+1}$ and $y = \ln(\frac{1}{x+3})$.

Q15. (non-calculator) Two curves are defined as graphs of the functions:

$$f(x) = 2ex^2 + 3ex + e, -1.1 \leq x \leq 0.1 \text{ and } g(x) = xe^{x^2} + e, -1.1 \leq x \leq 0.1.$$

(i) Show that point $P = (-1, 0)$ is a point of intersection of the curves.

(ii) Find $g'(x)$ and show that g is increasing over the whole domain.

(iii) Find $g''(x)$ and the coordinates of Q, the inflection point.

(iv) Show that Q is the other point of intersection of the curves.

(v) Knowing that P and Q are the only points of intersection of f and g sketch the curves in the same set of axes.

(vi) Find $\frac{d}{dx}e^{x^2}$ and hence find $\int g(x)dx$.

(vii) Find the area of the region enclosed completely by the curves.

Q16. (calculator) The region R is bounded by the curve $y = x^3 + 2x^2 - x - 2$ and the normal to the curve at the point where $x = 0$.

(i) Find the equation of the normal line.

(ii) Find the other two points of intersection.

(iii) Write down the expression representing the area of R.

(iv) Find the area of R.

Q17. (non-calculator) Points P and Q lie on the graph of $y = \frac{1}{x}$. The x-coordinate of Q is one unit larger than the x-coordinate of P and both are positive.
The region R_1 is bounded by the curve, the x-axis and the vertical lines passing through P and Q.
The region R_2 is bounded by the curve, the y-axis and the horizontal lines passing through P and Q.
Show that the areas of both regions are equal.

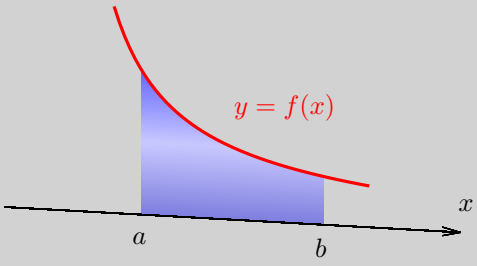
Q18. (non-calculator)

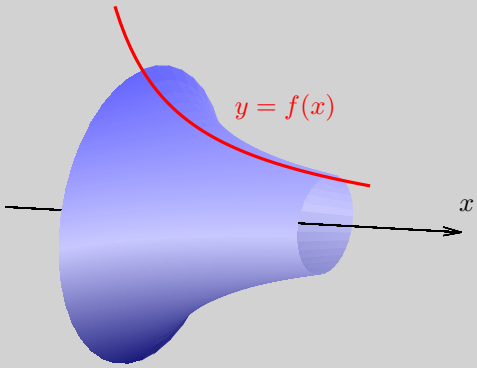
(a) Points P and Q lie on the graph of $y = x^2$. The x-coordinate of Q is one unit larger than the x-coordinate of P and both are positive.
The region R_1 is bounded by the curve, the x-axis and the vertical lines passing through P and Q.
The region R_2 is bounded by the curve, the y-axis and the horizontal lines passing through P and Q.
Find the ratio of the area of R_2 to the area of R_1.

(b) Points P and Q lie on the graph of $y = x^n, n \in \mathbb{Z}^+$. The x-coordinate of Q is one unit larger than the x-coordinate of P.
The region R_1 is bounded by the curve, the x-axis and the vertical lines passing through P and Q.
The region R_2 is bounded by the curve, the y-axis and the horizontal lines passing through P and Q.
Find the ratio of the area of R_2 to the area of R_1 in terms of n.

24.3 Volume of revolution

Theorem 179 (volume of revolution about x-axis).
Let $y = f(x)$ be an integrable function (i.e. one whose integral exists) defined on interval $a \le x \le b$.
When the region(-s) bounded by the graph of f, the x-axis and the lines $x = a$ and $x = b$ is rotated through 360° (2π) about the x-axis, a solid of revolution if formed.





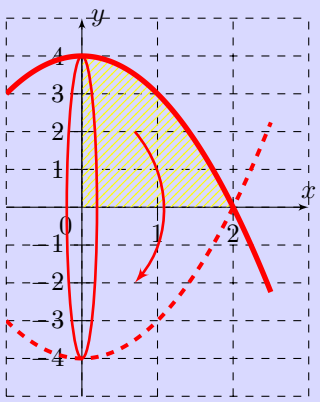
The volume of the solid is given by

$$V = \pi \int_a^b (f(x))^2 dx = \pi \int_a^b y^2 dx$$

Example 180 (volume of revolution about the y-axis).
The region bouded by the curve $y = 4 - x^2$ and the coordinate axes for $x \ge 0$ is rotated through 2π about the x-axis.
Find the volume of the solid of revolution formed.

The volume is given by

$$\pi \int_0^2 (4-x^2)^2 \, dx =$$
$$= \pi \int_0^2 16 - 8x^2 + x^4 \, dx =$$
$$= \pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2 =$$
$$= \pi \left(16(2) - \frac{8}{3}(2^3) + \frac{1}{5}(2^5) \right) - \pi(0) = \frac{256}{15}\pi$$



Questions 24.3.

Q19. The region bounded by the lines $x - 2y - 2 = 0$, $x = 4$, $x = 8$ and the x-axis is rotated through 2π about the x-axis. Find the volume of the solid formed.

Q20. The region bounded by the graph of $y = 1 - x^2$ and the x-axis is rotated through 2π about the x-axis. Find the volume of the solid formed.

Q21. Find the volume of the solid generated when the graph of $y = \sec x$ for $0 \leq x \leq \frac{\pi}{4}$ is rotated about the x-axis.

Q22. (i) Use appropriate formula for $\cos 2x$ to express $\sin^2 x$ in terms of $\cos 2x$.

(ii) Hence find an indefinite integral $\int \sin^2 x dx$.

(iii) The region formed by the graph of $y = \sin x$ and the x-axis between $x = 0$ and $x = \pi$ is rotated through 2π about the x-axis. Find the volume of the solid formed.

Q23. N is the normal to the graph of the function $f(x) = -e^{2x} + 5e^x - 6$ at the smallest positive x-intercept of the graph of f.

(i) Find the equation of N.

The region R is bounded by the graphs of f and N.

(ii) Find the area of R.

(iii) Find the volume generated when R is rotated through $360°$ about the x-axis.

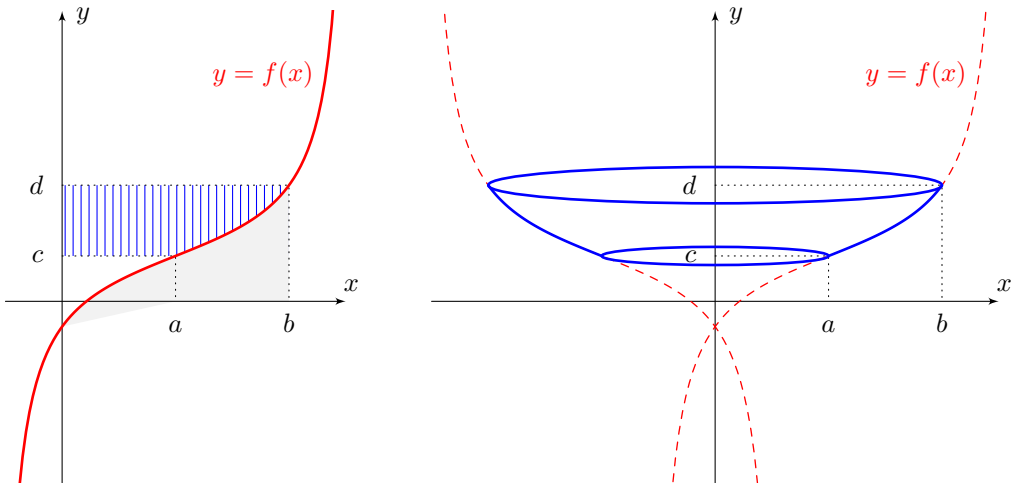
Q24. The graph of the function $y = \cos x$, $0 \leq x \leq \frac{\pi}{2}$ forms a region bounded by both x and y-axis.

(i) Find the area of the region.

(ii) Find the volume of the solid formed when the region is rotated through 2π about the x-axis.

Corollary 181 (to theorem 179).
Let $y = f(x)$ be an integrable function (i.e. one whose integral exists) defined on interval $a \leq x \leq b$.
When the region(-s) bounded by the graph of f, the y-axis and the lines $y = f(a)$ and $y = f(b)$ is rotated through $360°$ (2π) about the y-axis, a solid of revolution if formed.
The volume of the solid is given by

$$V = \pi \int_c^d (f^{-1}(y))^2 dy = \pi \int_c^d x^2 dy$$



EXAMPLE 182 (volume of revolution about the y-axis).
The region bouded by the curve $y = 4 - x^2$ and the coordinate axes for $x \geq 0$ is rotated through 2π about the y-axis.
Find the volume of the solid of revolution formed.

The volume is given by

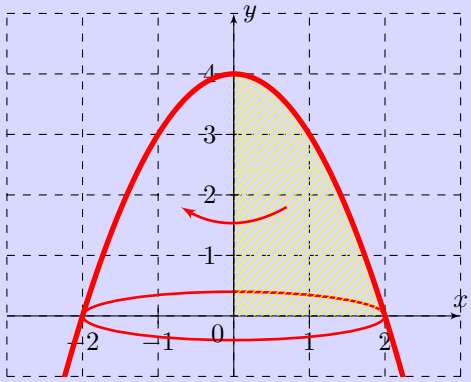
$$\pi \int_0^4 x^2 \, dy =$$

$$\boxed{y = 4 - x^2 \;\Rightarrow\; x^2 = 4 - y}$$

$$= \pi \int_0^4 4 - y \, dy =$$

$$= \pi \left[4y - \frac{1}{2} y^2 \right]_0^4 =$$

$$= \pi \left(4(4) - \frac{1}{2}(4^2) \right) - \pi(0) = 8\pi$$



QUESTIONS 24.4.

Q25. Region R is bounded by the graphs of $y = x$, $y = \sqrt{x - 1}$, the x-axis and the line $x = 5$. Find:

(i) the area of R,

(ii) the volume of the solid formed when R is rotated through 2π about the x-axis,

(iii) the volume of the solid formed when R is rotated through 2π about the y-axis.

Q26. (i) Find the area of the region bounded by the graph of $y = -2x^2 + 8x - 6$, its tangent at $x = 1.5$ and the x-axis.

(ii) Find the volume of the solid formed when the region is rotated through $360°$ about the x-axis.

(iii) Find the volume of the solid formed when the region is rotated through $360°$ about the y-axis.

Q27. The graph of the function $y = e^x - 1$, forms a region bounded by the y-axis and the line $y = 1$.

(i) Find the area of the region.

(ii) Find the volume of the solid formed when the region is rotated through 2π about the x-axis.

(iii) Find the volume of the solid formed when the region is rotated through 2π about the y-axis

Q28. Consider the region defined by the following inequalities:

$$y \geq \sin x, \quad y \leq 1, \quad 0 \leq x \leq \frac{\pi}{2}$$

The region is rotated through 2π about the y-axis. Find the volume of the solid formed.

24.4 Kinematics

We learned before that if s is the displacement covered by the particle in time t then:

- the rate of change of the displacement is the velocity ($v = \frac{ds}{dt}$),
- the rate of change of the velocity is the acceleration ($a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$)

In consequence we should be able to obtain:

- velocity as the integral of acceleration ($v = \int a\ dt$),
- displacement as the integral of velocity ($s = \int v\ dt$).

Let us consider an example.

Example 183 (integration in kinematics).
The velocity in ms^{-1} of a particle moving in a straight line is given by $v = \frac{3}{4}t^2 - 3$, where $0 \leq t \leq 4$ is time in seconds. Assuming that the initial displacement of the particle is 0, find:

1. *the acceleration of the particle after 2 seconds,*
2. *the time when the particle is at rest,*
3. *the displacement of the particle after 1 second, 2 seconds and 4 seconds,*
4. *the total distance travelled by the particle.*

1. $a = \frac{dv}{dt} = \frac{3}{2}t$, so at $t = 2$ the acceleration is $a = 3$.

2. the particle is at rest when $v = 0$;
$\frac{3}{4}t^2 - 3 = 0$
so $t = 2$

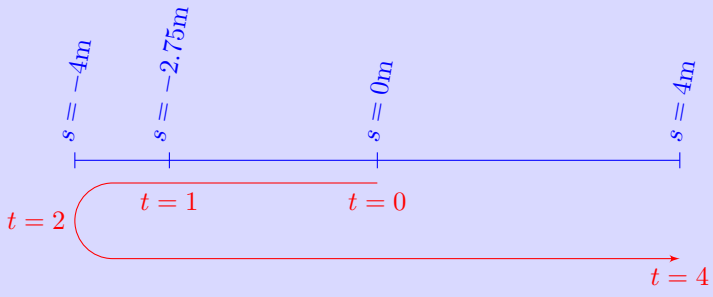
3.

$$s = \int v\ dt = \int \frac{3}{4}t^2 - 3\ dt$$
$$= \frac{1}{4}t^3 - 3t + c$$

As $s(0) = 0$, $c = 0$, so $s(t) = \frac{1}{4}t^3 - 3t$.
Hence:

- $s(1) = -2.75$. The displacement after 1 second is 2.75 metres.
- $s(2) = -4$. The displacement after 2 seconds is 4 metres.
- $s(4) = 4$. The displacement after 4 seconds is 4 metres.

It should be noted that $s(1) < 0$ and $s(2) < 0$ while $s(4) > 0$. It means that the particle changed the direction of movement (at time $t = 2$, when it was at rest). The positions of the particle at $t = 2$ and at $t = 4$ are 8 metres apart.
The diagram below shows the **displacement** of the particle as it changes in **time**.



The distances above can be evaluated as definite integrals of velocity:

- $s(1) = \int_0^1 v(t)\ dt = -2.75$
- $s(2) = \int_0^2 v(t)\ dt = -4$
- $s(4) = \int_0^4 v(t)\ dt = 4$

4. The total distance travelled by the particle can be easily seen in the diagram above. In the first two seconds it travelled 4 metres, then changed the direction of motion, went back to its initial position and travelled another 4 metres, which gives a total of 12 metres.
The total distance travelled can be evaluated as the definite integral of the abcolute value of the velocity:

$$\int_0^4 |v(t)|\ dt$$

As $v < 0$ for $0 \leq t < 2$ and $v > 0$ for $2 < t \leq 4$, the integral can be evaluated as follows:

$$\begin{aligned}\int_0^4 |v(t)|\ dt &= \int_0^2 (-v(t))\ dt + \int_2^4 v(t)\ dt = \\ &= \int_0^2 3 - \frac{3}{4}t^2\ dt + \int_2^4 \frac{3}{4}t^2 - 3\ dt = \\ &= \left[3t - \frac{1}{4}t^3\right]_0^2 + \left[\frac{1}{4}t^3 - 3t\right]_2^4 = \\ &= [(6-2)-0] + [(16-12)-(2-6)] = 4+8 = 12\end{aligned}$$

Note that 4 in the sum "$4 + 8$" above stands for the displacement in the first 2 seconds, while 8 - in the last 2.

Questions 24.5.

Q29. A car started moving from rest towards point A on the highway located $200km$ away. It moved with the constant acceleration of $20kmh^{-2}$.

(i) When did the car arrive at point A?
(ii) What was its velocity at that time?
(iii) When was the car $200km$ from A again?

Q30. A car was moving with the constant speed of $60kmh^{-1}$ towards point A on the highway. When the car was $200km$ away from the point it started accelerating with the constant acceleration of $15kmh^{-2}$.

(i) When did the car arrive at point A?
(ii) What was its velocity at that time?
(iii) When was the car $200km$ from A again?

Q31. A car was moving with the constant speed of $50kmh^{-1}$ towards point A on the highway. When the car was $200km$ away from the point it started accelerating with the constant acceleration of $(10 + 2t)kmh^{-2}$.

(i) When did the car arrive at point A?
(ii) What was its velocity at that time?
(iii) When was the car $200km$ from A again?

Q32. A car was moving with the constant speed of $50kmh^{-1}$ towards point A on the highway. When the car was $200km$ away from the point it started accelerating with the constant acceleration of $(10 - 4t)kmh^{-2}$.

(i) When did the car arrive at point A?
(ii) What was its velocity at that time?
(iii) When was the car $200km$ from A again?

Q33. Velocity in ms^{-1} of the particle moving in a straight line is given by $v = 4\sin(0.01\pi t)$, where $0 \leq t \leq 300$ is measured in seconds.

(i) What is the acceleration of the particle at $t = 100$ seconds?

(ii) What is the displacement of the particle in the whole motion observed?

(iii) What is the total distance covered by the particle?

Q34. The acceleration in kmh^{-2} of a particle moving in a straight line is given by the formula $a(t) = -1.5t + 3$ where t is time in hours, $0 \leq t \leq 10$.
The initial velocity of the particle was 9 kmh^{-1} while the initial displacement (from the observer) was 10 metres.

(i) When did the particle begin to decelerate?

(ii) When did the particle change the direction of motion?

(iii) What was the largest velocity of the particle when it was moving in the same direction as initially?

(iv) What was the largest velocity of the particle when it was moving in the opposite direction than initially?

(v) What was the largest displacement of the particle from the observer?

(vi) What was the final displacement of the particle from the observer?

(vii) What was the total distance covered by the particle?

Chapter 25

Further integration

25.1 Integration by parts

Task
part 1

Differentiate...	...and integrate.
(1) $\sin 2x$	$\cos 2x$
(2) $\cos 3x$	$\sin 3x$
(3) e^{-2x}	e^{-2x}

part 2

(1) Differentiate $x\cos x$ and **hence** integrate $x\sin x$.

(2) Differentiate $x\sin x$ and **hence** integrate $x\cos x$.

(3) Differentiate xe^{2x} and **hence** integrate xe^{2x}.

Introducing integration by parts

Rearranging the formula for a derivative of a product of two functions (i.e. the **product rule**) we get the following:

$$\begin{aligned} & (uv)' = u'v + uv' \\ \Leftrightarrow \quad & u'v = (uv)' - uv' \\ \Leftrightarrow \quad & \int u'v = \int (uv)' - \int uv' \\ \Leftrightarrow \quad & \int u'v = uv - \int uv' \end{aligned}$$

which is known as the **integration by parts** formula:

Theorem 184 (integration by parts).
For real functions u and v whose derivative exist:

1. $\int u'v = uv - \int uv'$

2. *or definitely:* $\int_a^b u'v = [uv]_a^b - \int_a^b uv'$

EXAMPLE 185 (integration by parts).
We are going to apply integration ***by parts*** *to evaluate* $\int x\cos 3x dx$.

FIRST ATTEMPT $\begin{bmatrix} u' = x & v = \cos 3x \\ u = \frac{x^2}{2} & v' = -3\sin 3x \end{bmatrix}$

Thus we get $\int uv' = \int -\frac{3}{2}x^2 \sin 3x dx$, which is even more complicated.

SECOND ATTEMPT $\begin{bmatrix} u' = \cos 3x & v = x \\ u = \frac{1}{3}\sin 3x & v' = 1 \end{bmatrix}$

$$\begin{aligned} &\int x\cos 3x dx = \\ =&\frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin 3x dx = \\ =&\frac{1}{3}x\sin 3x - \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right) + C = \\ =&\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C \end{aligned}$$

Similarly:
$\int x^2 e^x dx =$ $\begin{bmatrix} u' = e^x & v = x^2 \\ u = e^x & v' = 2x \end{bmatrix}$

$$= x^2e^x - 2\int xe^x dx =$$

$$\begin{bmatrix} u' = e^x & v = x \\ u = e^x & v' = 1 \end{bmatrix} \begin{aligned} =&x^2e^x - 2(xe^x - \int e^x dx) = \\ =&x^2e^x - 2(xe^x - e^x) + C = \\ =&e^x(x^2 - 2x + 2) + C \end{aligned}$$

QUESTIONS 25.1.

Q1. Integrate by parts.

(1) $\int x\cos x dx$
(2) $\int x\sin x dx$
(3) $\int xe^x dx$
(4) $\int x\sin 2x dx$
(5) $\int 3xe^{2x-1} dx$
(6) $\int x^2\cos 2x dx$
(7) $\int (x+2)e^{-x} dx$
(8) $\int (x^2 - x)\sin x dx$
(9) $\int (x^3 - 2x + 5)\sin x dx$
(10) $\int (x^2 - x + 3)\cos 2x dx$

FACTS, REMARKS, OBSERVATIONS - INTEGRATION BY PARTS - STANDARD APPROACH.
Integration by parts ($\int u'v = uv - \int uv'$) *is usually applied is when:*

- u *is a polynomial,*
- v' *is a sine, cosine or exponential function.*

Exceptions to standard approach

QUESTIONS 25.2.

Q2. Complete the following.

(1) $\int \ln x dx = \int (1)(\ln x)$ $\begin{bmatrix} u' = 1 & v = \ln x \\ u = x & v' = \frac{1}{x} \end{bmatrix}$

(2) $\int \frac{\ln x}{\sqrt{x}} dx = \int (\ln x)(x^{-\frac{1}{2}}) dx$ $\begin{bmatrix} u' = x^{-\frac{1}{2}} & v = \ln x \\ u = 2x^{\frac{1}{2}} & v' = \frac{1}{x} = x^{-1} \end{bmatrix}$

Q3. Integrate.

(1) $\int x \ln x dx$
(2) $\int x^2 \ln x dx$
(3) $\int x \arctan x dx$ [1]
(4) $\int x^3 \ln 2x dx$
(5) $\int \ln(x + x^2) dx$
(6) $\int \cos x \ln \sin x dx$

Repeated integration by parts

EXAMPLE 186 (repeated integration by parts).
Evaluate $\int e^x \sin 2x dx$.

$$I = \int e^x \sin 2x dx = \qquad \begin{bmatrix} u' = e^x & v = \sin 2x \\ u = e^x & v' = 2\cos 2x \end{bmatrix}$$

$$= e^x \sin 2x - 2\int e^x \cos 2x dx = \qquad \begin{bmatrix} u' = e^x & v = \cos 2x \\ u = e^x & v' = -2\sin 2x \end{bmatrix}$$

$$= e^x \sin 2x - 2(e^x \cos 2x + 2\int e^x \sin 2x dx) = \quad \textbf{!!!}$$

$$I = e^x \sin 2x - 2e^x \cos 2x - 4I$$

$$I = \tfrac{1}{5} e^x (\sin 2x - 2\cos 2x) + C$$

EXAMPLE 187 ($\int \sec^3 x\ dx$).
Use the fact that $\int \sec\theta\ d\theta = \ln(\sec\theta + \tan\theta) + c$ *(which will be shown later in example 196) to evaluate* $\int \sec^3\theta\ d\theta$.

$$\int \sec^3\theta\ d\theta = \int \sec^2\theta \sec\theta\ d\theta =$$

$$= \begin{bmatrix} u' = \sec^2\theta & v = \sec\theta \\ u = \tan\theta & v' = \sec\theta\tan\theta \end{bmatrix} =$$

$$= \tan\theta\sec\theta - \int \sec\theta \tan^2\theta\ d\theta =$$

$$= \tan\theta\sec\theta - \int \sec\theta\ (\sec^2\theta - 1) d\theta =$$

$$= \tan\theta\sec\theta - \int \sec^3\theta\ d\theta + \int \sec\theta\ d\theta =$$

$$= [\text{by example } 196] =$$

$$= \tan\theta\sec\theta - \int \sec^3\theta\ d\theta + \ln(\sec\theta + \tan\theta)$$

Finally:

$$\int \sec^3\theta\ d\theta = \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln(\sec\theta + \tan\theta) + c$$

QUESTIONS 25.3.

Q4. Integrate.

(1) $\int e^x \cos x\ dx$
(2) $\int e^{2x} \sin x\ dx$
(3) $\int e^{-x} \cos 3x\ dx$
(4) $\int \sin x \cos 2x\ dx$
(5) $\int \sin 2x e^{3x}\ dx$
(6) $\int \sin(\ln x) dx$
(7) $\int \sin x \ln(\cos x)\ dx$
(8) $\int \cos 2x \ln(\sin x) dx$

Q5. Evaluate. Give exact answers.

(1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin 2x\ dx$
(2) $\int_0^{\frac{\pi}{6}} x^2 \sin x dx$
(3) $\int_1^2 x \ln x dx$
(4) $\int_0^{\frac{\pi}{2}} \sin^2 2t \cos t dt$

[1] hint: $\frac{x^2}{1+x^2} = \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}$

25.2 Integration by substitution

Task

differentiate	integrate
$\frac{d}{dx}\ln(x^2+1)$	$\int \frac{2x}{x^2+1}dx$
$\frac{d}{dx}\ln(x^3-2)$	$\int \frac{3x^2}{x^3-2}dx$
$\frac{d}{dx}\ln(4x^3-x)$	$\int \frac{12x^2-1}{4x^3-x}dx$
$\frac{d}{dx}\ln(\sin x)$	$\int \frac{\cos x}{\sin x}dx$
	$\int \tan x dx$
$\frac{d}{dx}\ln f(x)$	$\int \frac{f'(x)}{f(x)}dx$

Corollary 188 (integral of a derivative over a function).
$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$

Example 189 (integral of tangent).
Evaluate $\int \tan x\,dx$.

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}dx = \\ &= -\int \frac{-\sin x}{\cos x}dx = \\ &= -\int \frac{\frac{d}{dx}\cos x}{\cos x}dx = -\ln\cos x + C = \ln\sec x + c\end{aligned}$$

Task

differentiate	integrate
$\frac{d}{dx}(x^2+3)^5$	$\int 2x(x^2+3)^4dx$
$\frac{d}{dx}(x^3+3x)^6$	$\int (x^2+1)(x^3+3x)^5dx$
$\frac{d}{dx}\sin^5 x$	$\int \cos x\sin^4 x dx$
	$\int \cos^3 x dx$
	$\int \sin^3 x dx$
	$\int \sin^4 x dx$

Theorem 190 (integration by substitution).
Let u and f be real functions defined such that the composition $f\circ u$ exists, $\frac{du}{dx}$ exists and $\int f(t)\,dt$ exists. Then

$$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$$

Example 191 (integration by substitution).
Show that by the substition $u = x^2+3$ an integral

$$\int \frac{x}{(x^2+3)^3}\,dx$$

can be transformed into $\frac{1}{2}\int u^{-3}\,du$. Hence evaluate the initial integral.

For $u = x^2+3$ the derivative $\frac{du}{dx} = 2x$.
Hence the integral considered can be expressed as:

$$\int \frac{x}{(x^2+3)^3}\,dx = \frac{1}{2}\int (x^2+3)^{-3}\,2x\,dx = \frac{1}{2}\int u^{-3}\,\frac{du}{dx}\,dx = \frac{1}{2}\int u^{-3}\,du$$

Then:

$$\frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\left(\frac{1}{-2}\right)u^{-2}+c = -\frac{1}{4}u^{-2}+c$$

Hence

$$\int \frac{x}{(x^2+3)^3}\,dx = \text{ (by substitution) } = -\frac{1}{4}u^{-2}+c = \text{ (again in terms of } x\text{) } -\frac{1}{4}(x^2+3)^{-2}+c$$

EXAMPLE 192 (integral of cotangent).
Evaluate $\int \cot x\,dx$.

$$\begin{aligned}\int \cot x\,dx &= \int \frac{\cos x}{\sin x}\,dx = \\ &= \left[\begin{matrix} u = \sin x \\ du = \cos x\,dx \end{matrix}\right] = \\ &= \int \frac{1}{u}\,du = \ln u + c = \ln(\sin x) + c\end{aligned}$$

QUESTIONS 25.4.

Q6. Integrate.

(1) $\int \frac{3x^2}{x^3+2}dx$
(2) $\int 3x(x^2+1)^5 dx$
(3) $\int \frac{1}{2x-3}\sqrt{\ln(2x-3)}dx$
(4) $\int \frac{2x-3}{x^2-3x+2}\sqrt{\ln(x^2-3x+2)}dx$
(5) $\int \sqrt{2x+1}dx$
(6) $\int x\sqrt{2x+1}dx$
(7) $\int 2x\sqrt{1-x}dx$
(8) $\int \cos\frac{1}{2}x\sqrt{3+\sin\frac{1}{2}x}dx$
(9) $\int \tan^2 3x\sec^2 3x dx$
(10) $\int 2xe^{x^2+1}dx$
(11) $\int 2x\sin 3x^2 dx$
(12) $\int \frac{\sin\ln(x+1)}{x+1}dx$
(13) $\int \frac{\sin 3x}{2-\cos 3x}dx$
(14) $\int \frac{x^2}{\sqrt{x^3+2}}dx$

Q7. Integrate.

(1) $\int \cos x\sin^3 x dx$
(2) $\int \tan x dx$
(3) $\int \tan x\sec^2 x dx$
(4) $\int \frac{\tan^3 x}{\cos^2 x}dx$
(5) $\int \sin x\cos^4 x dx$
(6) $\int \sin^3 x dx$
(7) $\int \cos^3 x dx$
(8) $\int \cos^5 x\,dx$
(9) $\int \tan^3 x\sec^4 x\,dx$
(10) $\int \tan^3 x\sec^3 x\,dx$
(11) $\int e^x e^{e^x}dx$
(12) $\int \frac{\ln^2 x}{x}dx$
(13) $\int (\sin x)(\sin x\cos x)dx$
(14) $\int (\sin x)(\sin\cos x)dx$
(15) $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx$
(16) $\int \frac{\cos\frac{1}{x}}{x^2}dx$
(17) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
(18) $\int \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$

Q8. Evaluate the following integrals:

(1) $\int \tan^2 x dx$
(2) $\int \tan x dx$
(3) $\int \tan^3 x dx$
(4) $\int \frac{\tan^3\ln x}{x}\,dx$
(5) $\int \tan^4 x + \tan^2 x\,dx$
(6) $\int \tan^4 x - \tan^2 x\,dx$
(7) $\int \cot^2 2x dx$

Definite integration by substitution

EXAMPLE 193 (definite integration by substitution).
Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\sin^2 x dx$.

METHOD 1.
Integrate **indefinitely**:

$$\int \cos x \sin^2 x dx = \left[\begin{array}{c} u = \sin x \\ du = \cos x dx \end{array}\right] =$$
$$= \int u^2 du = \frac{1}{3}u^3 + C =$$
$$= \frac{1}{3}\sin^3 x + C$$

and then find the definite value:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \sin^2 x dx = \left[\frac{1}{3}\sin^3 x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$
$$= \frac{1}{3} \times 1^3 - \frac{1}{3} \times \left(\frac{\sqrt{2}}{2}\right)^3 =$$
$$= \frac{1}{12}(4 - \sqrt{2})$$

METHOD 2.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \sin^2 x dx = \left[\begin{array}{c} u = \sin x \\ du = \cos x dx \\ x = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2} \\ x = \frac{\pi}{2} \Rightarrow u = 1 \end{array}\right] =$$
$$= \int_{\frac{\sqrt{2}}{2}}^{1} u^2 du = \left[\frac{1}{3}u^3\right]_{\frac{\sqrt{2}}{2}}^{1} =$$
$$= \frac{1}{3} \times 1^3 - \frac{1}{3} \times \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{1}{12}(4 - \sqrt{2})$$

Questions 25.5.

Q9. Evaluate.

(1) $\int_0^4 \frac{x dx}{\sqrt{x^2+9}}$

(2) $\int_{-\frac{\pi}{4}}^{0} \tan x dx$

(3) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x dx$

(4) $\int_1^e \frac{\ln x}{x} dx$

(5) $\int_1^2 x(3x^2 - 2)^4 dx$

(6) $\int_1^2 x \sin(\frac{\pi x^2}{2}) dx$

(7) $\int_0^1 x^2 e^{x^3 - 1} dx$

(8) $\int_{\frac{3\pi}{4}}^{\frac{\pi}{2}} \frac{\cot^2 x}{\sin^2 x} dx$

(9) $\int_{0.5}^1 \frac{\arcsin x}{\sqrt{1-x^2}} dx$

(10) $\int_1^3 \frac{3}{(2x-4)^2+4} dx$

(11) $\int_0^{\frac{\pi}{4}} x \tan^2 x dx$

(12) $\int_0^1 \arctan x dx$

(13) $\int_{-\frac{\pi}{6}}^0 x \sec^2 \ 2x dx$

(14) $\int_1^{\sqrt{3}} \frac{\sqrt{\arctan x}}{1+x^2} dx$

Trigonometric substitutions

Facts, remarks, observations - trigonometric substitutions.
When and integral to be evaluated involves the expression $(a^2 - x^2)$ or $(a^2 + x^2)$ or $(x^2 - a^2)$ the following substitutions can be used simplify the integrand:

expression	*substitution*
$a^2 - x^2$	$x = a \sin \theta$
$a^2 + x^2$	$x = a \tan \theta$
$x^2 - a^2$	$x = a \sec \theta$

EXAMPLE 194 (integration by substitution).
Use the substitution $x = 2\cos\theta$ *to evaluate* $\int_1^{\sqrt{3}} \sqrt{4-x^2}dx$.

Let us see that

$$\int_1^{\sqrt{3}} \sqrt{4-x^2}dx = \int_{x=1}^{x=\sqrt{3}} (\sqrt{4-x^2})dx$$

For substitution $x = 2\cos\theta$ we have to consider:

1. function integrated
2. dx - variable of integration
3. limits of integration

In all of the above x must be changed to θ.

Ad. 1. $\sqrt{4-x^2} = \sqrt{4-4\cos^2\theta} = \sqrt{4(1-\cos^2\theta)} = 2\sqrt{1-\cos^2\theta}$

Ad. 2. $x = 2\cos\theta$
$\frac{dx}{d\theta} = -2\sin\theta$
$dx = (-2\sin\theta)d\theta$

Ad. 3. $x = 2\cos\theta$

When $x = 1$,
$1 = 2\cos\theta$,
$\cos\theta = \frac{1}{2}$,
$\theta = \frac{\pi}{3}$,

when $x = \sqrt{3}$,
$\sqrt{3} = 2\cos\theta$,
$\cos\theta = \frac{\sqrt{3}}{2}$,
$\theta = \frac{\pi}{6}$

$$\therefore \int_{x=1}^{x=\sqrt{3}} (\sqrt{4-x^2})dx =$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} (2\sqrt{1-\cos^2\theta})(-2\sin\theta)d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} (2\sin\theta)(-2\sin\theta)d\theta =$$

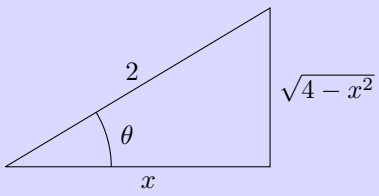
$$= -4\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} (\sin^2\theta)d\theta = \begin{bmatrix} \cos 2\theta = 1-2\sin^2\theta \\ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \end{bmatrix}$$

$$= -2\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} (1-\cos 2\theta)d\theta = -2\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{6}} =$$

$$= -2\left((\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}) - (\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3})\right) = \frac{\pi}{3}$$

EXAMPLE 195 (integration by substitution).
Use the substitution $x = 2\cos\theta$ *to evaluate* $\int \sqrt{4-x^2}\ dx$.

As we got in the previous example: $\int \sqrt{4-x^2}\ dx = \sin 2\theta - 2\theta + c$ where $x = 2\cos\theta$.
Then θ is an angle such that $\cos\theta = \frac{x}{2}$ which can be shown in the following triangle:



Thus:

$$\begin{cases} \cos\theta = \frac{x}{2} \\ \sin\theta = \frac{\sqrt{4-x^2}}{2} \\ \theta = \arccos\left(\frac{x}{2}\right) \end{cases}$$

and:

$$\begin{aligned} \int \sqrt{4-x^2}\ dx &= \sin 2\theta - 2\theta + c = \\ &= 2\left(\sin\theta\cos\theta - \theta\right) + c = \\ &= 2\left(\frac{\sqrt{4-x^2}}{2}x - \arccos\left(\frac{x}{2}\right)\right) + c = \\ &= x\sqrt{4-x^2} - 2\arccos\left(\frac{x}{2}\right) + c \end{aligned}$$

EXAMPLE 196 ($\int \sec x\ dx$).
Use the fact that $1 = \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$ *to evaluate* $\int \sec\theta\ d\theta$.

$$\begin{aligned} \int \sec\theta\ d\theta &= \int \sec\theta \times 1\ d\theta = \\ &= \int \sec\theta \times \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\ d\theta = \\ &= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\ d\theta = \\ &= \begin{bmatrix} u = \sec\theta + \tan\theta \\ du = (\sec^2\theta + \sec\theta\tan\theta)d\theta \end{bmatrix} = \\ &= \int \frac{du}{u} = \ln u + c = \ln(\sec\theta+\tan\theta) + c \end{aligned}$$

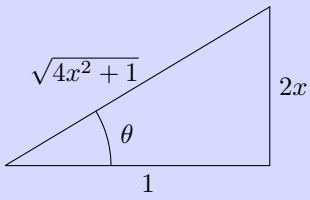
EXAMPLE 197 (integration by substitution).
Use the substitution $x = \frac{1}{2}\tan\theta$ *to evaluate* $\int \sqrt{4x^2+1}\ dx$.

$$\begin{aligned} \int \sqrt{4x^2+1}\ dx &= \begin{bmatrix} x = \frac{1}{2}\tan\theta \\ dx = \frac{1}{2}\sec^2\theta d\theta \end{bmatrix} = \\ &= \int \sqrt{4\left(\frac{1}{4}\tan^2\theta\right)+1} \times \frac{1}{2}\sec^2\theta\ d\theta = \\ &= \frac{1}{2}\int \sqrt{\tan^2\theta+1}\sec^2\theta\ d\theta = \\ &= \frac{1}{2}\int \sqrt{\sec^2\theta}\sec^2\theta\ d\theta = \\ &= \frac{1}{2}\int \sec\theta\sec^2\theta\ d\theta = \\ &= \frac{1}{2}\int \sec^3\theta\ d\theta = \\ &= [\text{by example 187}] = \\ &= \frac{1}{4}\tan\theta\sec\theta + \frac{1}{4}\ln(\sec\theta+\tan\theta) + c \end{aligned}$$

So:

$$\int \sqrt{4x^2+1}\ dx = \frac{1}{4}\tan\theta\sec\theta + \frac{1}{4}\ln(\sec\theta+\tan\theta) + c$$

where $\tan\theta = 2x$ which can be shown in the following triangle:



Thus:

$$\begin{cases} \tan\theta = 2x \\ \sec\theta = \sqrt{4x^2+1} \end{cases}$$

and

$$\begin{aligned} \int \sqrt{4x^2+1}\ dx &= \frac{1}{4}\tan\theta\sec\theta + \frac{1}{4}\ln(\sec\theta+\tan\theta) + c = \\ &= \frac{1}{4}(2x)\sqrt{4x^2+1} + \frac{1}{4}\ln(\sqrt{4x^2+1}+2x) + c = \\ &= \frac{1}{2}x\sqrt{4x^2+1} + \frac{1}{4}\ln(\sqrt{4x^2+1}+2x) + c \end{aligned}$$

EXAMPLE 198 (integration by substitution).
Use the substitution $x = \sqrt{5}\sec\theta$ to evaluate $\int \sqrt{x^2-5}dx$.

$$\begin{aligned} \int \sqrt{x^2-5}\ dx &= \begin{bmatrix} x = \sqrt{5}\sec\theta \\ dx = \sqrt{5}\sec\theta\tan\theta\ d\theta \end{bmatrix} = \\ &= \int \sqrt{5\sec^2\theta - 5}\sqrt{5}\sec\theta\tan\theta\ d\theta = \\ &= \int \sqrt{5}\sqrt{\sec^2\theta-1}\sqrt{5}\sec\theta\tan\theta\ d\theta = \int 5\tan\theta\sec\theta\tan\theta\ d\theta = \\ &= 5\int \tan^2\theta\sec\theta\ d\theta = 5\int (\sec^2\theta - 1)\sec\theta\ d\theta = \\ &= 5\int \sec^3\theta - \sec\theta\ d\theta = \\ &= [\text{by examples 187 and 196}] = \\ &= \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln(\sec\theta+\tan\theta) + \ln(\sec\theta+\tan\theta) + c = \\ &= \frac{1}{2}\tan\theta\sec\theta + \frac{3}{2}\ln(\sec\theta+\tan\theta) + c \end{aligned}$$

QUESTIONS 25.6.

Q10. Show that $\int_0^1 \frac{1}{(1+x^2)^2}dx = \frac{\pi}{8} + \frac{1}{4}$.

Q11. Evaluate $\int_0^1 \sqrt{1-x^2}dx$.

Q12. Evaluate $\int_{-2}^2 \sqrt{4-x^2}dx$.

Q13. Evaluate $\int_{0.5}^0 \sqrt{1-2x^2}dx$.

Q14. Evaluate $\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}dx$.

Q15. Integrate.

(1) $\int \frac{1}{\sqrt{1-x^2}}$ (2) $\int \frac{x}{\sqrt{1-x^2}}$ (3) $\int \frac{x^2}{\sqrt{1-x^2}}$ (4) $\int \frac{x^3}{\sqrt{1-x^2}}$

Q16. Using an appropriate substitution integrate $\int \sqrt{1-\frac{1}{9}x^2}dx$ and hence find the exact area of an ellipse defined by $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Q17. Evaluate the following integrals.

(1) $\int \sqrt{x^2-9}\,dx$
(2) $\int \sqrt{9-x^2}\,dx$
(3) $\int \sqrt{x^2+9}\,dx$
(4) $\int \frac{1}{\sqrt{x^2+4}}\,dx$
(5) $\int \frac{x}{\sqrt{x^2+4}}\,dx$
(6) $\int \frac{x^2}{\sqrt{x^2+4}}\,dx$
(7) $\int \frac{x}{(x^2+4)^{\frac{3}{2}}}\,dx$
(8) $\int \frac{1}{(x^2+4)^{\frac{3}{2}}}\,dx$
(9) $\int \frac{x}{(4-x^2)^{\frac{3}{2}}}\,dx$

Q18. Integrate.

(1) $\int \sqrt{x^2-1}dx$
(2) $\int \sqrt{x^2+1}dx$
(3) $\int \sqrt{4-x^2}dx$
(4) $\int \sqrt{x^2-4}dx$
(5) $\int \sqrt{x^2+4}dx$
(6) $\int \sqrt{9-4x^2}dx$
(7) $\int \sqrt{4x^2-9}dx$
(8) $\int \sqrt{4x^2+9}dx$
(9) $\int \frac{dx}{\sqrt{1-x^2}}$
(10) $\int \frac{dx}{\sqrt{1+x^2}}$
(11) $\int \frac{dx}{\sqrt{x^2-1}}$
(12) $\int \frac{dx}{\sqrt{4-x^2}}$
(13) $\int \frac{dx}{\sqrt{4+x^2}}$
(14) $\int \frac{dx}{\sqrt{x^2-4}}$
(15) $\int \frac{dx}{\sqrt{1-4x^2}}$
(16) $\int \frac{dx}{\sqrt{1+4x^2}}$
(17) $\int \frac{dx}{\sqrt{4x^2-1}}$

Q19. Evaluate.

(1) $\int \arcsin x dx$
(2) $\int x \arcsin x dx$ [2]
(3) $\int \arctan x dx$
(4) $\int \arccos x dx$

25.3 Integrating rational expressions

In this section we will be dealing with integrating expression in the form $\frac{\text{polynomial}}{\text{quadratic}}$.

Facts, remarks, observations.
Each integral of the form

$$\int \frac{P(x)}{Ax^2+Bx+C}dx,$$

where $\deg P \geq 2$ *and* $A, B, C \in \mathbb{R}$, $A \neq 0$
can be expressed in the form

$$Q(x) + k_1 \ln(Ax^2+Bx+C) + k_2 \int \frac{1}{Ax^2+Bx+C}dx,$$

where $Q(x)$ *is a polynomial such that* $\deg Q = \deg P - 1$ *and* $k_1, k_2 \in \mathbb{R}$.

Example 199 (simplifying an integral with a rational expression).
Simplify $\int \frac{2x^3+3x^2+8x+4}{x^2+2x+3}dx$.

Firstly, the top expression $(2x^3+3x^2+8x+4)$ can be divided by the bottom one (x^2+2x+3).

$$\int \frac{2x^3+3x^2+8x+4}{x^2+2x+3}dx =$$

the outcome is $2x-1$ and the remainder is $4x+7$

$$= \int 2x-1\,dx + \int \frac{4x+7}{x^2+2x+3}dx =$$

note that $\frac{d}{dx}(x^2+2x+3) = 2x+2$

$$= x^2 - x + \int \frac{2(2x+2)+3}{x^2+2x+3}dx =$$

split the integrals

$$= x^2+x+2\int \frac{2x+2}{x^2+2x+3}dx + \int \frac{3}{x^2+2x+3}dx =$$

$$= x^2 - x + 2\ln(x^2+2x+3) + \int \frac{3}{x^2+2x+3}dx$$

The question is, how to deal with integrals of the form

$$\int \frac{1}{Ax^2+Bx+C}dx \ ?$$

[2] hint: $\frac{x^2}{\sqrt{1-x^2}} = -\frac{1-x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$

FACTS, REMARKS, OBSERVATIONS.
The diagram below shows three different cases with three different outcomes. The examples below illustrate the methods that should be used.

discriminant	$\Delta = 0$	$\Delta < 0$	$\Delta > 0$
rearranged expression	$k\int \frac{1}{(x-p)^2}dx$	$k\int \frac{1}{(x-p)^2+a^2}dx$	$k\int \frac{1}{(x-a)(x-b)}dx$
outcome	$-\frac{k}{x-p}+c$	$\frac{k}{a}\arctan\left(\frac{x-p}{a}\right)$	$\int \frac{A}{x-a}+\frac{B}{x=b}dx$

EXAMPLE 200 (integral of a rational expression, $\Delta = 0$).
Evaluate $\int \frac{1}{3x^2+6x+3}dx$.

$$\begin{aligned}\int \frac{1}{3x^2+6x+3}dx &= \frac{1}{3}\int \frac{1}{x^2+2x+1}dx = \\ &= \frac{1}{3}\int \frac{1}{(x+1)^2}dx = \frac{1}{3}\int (x+1)^{-2}dx = \\ &= \frac{1}{3}(-1)(x+1)^{-1}+c = -\frac{1}{3(x+1)}+c\end{aligned}$$

EXAMPLE 201 (integral of a rational expression, $\Delta < 0$).
Evaluate $\int \frac{1}{4x^2+16x+25}dx$.

$$\begin{aligned}\int \frac{1}{4x^2+16x+25}dx &= \int \frac{1}{(2x+4)^2+9}dx = \int \frac{1}{(2x+4)^2+3^2}dx = \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\arctan\left(\frac{2x+4}{3}\right)+c = \\ &= \frac{1}{6}\arctan\left(\frac{2x+4}{3}\right)+c\end{aligned}$$

EXAMPLE 202 (integral of a rational expression, $\Delta > 0$: partial fractions).
Evaluate $\int \frac{1}{x^2+x-2}dx$.

When the quadratic expression in the bottom can be factorised:

$$\int \frac{1}{x^2+x-2}dx = \int \frac{1}{(x-1)(x+2)}dx$$

then the whole fraction can be expressed in *so-called* **partial fractions**.

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1}+\frac{B}{x+2}$$

To find constants A and B we add two fractions in the right hand side of the euation above

$$\begin{aligned}\frac{1}{(x-1)(x+2)} &= \frac{A(x+2)}{(x-1)(x+2)}+\frac{B(x-1)}{(x-1)(x+2)} \\ \frac{1}{(x-1)(x+2)} &= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}\end{aligned}$$

We need to find A and B such that:

$$A(x+2)+B(x-1) = 1 \text{ for every } x \in \mathbb{R}$$

Let us try $x = -2$ and $x = 1$:

$$\begin{aligned} x=-2: &\quad A(0)+B(-2-1) = 1 &\Rightarrow\ B = -\tfrac{1}{3} \\ x=1: &\quad A(1+2)+B(1-1) = 1 &\Rightarrow\ A = \tfrac{1}{3}\end{aligned}$$

Hence

$$\int \frac{1}{x^2+x-2}dx = \int \frac{1}{3}\left(\frac{1}{x-1}\right) - \frac{1}{3}\left(\frac{1}{x+2}\right)dx =$$
$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\ln(x+2) + c = \frac{1}{3}\ln\left(\frac{x-1}{x+2}\right) + c$$

Questions 25.7.

Q20. Evaluate.

(1) $\int \frac{dx}{2x^2+8}$
(2) $\int \frac{2dx}{x^2+3}$
(3) $\int \frac{dx}{(x+3)^2+1}$
(4) $\int \frac{dx}{(2x-1)^2+4}$
(5) $\int \frac{dx}{x^2+6x+10}$
(6) $\int \frac{dx}{x^2-8x+20}$
(7) $\int \frac{dx}{x^2+3x+3}$
(8) $\int \frac{dx}{4x-6-x^2}$
(9) $\int \frac{dx}{3x^2+12x+14}$
(10) $\int \frac{dx}{2x^2-10x+17}$
(11) $\int \frac{dx}{(3x+2)^2}$
(12) $\int \frac{dx}{(2-x)^2}$
(13) $\int \frac{dx}{x^2+10x+25}$
(14) $\int \frac{dx}{9x^2-12x+4}$
(15) $\int \frac{dx}{3x^2+6x2+6}$
(16) $\int \frac{dx}{8x-16x^2-1}$
(17) $\int \frac{dx}{(x+1)(x+2)}$
(18) $\int \frac{dx}{(x-1)(x+3)}$
(19) $\int \frac{dx}{x^2-x-2}$
(20) $\int \frac{dx}{2x^2+5x+2}$
(21) $\int \frac{dx}{3x^2+x-2}$

Q21. Evaluate.

(1) $\int \frac{4xdx}{2x^2+8}$
(2) $\int \frac{xdx}{x^2+3}$
(3) $\int \frac{x+3}{(x+3)^2+1}dx$
(4) $\int \frac{x-0.5}{(2x-1)^2+4}dx$
(5) $\int \frac{2xdx}{x^2+6x+10}$
(6) $\int \frac{xdx}{x^2-8x+20}$
(7) $\int \frac{xdx}{x^2+3x+3}$
(8) $\int \frac{xdx}{4x-6-x^2}$
(9) $\int \frac{xdx}{3x^2+12x+14}$
(10) $\int \frac{xdx}{2x^2-10x+17}$
(11) $\int \frac{xdx}{(3x+2)^2}$
(12) $\int \frac{xdx}{(2-x)^2}$
(13) $\int \frac{xdx}{x^2+10x+25}$
(14) $\int \frac{xdx}{9x^2-12x+4}$
(15) $\int \frac{xdx}{3x^2+6x2+6}$
(16) $\int \frac{xdx}{8x-16x^2-1}$
(17) $\int \frac{xdx}{(x+1)(x+2)}$
(18) $\int \frac{xdx}{(x-1)(x+3)}$
(19) $\int \frac{xdx}{x^2-x-2}$
(20) $\int \frac{xdx}{2x^2+5x+2}$
(21) $\int \frac{xdx}{3x^2+x-2}$

Q22. Evaluate.

(1) $\int \frac{x^2}{2x^2+8}dx$
(2) $\int \frac{2x^3+8x}{x^2+3}dx$
(3) $\int \frac{3x^3+17x^2+24x-8}{x^2+6x+10}dx$
(4) $\int \frac{3x^3+17x^2+26x-3}{x^2+6x+10}dx$
(5) $\int \frac{x^4-8x^3+21x^2-8x+26}{x^2-8x+20}dx$
(6) $\int \frac{x^4-8x^3+60x}{x^2-8x+20}dx$
(7) $\int \frac{2x^3+8x^2+12x+6}{x^2+3x+3}dx$
(8) $\int \frac{2x^3-11x^2+24x}{4x-6-x^2}dx$
(9) $\int \frac{3x^2-10}{3x^2+12x+14}dx$
(10) $\int \frac{(x-5)^2}{x^2-10x+26}dx$
(11) $\int \frac{3x+3}{(3x+2)^2}dx$
(12) $\int \frac{x^3-3x^2+3x}{(1-x)^2}dx$
(13) $\int \frac{x^2+25}{x^2+10x+25}dx$
(14) $\int \frac{3x^2-4x}{9x^2-12x+4}dx$
(15) $\int \frac{3x}{(x-2)(x+1)}dx$
(16) $\int \frac{3x}{(x+2)(2x+1)}dx$
(17) $\int \frac{xdx}{x^2+3x+2}$
(18) $\int \frac{(x^2dx)}{x^2+3x+2}$
(19) $\int \frac{xdx}{x^2+x-2}$
(20) $\int \frac{3x^3+3x^2+3x}{2x^2+5x+2}dx$
(21) $\int \frac{15x^2-10}{3x^2+x-2}dx$

25.4 More trigonometric substitutions

When integrating rational expressions involving trigonometric ratios we will use different type of *trigonometric substitutions* that eliminate the trigonometric ratios from the expressions.

Example 203 (integration by substitution).
Use the substitution $t = \cos x$ *to valuate* $\int \frac{dx}{\sin x}$.

When $t = \cos x$ then $dt = -\sin x\ dx$. Hence:

$$\begin{aligned} dx &= -\frac{dt}{\sin x} = \\ &= -\frac{dt}{\sqrt{1-\cos^2 x}} = \\ &= -\frac{dt}{\sqrt{1-t^2}} \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sin x} &= \int \frac{\left(-\frac{dt}{\sqrt{1-t^2}}\right)}{\sqrt{1-t^2}} = \int \frac{dt}{t^2-1} = \\ &= \frac{1}{2}\int \frac{1}{t-1} - \frac{1}{t+1}\ dt = \\ &= \frac{1}{2}(\ln(t-1) - \ln(t+1)) + c = \frac{1}{2}\ln\left(\frac{\cos x - 1}{\cos x + 1}\right) + c \end{aligned}$$

EXAMPLE 204 (integration by substitution).
Use the substitution $t = \tan x$ to evaluate $\int \frac{dx}{1+2\sin^2 x+3\cos^2 x}$.

When $t = \tan x$ then $dt = \sec^2 x\ dx$. Then

$$\begin{aligned} \int \frac{dx}{1+2\sin^2 x+3\cos^2 x} &= \int \frac{\sec^2 x\ dx}{(1+2\sin^2 x+3\cos^2 x)\sec^2 x} = \\ &= \int \frac{\sec^2 x\ dx}{\sec^2 x + 2\tan^2 x + 3} = \\ &= \int \frac{\sec^2 x\ dx}{3\tan^2 x + 4} = \\ &= \int \frac{dt}{3t^2+4} = \frac{1}{2\sqrt{3}}\arctan\left(\frac{\sqrt{3}\tan x}{2}\right) + c \end{aligned}$$

EXAMPLE 205 (integration by substitution).
Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to evaluate $\int \frac{dx}{2\sin x+\cos x+1}$.

When $t = \tan\left(\frac{x}{2}\right)$ then $dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\ dx$.
As the substitution uses $\left(\frac{x}{2}\right)$, we will start with changing all x's to $\left(\frac{x}{2}\right)$. Then

$$\begin{aligned} &&\int \frac{dx}{2\sin x + \cos x + 1} = \\ \text{use double angel identities} &&= \int \frac{dx}{4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) + 2\cos^2\left(\frac{x}{2}\right) - 1 + 1} = \\ \text{multiply top and bottom by } \frac{1}{2}\sec^2\left(\frac{x}{2}\right) &&= \int \frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right)\ dx}{\left(4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) + 2\cos^2\left(\frac{x}{2}\right)\right)\frac{1}{2}\sec^2\left(\frac{x}{2}\right)} = \\ \text{simplify} &&= \int \frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right)\ dx}{2\tan\left(\frac{x}{2}\right) + 2} = \\ \text{change } x \text{ to } t &&= \int \frac{dt}{2t+2} = \frac{1}{2}\ln(t+1) + c = \\ \text{and go back to } x &&= \frac{1}{2}\ln\left(\tan\left(\frac{x}{2}\right) + 1\right) + c \end{aligned}$$

QUESTIONS 25.8.

Q23. Use the substitution $u = \cos x$ to integrate the following.

(1) $\frac{\cos^2 x}{\sin x}$ (2) $\frac{\sin x}{1+\cos^2 x}$ (3) $\frac{\sin x}{1+\sin^2 x}$

Q24. Use the substitution $u = \sin x$ to integrate the following.

(1) $\frac{1}{\cos x}$ (2) $\frac{\sin^2 x}{\cos x}$ (3) $\frac{\cos x}{1+\cos^2 x}$ (4) $\frac{\cos x}{1+\sin^2 x}$

Q25. Use the substitution $u = \tan x$ to integrate the following.

(1) $\frac{1}{1+\sin^2 x}$
(2) $\frac{1}{1+\cos^2 x}$
(3) $\frac{1}{3+6\sin^2 x+\cos^2 x}$
(4) $\frac{1}{2+2\sin^2 x+7\cos^2 x}$
(5) $\frac{1}{3-2\sin^2 x+6\cos^2 x}$
(6) $\frac{1}{5-\sin^2 x-4\cos^2 x}$
(7) $\frac{1}{2+\sin^2 x+2\cos^2 x}$
(8) $\frac{1}{1+\cos^2 x}$
(9) $\frac{1}{1+2\sin^2 x}$
(10) $\frac{1}{16-13\cos^2 x}$

Q26. Use the substitution $u = \tan\left(\frac{x}{2}\right)$ to integrate the following.

(1) $\frac{1}{1+\cos x+\sin x}$
(2) $\frac{1}{3+\sin x+2\cos x}$
(3) $\frac{1}{5-3\sin x+4\cos x}$
(4) $\frac{2}{13-12\sin x+5\cos x}$
(5) $\frac{2}{1-2\sin x-\cos x}$
(6) $\frac{1}{1+3\sin x-\cos x}$
(7) $\frac{1}{5+4\cos x}$
(8) $\frac{1}{17+15\sin x-8\cos x}$
(9) $\frac{1}{6+2\cos x}$
(10) $\frac{1}{13-12\cos x}$
(11) $\frac{2}{5-3\cos x}$
(12) $\frac{1}{2+2\sin x}$
(13) $\frac{2}{-3-5\cos x}$
(14) $\frac{2}{1+\sin x-\cos x}$

FACTS, REMARKS, OBSERVATIONS - TRIGONOMETRIC SUBSTITUTIONS.

Let $R(\sin x, \cos x)$ be a rational expression of $\sin x$ and $\cos x$. We will use one of the following substitutions:

	substitution	*when and how should be applied*
1.	$t = \cos x$	*when R is **odd** with respect to $\sin x$,* *i.e. $R(-\sin x, \cos x) = -R(\sin x, \cos x)$*
		$\sin x = \sqrt{1-t^2}, \quad dx = \frac{-1}{\sqrt{1-t^2}}dt$
2.	$t = \sin x$	*when R is **odd** with respect to $\cos x$,* *i.e. $R(\sin x, -\cos x) = -R(\sin x, \cos x)$*
		$\cos x = \sqrt{1-t^2}, \quad dx = \frac{1}{\sqrt{1-t^2}}dt$
3.	$t = \tan x$	*when R is **even** with respect to both $\sin x$ and $\cos x$,* *i.e. $R(-\sin x, -\cos x) = R(\sin x, \cos x)$*
		$\cos x = \frac{1}{\sqrt{t^2+1}}, \quad \sin x = \frac{t}{\sqrt{t^2+1}}, \quad dx = \frac{1}{1+t^2}dt$
4.	$t = \tan(\frac{x}{2})$	*more universally*
		$\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2}{1+t^2}dt$

25.5 DIFFERENTIAL EQUATIONS

What is a differential equation?

We are going to deal with a very simple type of differential equations only.

A **differential equation** is an equation in which an unknown to be found is not a number but a function. A differential equation involves derivatives; in our case only the first derivative.

EXAMPLE 206 (differential equation).

Given that $\frac{dy}{dx} = 6x^2 - x - 3$ find an expression for y in terms of x.

To find y we will use integration:

$$y = \int \frac{dy}{dx}dx = \int 6x^2 - x - 3dx = 2x^3 - \frac{1}{2}x^2 - 3x + C.$$

As we can see, a solution in this example as a **family of functions** rather than a **function**.

A solution like this is called a **general solution** of a differential equation.

When we are given certain **boundary conditions** we can find the **particular solution**.

EXAMPLE 207 (differential equation).

Given that $\frac{dy}{dx} = 6x^2 - x - 3$ and knowing that $y = -10$ when $x = -2$ find an expression for y in terms of x.

From **Example 206** we see that $y = 2x^3 - \frac{1}{2}x^2 - 3x + C$.
Substituting $x = -2$ and $y = -10$ we get $C = 2$, therefore $y = 2x^3 - \frac{1}{2}x^2 - 3x + 2$ is the only solution.

EXAMPLE 208 (differential equation).
Given that $\frac{dy}{dx} = -\frac{1}{2}y$ *and that* $y = 200$ *for* $x = 0$ *find the value of* y *for* $x = 20$.

Introducing a method

I. 1^{st} **attempt**
If we try to integrate $\frac{dy}{dx}$ with respect to x (as we successfully did in previous examples), well get:

$$y = \int \frac{dy}{dx}dx = -\frac{1}{2}\int y dx$$

and we cannot integrate y with respect to x not knowing what is y in terms of x.

II. 2^{nd} **attempt**
The equation $\frac{dy}{dx} = \frac{y}{-2}$ can be changed into $\frac{dx}{dy} = \frac{-2}{y}$. Now we can integrate it with respect to y:

$$x = \int \frac{dy}{dx}dy = \int \frac{-2}{y}dy = -2\ln y + C$$

From $x = -2\ln y + C$ we easily get $\ln y = \frac{C-x}{2}$ and thus $y = e^{\frac{C-x}{2}}$.
The expression can be simplified:

$$y = e^{\frac{C-x}{2}} = e^{\frac{C}{2}}e^{-\frac{1}{2}x} = ke^{-\frac{1}{2}x},$$

where $k = e^{\frac{C}{2}}$ is a constant.
To find the constant:
$200 = ke^0, \therefore k = 200$ and finally $y = 200e^{-\frac{1}{2}x}$.
Now we can find $y(20) = 200e^{-10} \approx 0.00908$.

III. **different method**
Same answer $(y = ke^{(}-\frac{1}{2}x))$ can be achieved in a different way.

$$\frac{dy}{dx} = -\frac{1}{2}y \qquad (1)$$
$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{2} \qquad (2)$$

Let us integrate both sides with respect to x:

$$\int \frac{1}{y}\frac{dy}{dx}dx = \int -\frac{1}{2}dx \qquad (3)$$
$$\int \frac{1}{y}dy = \int -\frac{1}{2}dx \qquad (4)$$

Thus:

$$\ln y = -\frac{1}{2}x + C$$

and then the rest of the solution is almost the same.

IV. **simplification**
To simplify and shorten the process we will transform the equation

$$\frac{dy}{dx} = -\frac{1}{2}y \qquad (1)$$

into

$$\frac{1}{y}dy = -\frac{1}{2}dx$$

and then simply write the integral sign in front of expressions in both sides of the equation to get

$$\int \frac{1}{y}dy = \int -\frac{1}{2}dx \qquad (4)$$

FACTS, REMARKS, OBSERVATIONS - SEPERATING VARIABLES.
Method: separating variables

(1) *Separate variables to get:*

$$g(y)\frac{dy}{dx} = f(x) \text{ or } g(y)dy = f(x)dx$$

(2) *Integrate:*

$$\int g(y)\frac{dy}{dx}dx = \int f(x)dx \text{ or } \int g(y)dy = \int f(x)dx$$

(3) *Apply boundary conditions if appropriate.*

Example 209 (differential equation).
Solve the differential equation $\frac{dy}{dx} + x^2\frac{dy}{dx} = \frac{\sqrt{y}}{\pi}$ *knowing that* $y(\sqrt{3}) = \frac{1}{9}$.

Let us transform the equation into the following form:

$$(1+x^2)\frac{dy}{dx} = \frac{\sqrt{y}}{\pi}$$

and then separate variables:

$$\frac{1}{\sqrt{y}}dy = \frac{1}{\pi}\frac{1}{1+x^2}dx$$

and integrate:

$$\int y^{-\frac{1}{2}}dy = \frac{1}{\pi}\int \frac{dx}{1+x^2}$$
$$2y^{\frac{1}{2}} = \frac{1}{\pi}\arctan x + C$$

As $y(\sqrt{3}) = \frac{1}{9}$:

$$2\sqrt{\frac{1}{9}} = \frac{1}{\pi}\arctan\sqrt{3} + C$$
$$\frac{2}{3} = \frac{1}{3} + C,$$
$$\therefore C = \frac{1}{3} \text{ and thus}$$
$$y^{\frac{1}{2}} = \frac{1}{2\pi}\arctan x + \frac{1}{6}$$
$$y = \left(\frac{1}{2\pi}\arctan x + \frac{1}{6}\right)^2$$

Example 210 (application of differential equations).
The rate of replication of viral particles (measured in millions per hour) is numerically equal to 1% of the number. If the initial number is 2000000, after what time is it going to double?

Let N be the number of viral particles in millions and t be time in hours.

$$\frac{dN}{dt} = 0.01N$$
$$\frac{1}{N}dN = 0.01dt$$
$$\int \frac{1}{N}dN = \int 0.01dt$$
$$\ln N = 0.01t + C$$
$$N = ke^{0.01t}$$

As $N(0) = 2$:

$$2 = ke^0 \therefore k = 2 \text{ and } N = 2e^{0.01t}.$$

We are looking for the time t such that:

$$2e^{0.01t} = 4$$
$$e^{0.01t} = 2$$
$$t = 100\ln 2 \approx 69.3 \text{ hours.}$$

Questions 25.9.

Q27. Verify that $y = \tan x$ is a solution of a differential equation $\frac{dy}{dx} = 1 + y^2$.

Q28. Check whether $y = \sin^2 x$ is a solution of a differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2} = 2 - 4y^2$.

Q29. Find the general solution of the following differential equations.

(1) $x\frac{dy}{dx} = \csc 2y$ (3) $x\frac{dy}{dx} = x^3\sqrt{y}$ (5) $2x\frac{dy}{dx} + \frac{dy}{dx} = 3 - y$ (7) $\frac{1}{1+y^2}\frac{dy}{dx} = \cos^2 x$

(2) $e^{x+y}\frac{dy}{dx} = 3$ (4) $\tan y = \cot x\frac{dy}{dx}$ (6) $2\sqrt{1-x^2}dy = y^3dx$ (8) $\frac{1}{1+x^2}\frac{dy}{dx} = \cos^2 y$

Q30. Solve the following differential equations.

(1) $x\frac{dy}{dx} = 2y; x = e$ for $y = e^2$ (3) $y^2\sqrt{x}\frac{dx}{dy} = x; y(1) = 1$

(2) $2\sin x\sin^2 y\frac{dx}{dy} = 1; y(0) = \frac{\pi}{4}$ (4) $e^{2x-y}dy = 2dx; y(0) = -\ln 2$

Q31. Normal to a curve at any point (x, y) passes through point $(x + \frac{1}{2}, 0)$. Find the equation of the curve knowing that it passes through the origin.

25.6 Mixed exercise

Questions 25.10.

Q32. What method is appropriate in the following cases?
(1) $\int \sin 2x \sin 3x dx$ (2) $\int \sin 2x \cos 3x dx$ (3) $\int \sin 3x \cos 3x dx$ (4) $\int \sin 3x \sin 3x dx$

Q33. Use the substitution $u^2 = x + 2$ to evaluate $\int_{-1}^{2} \frac{x^2}{\sqrt{x+2}}dx$.

Q34. Find the values of A and B such that $\frac{x}{x^2-3x+2} \equiv \frac{A}{x-1} + \frac{B}{x-2}$.

Hence show that $\int_{-1}^{0} \frac{x}{x^2-3x+2}dx = \ln\frac{p}{q}$, where p and q are integers to be found.

Q35. (1) Find the values of the constants A, B and C such that

$$\frac{x^2+1}{6+x-x^2} \equiv A + \frac{B}{x+2} + \frac{C}{x-3}$$

(2) Evaluate

$$\int_{-1}^{2} \frac{x^2+1}{6+x-x^2}dx,$$

giving your answer in the form $p + q\ln 2$, where p and q are integers.

Q36. Evaluate

(1) $\int_0^1 x\sqrt{1-x}dx$ (3) $\int_0^2 \frac{x^2}{x^2+4}dx$ (5) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2\cos x + \sec x)^2 dx$

(2) $\int_{-1}^{2} \frac{x+1}{x+2}dx$ (4) $\int_0^{\frac{\pi}{2}} \cos x(1+\sin x)^3 dx$ (6) $\int_0^{\ln 2} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx$

(7) $\int_0^{\frac{\pi}{3}} \sec^2 x \ln(\cos x)\ dx$

Q37. Use the substitution $u = 1 - \sqrt{x}$ to find $\int \frac{1}{1-\sqrt{x}}dx$.

Q38. Find $\int e^{2x}\sin 3x dx$.

Q39. Integrate using suitable method.

(1) $\int \frac{x^2-2x}{x^3-3x^2+6}dx$ (4) $\int_0^{\frac{\pi}{6}} \cos^3 x dx$ (7) $\int e^x \cos x dx$

(2) $\int \tan^3 x \sec^2 x dx$ (5) $\int (x+1)\sin 2x dx$ (8) $\int_{-\frac{\pi}{2}}^{0} \sin^2 x dx$

(3) $\int x^2\sqrt{1-x}dx$ (6) $\int_0^{\ln 2} x^2 e^x dx$ (9) $\int x\sec^2(2x)$

Q40. Use the substitution $x = \frac{1}{u}$ to evaluate $\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}dx$.

Q41. Evaluate $\int_{-1}^{2} \frac{x^2}{\sqrt{x+2}}dx$ using the substitution $u^2 = x + 2$.

Q42. Evaluate $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx$ using the substitutions: $x = \tan\theta$ and $u = x^2 + 1$.

Part VII

VECTORS

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Chapter 26

Introduction to vectors

26.1 Basics

Notation.
Let O be the origin. *The following notation and terminology will be used.*

For points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ their **position vectors** are:

$$\vec{a} = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} = x_A\vec{i} + y_A\vec{j} + z_A\vec{k} \text{ and } \vec{b} = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix} = x_B\vec{i} + y_B\vec{j} + z_B\vec{k}$$

where $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then

$$\vec{AB} = \vec{AO} + \vec{OB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$$

Facts, remarks, observations.

- *The midpoint of a line segment AB is:*

$$M = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2} \right)$$

- *For a vector $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ its* ***magnitude*** *is $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$*
- *Hence the distance between A and B is*

$$|AB| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

- ***unit vector*** *is a vector $\vec{u}$ whose magnitude is 1, $|\vec{u}| = 1$*
- *a unit vector in the direction of $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$*

Questions 26.1.

Q1. Given $A(-3,2,-8)$ and $B(7,-9,-6)$ find:

(1) $\vec{AB}$,

(2) the unit vector in the direction of $\vec{AB}$,

(3) a vector of magnitude 7 in the direction of $\vec{AB}$.

Q2. Given $A(-7,4,5)$ and $B(5,0,8)$ find:

(1) $\vec{AB}$,

(2) the unit vector in the direction of $\vec{AB}$,

(3) a vector of magnitude 5 in the direction of $\vec{AB}$.

Q3. Given the points: $A(-3,-7,1)$, $B(8,-1,11)$ and $C(6,12,4)$ find the coordinates of:

(1) point D such that quadrilateral $ABCD$ is a parallelogram,

(2) point S, the intersection of diagonals of $ABCD$.

Q4. Consider the points: $A(-6,7,-10)$, $B(4,-3,13)$ and $P(2,-1,4)$.

(1) find a vector $\vec{v}$ in the direction of $\vec{AP}$ with magnitude equal to that of $\vec{AB}$,

(2) find the coordinates of point C such that $\vec{OC} = \vec{OA} + \vec{v}$,

(3) find the coordinates of point M, the midpoint of BC,

(4) hence find the area of triangle ABC.

(5) Is the triangle equilateral?

26.2 Scalar product

Definition 211 (scalar product).

For vectors $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ *their* ***scalar product*** *(or dot product) is defined:*

(1) $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$,

where θ *is the angle between* $\vec{u}$ *and* $\vec{v}$. **or**

(2) $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Definitions (1) and (2) are equivalent.

Properties of the scalar product.

From the above definitions it follows easily that:

- *for the angle* θ *between vectors* $\vec{u}$ *and* $\vec{v}$*:*

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}||\vec{v}|}$$

- $\vec{u} \perp \vec{v} \quad \Rightarrow \vec{u} \cdot \vec{v} = 0$
- $\vec{u} \parallel \vec{v} \quad \Rightarrow |\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}|$

Questions 26.2.

Q5. Given $A = (-3,5,1)$, $B = (-1,9,5)$, $C = (-5,14,-5)$

(1) Find vectors AB and AC.

(2) Find the angle between AB and AC.

(3) Hence calculate the area of triangle ABC.

Q6. Given $A = (4, -3, 2)$, $B = (-6, 2, -8)$, $C = (7, -5, -4)$

(1) Find vectors AB and AC.
(2) Find the angle between AB and AC.
(3) Hence calculate the area of triangle ABC.

Q7. Given $A = (1, -2, -2)$, $B = (-5, 8, 2)$, $C = (10, -17, -8)$

(1) Find vectors AB and AC.
(2) Find the angle between AB and AC.
(3) Hence calculate the area of triangle ABC.

Q8. Find all angles in the triangle ABC, where $A = (1, -3, 2)$, $B = (-5, -2, 4)$, $C = (-1, 5, -2)$.

Q9. Given vectors $\vec{u} = 3a\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{v} = 2\vec{i} - a\vec{j} + 2\vec{k}$ find the value of a for which they are perpendicular.

Q10. Given vectors $\vec{u} = 2b\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{v} = 2b\vec{i} - (b+4)\vec{j} - 2\vec{k}$ find the value of b for which they are perpendicular.

Q11. Given vectors $\vec{u} = 3c\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{v} = (1-d)\vec{i} - d\vec{j} + 2\vec{k}$ find the value of c and d for which they are parallel.

Q12. Given $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$ find $\vec{u}$ such that $\vec{u} || \vec{v}$ and $|\vec{u}| = 4$.

Q13. The vectors $\vec{a} = 13\vec{i} - 7\vec{j} + 8\vec{k}$ and $\vec{b} = \vec{i} - x\vec{j} - 16\vec{k}$ are of equal length. Find x.

Q14. P, Q and R are three points with coordinates $(4, -1, 3)$, $(-3, 4, 7)$ and $(1, 1, -1)$ respectively.

(1) Find $\vec{PQ} \cdot \vec{PR}$.
(2) Find angle P in triangle PQR.

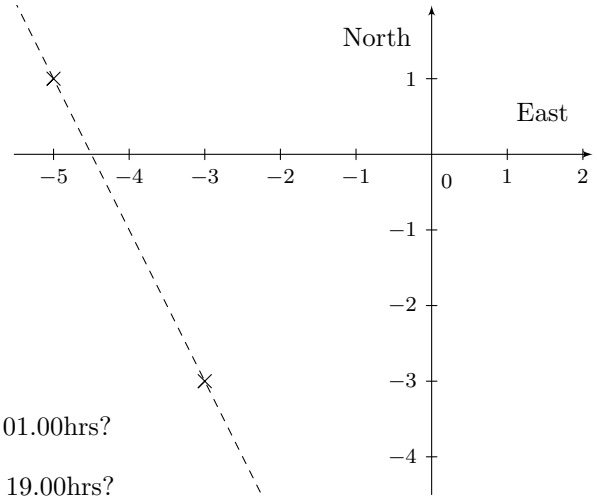
Q15. Find the values of x and y if $\vec{u} = x\vec{i} + 3y\vec{j} + 3\vec{k}$ is perpendicular to both $\vec{v} = 3\vec{i} - 2\vec{j} + 5\vec{k}$ and $\vec{w} = 6\vec{i} + 3\vec{j} - 4\vec{k}$.

Chapter 27

Straight lines

Task

A ship has been sailing through the Endless Sea. It was observed that on 1^{st} of June at 00.00hrs the ship's position was 5 miles West and 1 mile North from the Observer. Later, at 01.00hrs the new position of the ship was observed. It was 3 miles West and 3 miles South from the Observer. Assuming that the ship has sailed at a constant speed along a straight line answer the following questions.



- What was the position of the ship at 02.00hrs?
- What was the position of the ship at 13.00hrs?
- What was the position of the ship on 2^{nd} of June at 01.00hrs?
- What was the position of the ship on 31^{st} of May at 19.00hrs?
- When was the ship on the North-South line from the Observer? At what distance from the observer?

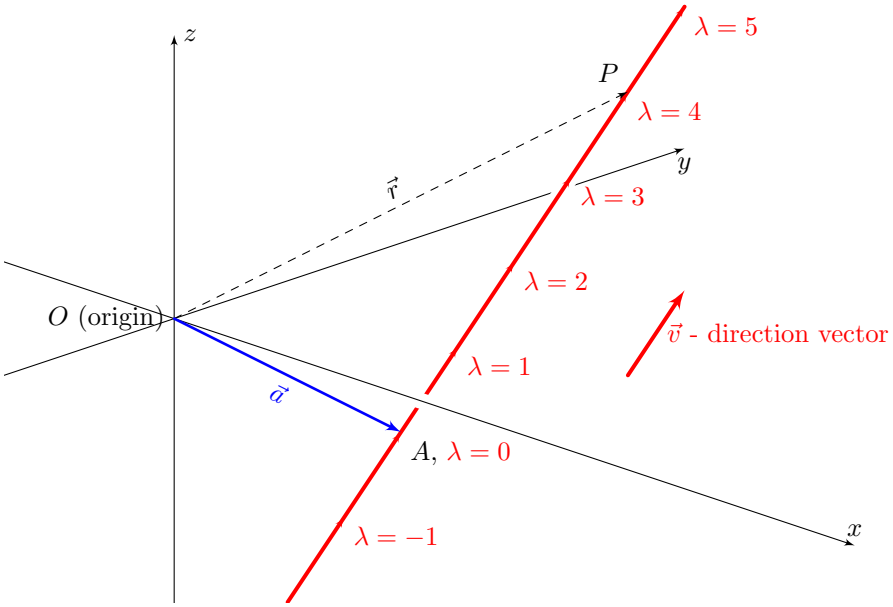
27.1 Equations of straight lines

To describe position of a straight line we need to take into consideration two factors:

- position of any chosen point on the line and
- direction of the line.

Let A be the point on the line with position vector $\vec{a}$, i.e. $\vec{OA} = \vec{a}$, where O is the origin.
Let $\vec{v}$ be the direction vector of the line.
To describe position of any point P on the line (i.e. to find its position vector $\vec{r}$) we will:

1. move from O to A with vector $\vec{a}$ and then
2. move along the line with vector $\vec{v}$.



Let λ denote the number of times that vector $\vec{v}$ must be used to go from A to P. Then the position vector of point p is given by:

$$\vec{r} = \vec{a} + \lambda\vec{v}$$

For $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, this could be written as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \text{or as} \qquad \begin{cases} x = a_1 + \lambda v_1 \\ y = a_2 + \lambda v_2 \\ z = a_3 + \lambda v_3 \end{cases}.$$

The two are called a **vector equation** of the line and a **parametric equation** of the line respectively (see definitions below).

Questions 27.1.

Q1. Consider the line L with the equation $\begin{cases} x = 3 - 3\lambda \\ y = -1 + \lambda \\ z = 2 + 4\lambda \end{cases}$.

(i) Find the point on L for $\lambda = 5$.

(ii) Given that point $B(-3, 1, 10)$ lies on L, find the value of a parameter λ for B.

(iii) Check whether the following points lie on L.

(1) $C(9, -3, -6)$ (2) $D(6, -2, 4)$ (3) $E(-\frac{3}{2}, -\frac{5}{2}, -4)$

(iv) Point $F(0, p, q)$ lies on L. Find the value of a parameter λ for F. Find p and q.

(v) Write the equation of L in the form $\begin{cases} x = a_1 + \lambda v_1 \\ y = a_2 + \lambda v_2 \\ z = a_3 + \lambda v_3 \end{cases}$, so that:

(1) $a_1 = 0, v_1 = 3$ (2) $a_1 = 9, v_1 = 6$ (3) $a_1 = 6, v_2 = 2$

Q2. Transform the following parametric equations of the lines into Cartesian form.

(i) $\begin{cases} x = 2\lambda \\ y = 1 + \lambda \end{cases}$ (ii) $\begin{cases} x = 3 - 2\lambda \\ y = -1 + 3\lambda \end{cases}$ (iii) $\begin{cases} x = -4 - 3\lambda \\ y = 1 + 2\lambda \end{cases}$ (iv) $\begin{cases} x = 2 + 6\lambda \\ y = -3 - 4\lambda \end{cases}$

Q3. Transform the following Cartesian equations of the lines into parametric form.

(i) $y = 2x - 3$ (ii) $3x - 4y - 2 = 0$ (iii) $y = -\frac{2}{3}x + 1$ (iv) $2x - 5y + 4 = 0$

Q4. Given a straight line $\begin{cases} x = 2\lambda \\ y = 1 + \lambda \end{cases}$

(i) write down the coordinates of the points for given values of λ.

λ	-2	-1	0	1	2
point					

(ii) Sketch the line.

(iii) Write down a direction vector of the line.

Q5. For a line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$:

(i) write down the coordinates of 5 points on the line.

(ii) Complete the equations below to express the line in a different form.

$$\begin{cases} x = 5 - 2t \\ y = \\ z = \end{cases}$$

(iii) From each of the equations above find t in terms of x, y and z respectively.
$t =$
$t =$
$t =$

(iv) Complete the equation below.

$$\frac{x - (\quad)}{(\quad)} = \frac{y - (\quad)}{(\quad)} = \frac{z - (\quad)}{(\quad)}$$

DEFINITION 212 (equations of a line).
We will use three types of equations of the line passing through point (a_1, a_2, a_3) with direction vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

- ***vector equation***
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
or $\vec{r} = \vec{a} + \lambda\vec{v}$

- ***parametric equation***
$\begin{cases} x = a_1 + \lambda v_1 \\ y = a_2 + \lambda v_2 \\ z = a_3 + \lambda v_3 \end{cases}$

- ***Cartesian equation***
$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$

Note that when a coordinate of a direction vector $\vec{v}$ equals 0 then a respective coordinate of points on the line is constant.

For example when $v_2 = 0$, i.e. $\vec{v} = \begin{pmatrix} v_1 \\ 0 \\ v_3 \end{pmatrix}$ then the equations are $\begin{cases} \frac{x - a_1}{v_1} = \frac{z - a_3}{v_3} \\ y = a_2 \end{cases}$.

QUESTIONS 27.2.

Q6. Change the following vector equations of the line into a Cartesian form.

(1) $\vec{r} = \vec{i} - 3\vec{j} - 6\vec{k} + \lambda(\vec{i} + 2\vec{j} + 5\vec{k})$
(2) $\vec{r} = -3\vec{i} + 2\vec{j} + \vec{k} + \mu(-2\vec{i} + \vec{j} - \vec{k})$
(3) $\vec{r} = 2\vec{i} - \vec{j} - 5\vec{k} + \lambda(-\vec{i} + 2\vec{j} - 3\vec{k})$
(4) $\vec{r} = 3\vec{i} - \vec{j} + 2\vec{k} + \mu(3\vec{i} - \vec{k})$
(5) $\vec{r} = 3\vec{i} - \vec{j} + \vec{k} + \lambda(\vec{i} + \vec{j} - \vec{k})$
(6) $\vec{r} = 3\vec{j} - 2\vec{k} + \mu(2\vec{i} + 5\vec{j})$
(7) $\vec{r} = \vec{k} + \lambda\vec{j}$

Q7. Change the following Cartesian equations of the line into a vector form.

(1) $\frac{x-2}{3} = 1 - y = \frac{z-7}{2}$

(2) $2 - x = y + 1 = -0.5z$

(3) $x + 1 = -y - 2 = z - 5$

(4) $\frac{2x+3}{4} = \frac{y-1}{3} = 2 - z$

(5) $\begin{cases} \frac{x+4}{3} = \frac{3-z}{2} \\ y + 1 = 0 \end{cases}$

Q8. Given $A = (3, 5, 1)$, $B = (1, 9, 5)$, $C = (5, 14, 5)$

(i) Find the Cartesian equation of line L_1 that passes through points B and C.

(ii) Find the Cartesian equation of line L_2 that is parallel to L_1 and passes through A.

Q9. Given $A = (4, 3, 2)$, $B = (6, 2, 8)$, $C = (7, 5, 4)$

(i) Find the Cartesian equation of line L_1 that passes through points B and C.

(ii) Find the Cartesian equation of line L_2 that is parallel to L_1 and passes through A.

Q10. Given $A = (1, 2, 2)$, $B = (5, 8, 2)$, $C = (10, 17, 8)$

(i) Find the Cartesian equation of line L_1 that passes through points B and C.

(ii) Find the Cartesian equation of line L_2 that is parallel to L_1 and passes through A.

27.2 Intersections of straight lines

To find the intersection of straight lines:

I. transform their equations to parametric form using two different parameters (e.g. λ and μ),

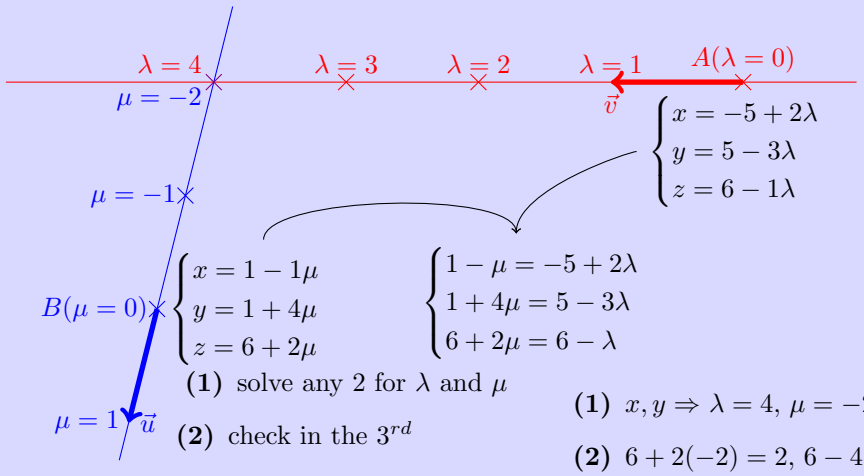
II. set x, y and z from both equations equal to form system of three equations and 2 variables (λ and μ),

III. solve any two of the equations and check your solution in the third one,

IV. use the obtained value of λ or μ to find the coordinates of intersection.

Example 213 (intersection of lines).

Find the intersection of the lines $\begin{cases} x = -5 + 2\lambda \\ y = 5 - 3\lambda \\ z = 6 - 1\lambda \end{cases}$ *and* $\begin{cases} x = 1 - 1\mu \\ y = 1 + 4\mu \\ z = 6 + 2\mu \end{cases}$.



Questions 27.3.

Q11. Consider the following lines:

$$l_1 : \vec{r} = \begin{pmatrix} 10 \\ -7 \\ -21 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 3 \\ 12 \end{pmatrix}$$

$$l_2 : \frac{x+4}{6} = \frac{-y}{3} = \frac{7-z}{12}$$

$$l_3 : \vec{r} = \begin{pmatrix} -10 \\ 3 \\ -19 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ -8 \end{pmatrix}$$

$$l_4 : \vec{r} = \begin{pmatrix} -1 \\ 3 \\ -14 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$$l_5 : \frac{x+4}{-2} = y, z = 7$$

$$l_6 : \vec{r} = \begin{pmatrix} -11 \\ 5 \\ 16 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -5 \\ -9 \end{pmatrix}$$

(i) Write down the equations of all lines in parametric form.

(ii) Fill in the table with information about resective position of the lines (coincident, parallel, intersecting, skew). If the lines intersect at a unique point, find the intersection.

?	l_1	l_2	l_3	l_4	l_5
l_6					
l_5					
l_4					
l_3					
l_2					

Q12. Find the point of intersection of the lines L_1 and L_2 (if it exists). Find the angle between the lines.

(1) $L_1 : \frac{x+1}{-3} = \frac{y}{2} = \frac{9-z}{4}$
$L_2 : \frac{x-2}{3} = 1 - y = \frac{z-7}{2}$

(2) $L_1 : \vec{r} = \begin{pmatrix} 1 \\ -3 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$
$L_2 : \vec{r} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

(3) $L_1 : 2 - x = y + 1 = -0.5z$
$L_2 : x + 1 = -y - 2 = z - 5$

(4) $L_1 : \vec{r} = 2\vec{i} - \vec{j} - 5\vec{k} + \mu(-\vec{i} + 2\vec{j} - 3\vec{k})$
$L_2 : \vec{r} = 3\vec{i} - \vec{j} + 2\vec{k} + \lambda(3\vec{i} - \vec{k})$

(5) $L_1 : \vec{r} = 3\vec{i} - \vec{j} + 2\vec{k} + \lambda(2\vec{i} - \vec{j} + \vec{k})$
$L_2 : \vec{r} = -4\vec{i} + \mu(\vec{i} - 3\vec{j} + 2\vec{k})$

Chapter 28

Matrices

28.1 Basics

Definition 214 (matrix).
Informally speaking a ***matrix*** *(pl. matrices) is a rectangular array of numbers, e.g.*

$$\begin{pmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \end{pmatrix}$$

is an example of 2×3 *matrix.*
A matrix with n *rows and* m *columns is said to be of* order $n \times m$.
When $n = m$ *a matrix is said to be of order* n.

Properties of matrices.

1. *Matrices of the same* order *can be* ***added*** *or* ***subtracted*** *element by element.*

2. *Two matrices can be* ***multiplied*** *if the number of columns in the first one is the same as the number of rows in the second, i.e. for*

$$A = \underset{2 \times \boldsymbol{3}}{\begin{pmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \end{pmatrix}} \qquad B = \underset{\boldsymbol{3} \times 4}{\begin{pmatrix} 2 & 0 & -1 & 3 \\ 4 & -2 & 1 & 2 \\ 0 & -3 & 2 & -1 \end{pmatrix}}$$

   the product AB *exists (and it is* 2×4 *matrix), however the product* BA ***does not*** *exist.*

3. *Multiplication of matrices is* ***associative*** *but NOT* ***commutative****.*

$$\boxed{A(BC) = (AB)C}$$
$$\boxed{AB \neq BA}$$

4. *There is no division of matrices.*

28.2 Multiplication of matrices

Example 215 (mutiplication of matrices).
Consider the matrices

$$A = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & -1 & 3 \\ 4 & -2 & 1 & 2 \\ 0 & -3 & 2 & -1 \end{pmatrix}$$

Find the product AB.

The product AB is going to be a 2×4 matrix:

$$AB = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & 3 \\ 4 & -2 & 1 & 2 \\ 0 & -3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \end{pmatrix}$$

In the resultant matrix $m_{r,k}$ denotes an element in the r^{th} row and k^{th} column.
It is defined as a scalar product of the r^{th} row of matrix A and k^{th} column of matrix B, e.g. $m_{1,2}$ is obtained as a scalar product of the 1^{st} row of matrix A and 2^{nd} column of matrix B:

$$AB = \begin{pmatrix} \boxed{3 & -1 & 4} \\ 0 & 2 & -5 \end{pmatrix} \begin{pmatrix} 2 & \boxed{0} & -1 & 3 \\ 4 & \boxed{-2} & 1 & 2 \\ 0 & \boxed{-3} & 2 & -1 \end{pmatrix} = \begin{pmatrix} m_{1,1} & \boxed{m_{1,2}} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \end{pmatrix}$$

that is:

$$m_{1,2} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix} = (3)(0) + (-1)(-2) + (4)(-3) = -10$$

Eventually, after evaluating 8 such scalar products we get:

$$AB = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & 3 \\ 4 & -2 & 1 & 2 \\ 0 & -3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -10 & 4 & 3 \\ 8 & 11 & -8 & 9 \end{pmatrix}$$

Questions 28.1.

Q1. Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 0 & 2 \\ 0 & 1 & -4 \end{pmatrix}$. Find one of the products: AB or BA.

Q2. For the matrix $M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$:

(i) find M^2, M^3 and M^4,
(ii) state a conjecture for M^n, $n \in \mathbb{N}$,
(iii) prove the conjecture by mathematical induction.

Q3. For the matrices $M = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$, $N = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} -2 & 0.5 \\ 1 & 2 \end{pmatrix}$ find the products:

(i) MN (iii) NP (v) $(MN)P$
(ii) NM (iv) PN (vi) $M(NP)$

Q4. Evaluate:

(i) $\begin{pmatrix} 3 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & -2 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 7 & -3 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}$

(iv) $\begin{pmatrix} 4 & 2 & 1 \\ 5 & 1 & -3 \\ -2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ -3 & 0 & 1 \\ 2 & 0 & -4 \end{pmatrix}$

(v) $\begin{pmatrix} 4 & -2 & 3 \\ -2 & 0 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(vi) $\begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$

(vii) $\begin{pmatrix} 7 & 0 & -8 \\ -3 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ -7 & 8 \\ 2 & 4 \end{pmatrix}$

(viii) $\begin{pmatrix} 2 & 5 \\ -7 & 8 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 7 & 0 & -8 \\ -3 & 1 & 3 \end{pmatrix}$

Q5. Consider matrices:
$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
and vectors:
$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$
By finding the products of all matrices with all vectors describe, what geometric transformations are applied to the vectors by use of the matrices.

Q6. Find a matrix I such that for any 2×2 matrix A: $AI = IA = A$.

Q7. Find a matrix I such that for any 3×3 matrix A: $AI = IA = A$.

DEFINITION 216 (identity matrix).
*The **identity matrix** is a square $(n \times n)$ matrix I such that for any other $n \times n$ matrix A:*

$$AI = IA = A$$

EXAMPLE 217 (identity matrices).
For 2×2 matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For 3×3 matrices:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

28.3 Determinant of a matrix, inverse matrix

DEFINITION 218 (inverse matrix).
*The **inverse matrix** of a square matrix A is a square matrix B such that*

$$AB = BA = I$$

*(where I is **the identity matrix**).*

NOTATION.

$$B = A^{-1}, \text{ so}$$
$$AA^{-1} = A^{-1}A = I$$

Questions 28.2.

Q8. Consider $A = \begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$.

(i) Find the product AB.

(ii) **Hence**, find A^{-1}.

Q9. Consider $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & -1 \\ -4 & -1 & 3 \\ -5 & -5 & 5 \end{pmatrix}$.

(i) Find the product AB.
(ii) **Hence**, find A^{-1}.

Q10. Show that matrix $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ has no inverse.

Q11. Given a square matrix A, prove that if A^{-1} exists then $(A^{-1})^{-1} = A$.

Q12. Given square matrices A and B prove that if A^{-1} and B^{-1} exist then $(AB)^{-1} = B^{-1}A^{-1}$.

DEFINITION 219 (determinant of a square 2×2 matrix).
[1]*Determinant of a square* 2×2 *matrix is defined by the following formula:*

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

NOTATION.

$$\det A = |A|$$
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

THEOREM 220 (inverse of a 2×2 matrix).
When

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

QUESTIONS 28.3.

Q13. For each of the following matrices:

(i) find the inverse matrix;
(ii) check that the result is the inverse.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 5 \\ 2 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ (e) $\begin{pmatrix} -3 & -3 \\ -2 & -4 \end{pmatrix}$ (f) $\begin{pmatrix} -5 & -3 \\ 3 & 2 \end{pmatrix}$ (g) $\begin{pmatrix} 3 & 1 \\ 5 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$ (i) $\begin{pmatrix} 3 & -7 \\ -2 & 3 \end{pmatrix}$ (j) $\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$ (k) $\begin{pmatrix} 2 & -3 \\ -5 & 6 \end{pmatrix}$ (l) $\begin{pmatrix} \frac{1}{3} & 2 \\ -1 & 1.5 \end{pmatrix}$

DEFINITION 221 (singular matrix).
A ***singular*** *matrix is a matrix* A *such that* $\det A = 0$.

THEOREM 222 (singular matrices are non-invertible).
When a square matrix A *(of any order) is* ***singular*** *then* A^{-1} *does not exist, i.e.* A *is non-invertible.*

Determinant of a square 3×3 matrix
For evaluating the determinant of a 3×3 matrix we will a use recursive rule (*Laplace expansion* or *cofactor expansion*) that can be used for higher order matrices, too. Let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix},$$

i.e. $a_{k,m}$ is an element in the k^{th} row and m^{th} column.
Let then $A_{k,m}$ denote a 2×2 matrix obtained from A by crossing out the k^{th} row and m^{th} column, e.g.

[1]formally, it is not a definition af a determinant, but a method of evaluating it only

$$A_{1,2} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \begin{pmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{pmatrix}$$

Then, we have to choose a single column or single row from matrix A and perform the *cofactor expansion along the chosen row or column.*
For the choice of a single column (m^{th}) the expansion is as follows:

$$\det A = \sum_{k=1}^{3} \left((-1)^{k+m} a_{k,m} \det A_{k,m} \right)$$

or if we choose a single row (k^{th}):

$$\det A = \sum_{m=1}^{3} \left((-1)^{k+m} a_{k,m} \det A_{k,m} \right)$$

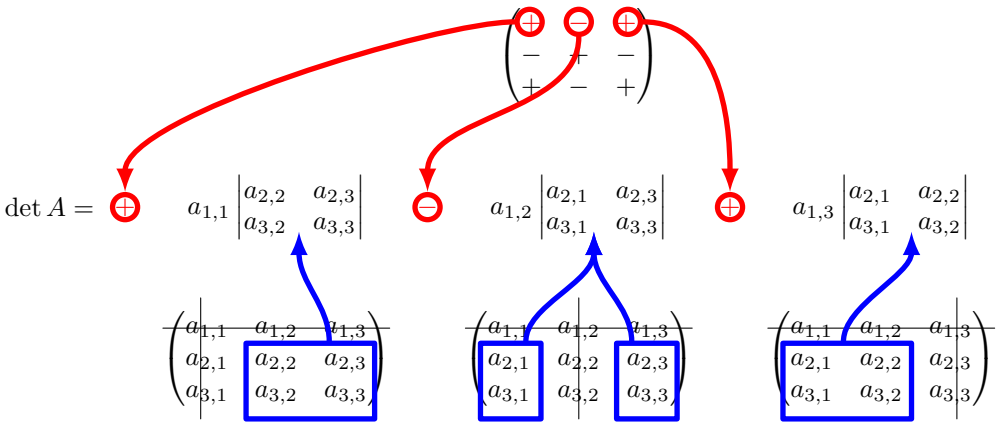
Let us see how this works.
Firstly, the power $(-1)^{k+m} = (-1)^{\text{number of row+number of column}}$ gives values of -1 and $+1$ only:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

When we choose e.g. the 1^{st} row, the expansion looks as follows:

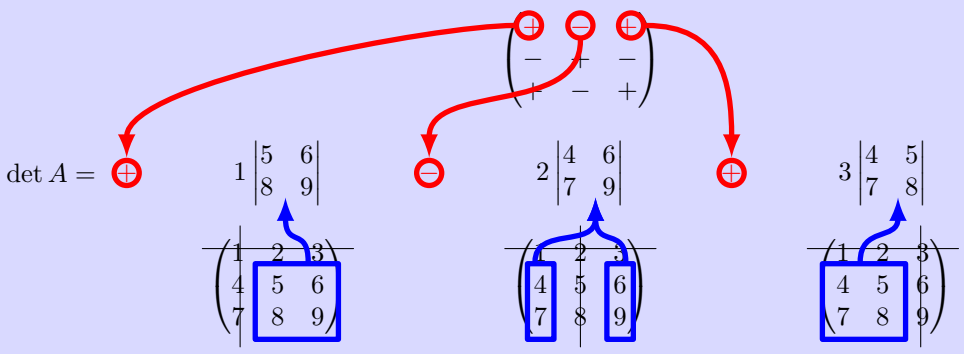
$$\det A = (+1)a_{1,1} \det A_{1,1} + (-1)a_{1,2} \det A_{1,2} + (+1)a_{1,3} \det A_{1,3}$$

or:



Similar pattern can be applied to any row or column.

EXAMPLE 223 (determinant of a 3×3 matrix).
Evaluate the determinant of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.



So,

$$\begin{aligned}\det A &= +1\begin{vmatrix}5 & 6\\ 8 & 9\end{vmatrix} - 2\begin{vmatrix}4 & 6\\ 7 & 9\end{vmatrix} + 3\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix} = \\ &= 1(5\times 9 - 6\times 8) - 2(4\times 9 - 6\times 7) + 3(4\times 8 - 5\times 7) = \\ &= (45-48) - 2(36-42) + 3(32-35) = -3 - 2(-6) + 3(-3) = \\ &= -3 + 12 - 9 = \\ &= 0\end{aligned}$$

Eventually, $\det A = 0$, hence matrix A is **singular**.

COROLLARY 224 (evaluation of a determinant of a 3×3 matrix).

- $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3}\\ a_{2,1} & a_{2,2} & a_{2,3}\\ a_{3,1} & a_{3,2} & a_{3,3}\end{pmatrix} \Rightarrow \det A = a_{1,1}\begin{vmatrix}a_{2,2} & a_{2,3}\\ a_{3,2} & a_{3,3}\end{vmatrix} - a_{1,2}\begin{vmatrix}a_{2,1} & a_{2,3}\\ a_{3,1} & a_{3,3}\end{vmatrix} + a_{1,3}\begin{vmatrix}a_{2,1} & a_{2,2}\\ a_{3,1} & a_{3,2}\end{vmatrix}$
- $A = \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & k\end{pmatrix} \Rightarrow \det A = a\begin{vmatrix}e & f\\ h & k\end{vmatrix} - b\begin{vmatrix}d & f\\ g & k\end{vmatrix} + c\begin{vmatrix}d & e\\ g & h\end{vmatrix}$

QUESTIONS 28.4.

Q14. Given $A = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$

(i) use cofactor expansion along the 2^{nd} row to show that A is singular;

(ii) use cofactor expansion along the 3^{rd} column to show that A is singular;

Q15. Evaluate determinants of the following matrices. Which of them are singular?

(i) $\begin{pmatrix}3 & 2 & 2\\ -1 & -2 & 4\\ 7 & 5 & 4\end{pmatrix}$ (ii) $\begin{pmatrix}4 & 1 & -2\\ 3 & -1 & 3\\ 8 & 0 & 2\end{pmatrix}$ (iii) $\begin{pmatrix}-7 & 1 & 0\\ 0 & -5 & 12\\ 11 & 0 & -4\end{pmatrix}$ (iv) $\begin{pmatrix}-3 & 3 & -2\\ 1 & -7 & 0\\ 2 & -5 & 1\end{pmatrix}$ (v) $\begin{pmatrix}-4 & -1 & 2\\ 2 & -4 & -5\\ 3 & 5 & 1\end{pmatrix}$ (vi) $\begin{pmatrix}3 & 4 & 1\\ -4 & -2 & -8\\ 1 & 3 & -3\end{pmatrix}$

Q16. Let $A = \begin{pmatrix}5 & -3\\ 3 & -1\end{pmatrix}$, $B = \begin{pmatrix}6 & -4\\ -5 & 3\end{pmatrix}$ and $C = \begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}$.

(i) Find the determinants of A,B and C.

(ii) **Hence**, find the determinant of $AC^{-1}B$.

(iii) Find the inverse of A.

(iv) Find the inverse of B.

(v) Find the inverse of AB.

(vi) Find the inverse of BA.

PROPERTIES OF INVERSES AND DETERMINANTS.

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $\det(AB) = (\det A)(\det B)$
- $\det A^{-1} = \frac{1}{\det A}$

Questions 28.5.

Q17. Find x for which matrix $M = \begin{pmatrix} -1 & 3 & -4 \\ 4 & 0 & 1 \\ 2 & x & 3 \end{pmatrix}$ is singular.

Q18. Let $M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where $a \neq 0$ and $b \neq 0$.

(i) Show that M is non-singular.

(ii) Find M^2.

(iii) Show that $\det(M^2) > 0$.

Q19. Given that $A = \begin{pmatrix} 3 & -2 \\ 6 & 4 \end{pmatrix}$ and that $A^2 - 7A + mI = 0$ find m.

Q20. Given that $B = \begin{pmatrix} -7 & 1 \\ 3 & 2 \end{pmatrix}$ and that $B^2 + kB - 17I = 0$ find k.

Q21. Given that $C = \begin{pmatrix} 4 & -3 \\ -7 & -5 \end{pmatrix}$ and that $A^2 + A + tI = 0$ find t.

Q22. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ be the matrix such that $A^2 + kA + sI = 0$.

(i) Find the values of k and s.

(ii) Express A^4 in the form $mA + tI$.

(iii) Solve the above for $A = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$.

Q23. Consider $M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & 0 \end{pmatrix}$.

(i) Find the values of λ for which $M - \lambda I$ is singular.

(ii) For each of the values of λ found in (i) find a vector $\vec{v}$ such that $M\vec{v} = \lambda\vec{v}$.

Q24. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

(i) Show that the determinant of the matrix $\det(A - tI) = (t-1)^2$.

(ii) Find the values of t for which the matrix $(A - tI)^2$ is singular.

Q25. Let A, B, C, D be square matrices of the same order.
Which of the following statements are correct and which are incorrect?
For every incorrect statement point out a mistake and write down a correct version.

(i) $(A+B)^2 = A^2 + B^2 + 2AB$

(ii) $ADB = C \Leftrightarrow D = A^{-1}B^{-1}C$

(iii) $(AB)DC^{-1} = I \Leftrightarrow D = B^{-1}(A^{-1}C)$

(iv) $AB = I \Rightarrow B = \frac{I}{A}$

(v) $(A+2I)^2 = A^2 + 4A + 4$

(vi) $\frac{|A||B||D|}{|C|} = |ADC^{-1}B|$

(vii) $(A+I)^3 = A^3 + 3A^2 + 3A + I$

(viii) $\det(-A) = -\det A$

(ix) $\det(-A) = \det A$

28.4 Solving simultaneous equations - intersections of the planes

Example 225 (solution of a system of equations with an inverse matrix).
A system of simultaneous equations

$$\begin{cases} x+2y+3z=9 \\ 4x+5y+6z=21 \\ 5x+7y+8z=27 \end{cases}$$

describes three planes, no two of them being parallel.

The planes intersect in a single point - solution of the system. The system can be rewritten in a matrix form:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 8 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 21 \\ 27 \end{pmatrix}$$

or $AX = B$, where $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 8 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 9 \\ 21 \\ 27 \end{pmatrix}$.
To solve the equation $AX = B$ for X we have to get rid of A using its inverse:

$$\begin{aligned} & A^{-1}(AX) = A^{-1}B \\ \Leftrightarrow\ & (A^{-1}A)X = A^{-1}B \\ \Leftrightarrow\ & X = A^{-1}B \end{aligned}$$

Using GDC: $X = \frac{1}{3}\begin{pmatrix} -2 & 5 & -3 \\ -2 & -7 & 6 \\ 3 & 3 & -3 \end{pmatrix}\begin{pmatrix} 9 \\ 21 \\ 27 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.
Hence intersection of the planes is $(2, -1, 3)$.

Note that such unique point of intersection exists if and only if matrix A is **non-singular**.

Questions 28.6.

Q26. Given the matrices: $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 2 \\ 5 & -1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -9 & -4 & 5 \\ -3 & 1 & 1 \\ 6 & 3 & -3 \end{pmatrix}$ find the product AB and hence solve the system of simultaneous equations.

$$\begin{cases} 2x-y+3z &= 2 \\ x+y+2z &= -1 \\ 5x-y+7z &= 3 \end{cases}$$

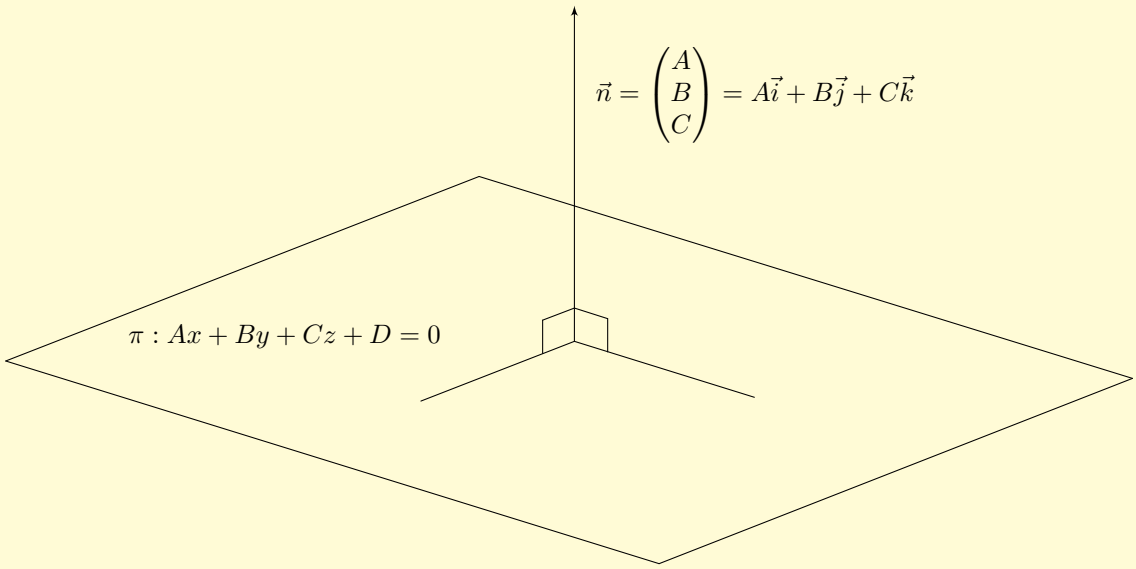
Q27. Given matrix $A = \begin{pmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$ and vector $\vec{v} = \begin{pmatrix} 2y-z \\ 4x+y \\ 3y-2z \end{pmatrix}$ find the values of x, y and z such that $A\vec{v} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$.

Chapter 29

Planes in three-dimensional space

Definition 226 (Cartesian equation of a plane).
Equation $Ax + By + Cz + D = 0$ describes a plane in 3-dimensional real space.

Properties of the Cartesian equation of a plane.
For the plane defined by Cartesian equation $Ax + By + Cz + D = 0$ a vector formed by coefficients of x, y and z i.e. $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is normal to the plane.



29.1 Intersections of the planes

A system of simultaneous equations

$$\begin{cases} x - 6y + 2z = 1 \\ 4x + 2y + 3z = 6 \\ 3x + 4y + 5z = -5 \end{cases}$$

describes three planes, no two of them being parallel. The planes intersect in a single point whose coordinates are solution of the system.
We will express the system of simultaneous equations in the simplified *matrix* form:

$$\begin{cases} x - 6y + 2z = 1 \\ 4x + 2y + 3z = 6 \\ 3x + 4y + 5z = -5 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & -6 & 2 & 1 \\ 4 & 2 & 3 & 6 \\ 3 & 4 & 5 & -5 \end{array}\right)$$

or more generally:

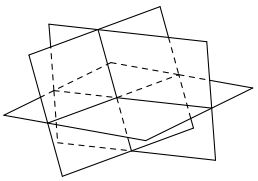
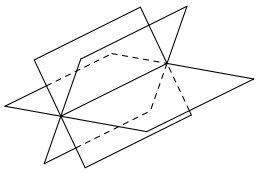
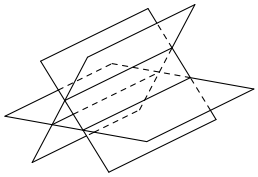
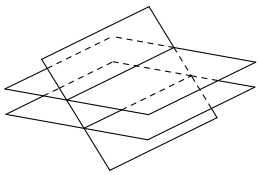
$$\begin{cases} A_1x + B_1y + C_1z = E_1 \\ A_2x + B_2y + C_2z = E_2 \\ A_3x + B_3y + C_3z = E_3 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & E_1 \\ A_2 & B_2 & C_2 & E_2 \\ A_3 & B_3 & C_3 & E_3 \end{array}\right)$$

The matrix above is called the ***augmented matrix***.
We will use a method called ***Gaussian elimination*** or ***row reduction*** to change the system into an equivalent one in one of the following forms:

(1) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$ **(2)** $\left(\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$ **(3)** $\left(\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & \# \end{array}\right)$ **(4)** $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right)$

where $*$ denotes a real number and $\#$ denotes non-zero real number.
All matrices shown above are said to be in ***reduced row echelon form*** (i.e. with 1 on the diagonal or 0 if it is not possible to get 1 and zeroes everywhere below and above ones).

What is the meaning of such results?
Let $A = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$ be the matrix of coefficients of the equations.
The table below shows the main cases to be considered.

	(1)	**(2)**	**(3)**
rref [1]	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$	$\left(\begin{array}{ccc\|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & \# \end{array}\right)$
(matrix A) [2]	non-singular $\det A \neq 0$	singular $\det A = 0$	singular $\det A = 0$
			
1 - reduced row echelon form 2 - non-obligatory			

Gaussian elimination (row reduction)

Gaussian elimination is a method to transform a system of equations (given in a matrix form) into reduced row echelon form.
OPERATIONS ALLOWED
The following operations on equations (rows in a matrix) will be conducted:

I. multiplying a row by a non-zero constant $t \in \mathbb{R}$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} A & B & C & E \\ * & * & * & * \\ * & * & * & * \end{array}\right) \begin{array}{c} tR_1 \\ R_2 \\ R_3 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} tA & tB & tC & tE \\ * & * & * & * \\ * & * & * & * \end{array}\right)$$

II. replacing a row with a sum of two rows

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}A_1&B_1&C_1&E_1\\A_2&B_2&C_2&E_2*&*&*&*\end{array}\right)\begin{matrix}R_1\\R_1+R_2\\R_3\end{matrix}\Leftrightarrow\left(\begin{array}{ccc|c}A_1&B_1&C_1&E_1\\A_1+A_2&B_1+B_2&C_1+C_2&E_1+E_2*&*&*&*\end{array}\right)$$

III. combination of both operations

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}A_1&B_1&C_1&E_1\\A_2&B_2&C_2&E_2*&*&*&*\end{array}\right)\begin{matrix}R_1\\tR_1+R_2\\R_3\end{matrix}\Leftrightarrow\left(\begin{array}{ccc|c}A_1&B_1&C_1&E_1\\tA_1+A_2&tB_1+B_2&tC_1+C_2&tE_1+E_2*&*&*&*\end{array}\right)$$

METHOD

$$\left(\begin{array}{ccc|c}*&*&*&**&*&*&**&*&*&*\end{array}\right)\Leftrightarrow\left(\begin{array}{ccc|c}*&*&*&*\\\boxed{0}&*&*&*\\0&*&*&*\end{array}\right)\Leftrightarrow\left(\begin{array}{ccc|c}*&*&*&*\\\boxed{0}&*&*&*\\\boxed{0}&0&*&*\end{array}\right)\Leftrightarrow\left(\begin{array}{ccc|c}*&*&\boxed{0}&*\\0&*&\boxed{0}&*\\0&0&*&*\end{array}\right)\Leftrightarrow\left(\begin{array}{ccc|c}*&\boxed{0}&0&*\\0&*&0&*\\0&0&*&*\end{array}\right)$$

EXAMPLE 227 (Gaussian elimination).
Solve the equations simultaneously to find the intersection of the planes.

$$\begin{cases}x-6y+2z=1\\4x+2y+3z=6\\3x+4y+5z=-5\end{cases}$$

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}1&-6&2&1\\4&2&3&6\\3&4&5&-5\end{array}\right)\begin{matrix}R_1\\R_2-4R_1\\R_3-3R_1\end{matrix}\Leftrightarrow$$

to get zeroes **below** the diagonal in the **first** column we will change R_2 and R_3 with use of R_1 as follows:
$(R_2-4R_1)\to R_2$;
$(R_3-3R_1)\to R_3$

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}1&-6&2&1\\0&26&-5&2\\0&22&-1&-8\end{array}\right)\begin{matrix}R_1\\R_2\\R_3-\frac{22}{26}R_2\end{matrix}\Leftrightarrow$$

to get zero **below** the diagonal in the **second** column we will change R_3 with use of R_2 as follows:
$(R_3-\frac{22}{26}R_2)\to R_3$

$$\left(\begin{array}{ccc|c}1&-6&2&1\\0&26&-5&2\\0&0&-1-\frac{22}{26}(-5)&-8-\frac{22}{26}\times 2\end{array}\right)\begin{matrix}R_1\\R_2\\R_3\end{matrix}\Leftrightarrow\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}1&-6&2&1\\0&26&-5&2\\0&0&\frac{42}{13}&-\frac{126}{13}\end{array}\right)\begin{matrix}R_1\\R_2\\\frac{13}{42}R_3\end{matrix}\Leftrightarrow$$

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}1&-6&2&1\\0&26&-5&2\\0&0&1&-3\end{array}\right)\begin{matrix}R_1+2R_3\\R_2+5R_3\\R_3\end{matrix}\Leftrightarrow$$

to get zeroes **above** the diagonal in the **third** column we will change R_1 and R_2 with use of R_3 as follows:
$(R_1-2R_3)\to R_1$;
$(R_2+5R_3)\to R_2$)

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}1&-6&0&7\\0&26&0&-13\\0&0&1&-3\end{array}\right)\begin{matrix}R_1\\\frac{1}{26}R_2\\R_3\end{matrix}\Leftrightarrow$$

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}1&-6&0&7\\0&1&0&-\frac{1}{2}\\0&0&1&-3\end{array}\right)\begin{matrix}R_1+6R_2\\R_2\\R_3\end{matrix}\Leftrightarrow$$

to get zero **above** the diagonal in the **second** column we will change R_1 with use of R_2 as follows:
$(R_1+6R_2)\to R_1$

$\left(\begin{array}{ccc|c}1&0&0&4\\0&1&0&-\frac{1}{2}\\0&0&1&-3\end{array}\right)$ i.e. $x=4$, $y=-\frac{1}{2}$, $z=-3$ therefore the planes intersect at $(4,-\frac{1}{2},-3)$.

EXAMPLE 228 (Gaussian elimination).
Solve the equations simultaneously to find the intersection of the planes.

$$\begin{cases}3x-2y-7z=1\\-2x+3y+8z=6\\x+y+z=7\end{cases}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|c} 3 & -2 & -7 & 1 \\ -2 & 3 & 8 & 6 \\ 1 & 1 & 1 & 7 \end{array}\right)\begin{matrix} R_3 \\ R_2 \\ R_1 \end{matrix} \Leftrightarrow$$

Let's firstly swap R_1 and R_3 to get 1 as a leading coefficient in the first row.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ -2 & 3 & 8 & 6 \\ 3 & -2 & -7 & 1 \end{array}\right)\begin{matrix} R_1 \\ R_2+2R_1 \\ R_3-3R_1 \end{matrix} \Leftrightarrow \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 5 & 10 & 20 \\ 0 & -5 & -10 & -20 \end{array}\right)\begin{matrix} R_1 \\ \frac{1}{5}R_2 \\ R_3+R_2 \end{matrix} \Leftrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This form shows that there is **no unique** solution. However,the solution exists. The three planes intersect in a line.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{matrix} R_1-R_2 \\ R_2 \\ R_3 \end{matrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x-z &= 3 \\ y+2z &= 4 \end{cases} \Leftrightarrow \begin{cases} x = 3+1z \\ y = 4-2z \\ z = 0+1z \end{cases}$$

Substituting $z = \lambda$ we get:

$$\begin{cases} x = 3+1\lambda \\ y = 4-2\lambda \\ z = 0+1\lambda \end{cases} \quad \text{OR } \vec{r} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

which is the equation of the line of intersection of the three planes.

Questions 29.1.

Q1. Solve the following systems of equations. Describe clearly, whether the planes intersect and what is their intersection.

(1) $\begin{cases} 2x-y+3z=1 \\ 5x+y-z=3 \\ x+3y-7z=1 \end{cases}$

(2) $\begin{cases} 2x+y-3z=5 \\ 3x-4y+z=2 \\ -4x-2y+6z=-10 \end{cases}$

(3) $\begin{cases} x+2y-z=-2 \\ -3x+y=4 \\ -2x+3y-z=2 \end{cases}$

(4) $\begin{cases} 2x-3y-z=-23 \\ -3x+y=18 \\ x+2y+z=-5 \end{cases}$

(5) $\begin{cases} x-y-2z=2 \\ 3x-2y-z=3 \\ -3x+y-3z=-1 \end{cases}$

(6) $\begin{cases} 2x+3y-4z=0 \\ 3x-2y-5z=7 \\ -x+y+3z=-2 \end{cases}$

(7) $\begin{cases} 2x-3y+3z=5 \\ -x+y-z=-1 \\ -2x-y+3z=2 \end{cases}$

(8) $\begin{cases} 2x+y-7z=7 \\ 4x-3y-9z=-1 \\ x-y-2z=-1 \end{cases}$

(9) $\begin{cases} 2x-3y+3z=22 \\ -x+y-z=-9 \\ x-4y+4z=21 \end{cases}$

(10) $\begin{cases} 4x+2y+4z=-11 \\ 2x+4y-7z=-4 \\ 4x+4y-2z=-10 \end{cases}$

(11) $\begin{cases} -3x+3y+z=-22 \\ 2x-2y-z=17 \\ x-y-2z=19 \end{cases}$

(12) $\begin{cases} 4x-2y+3z=5 \\ 2x-y=-2 \\ -3x+2y+z=1 \end{cases}$

(13) $\begin{cases} x+2y+3z=1 \\ -2x+z=-5 \\ 2y+4z=-2 \end{cases}$

(14) $\begin{cases} x-y+z=5 \\ 4x+2y-3.5z=-19 \\ -2x-y+2z=10 \end{cases}$

(15) $\begin{cases} x-3y+z=-6 \\ -3x+y=7 \\ x+2y-z=1 \end{cases}$

(16) $\begin{cases} -x-y-z=0 \\ -3x-2y-3z=1 \\ -7x-5y-8z=4 \end{cases}$

(17) $\begin{cases} 5x-2y+z=30 \\ -4x+y+3z=-19 \\ 2x-y+6z=18 \end{cases}$

(18) $\begin{cases} \frac{1}{2}x-\frac{1}{3}y+\frac{5}{6}z=-\frac{2}{3} \\ 2x+y+\frac{3}{2}z=8 \\ \frac{1}{6}x+y-\frac{1}{3}z=\frac{13}{3} \end{cases}$

(19) $\begin{cases} -2x+4y+6z=-4 \\ x-2y-3z=2 \\ 3x-6y-9z=6 \end{cases}$

(20) $\begin{cases} 2x+3y-5z=1 \\ x-2y+3z=0 \\ 3x+y-2z=2 \end{cases}$

(21) $\begin{cases} 2x+3y-5z=1 \\ 5x-7y+2z=2 \\ x-13y+12z=0 \end{cases}$

$$(22)\quad \begin{cases} -x+y-2z=2 \\ 2x-2y+z=1 \\ 2x-2y+4z=7 \end{cases}$$

Q2. Find the intersection of the line and the plane.

(1) $\vec{r}=\vec{i}-3\vec{j}-6\vec{k}+\lambda(\vec{i}+2\vec{j}+5\vec{k})$ and $2x+3y-z=2$

(2) $\vec{r}=2\vec{i}-\vec{j}+3\vec{k}+\lambda(2\vec{i}-\vec{k})$ and $3x-y+2z=5$

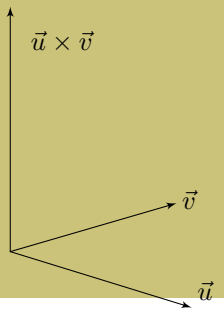
(3) $\frac{x+1}{-3}=\frac{y-1}{2}=\frac{9-z}{4}$ and $x-y+z+2=0$

(4) $\frac{x+1}{2}=y-2=\frac{z-2}{3}$ and $2x-y-z+6=0$

29.2 Vector product

Definition 229 (vector product).
*The **vector product** (cross product) of two 3-dimensional vectors $\vec{u}$ and $\vec{v}$ is the third vector denoted as $\vec{u}\times\vec{v}$ such that:*

1. *$\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$*
2. *$|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$*
3. *orientation of $\vec{u}\times\vec{v}$ is determined by the so-called **right-hand rule** (if a thumb points direction of $\vec{u}$, an index finger - direction of $\vec{v}$ then the middle finger points the direction of $\vec{u}\times\vec{v}$)*



Evaluation of a vector product

To evaluate the vector product of two given vectors we can use the determinant representation[3].

Theorem 230 (determinant form of a vector product).
For vectors $\vec{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ their vector product $\vec{u}\times\vec{v}$ can be evaluated as follows:

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}=\vec{i}\begin{vmatrix}u_2&u_3\\v_2&v_3\end{vmatrix}-\vec{j}\begin{vmatrix}u_1&u_3\\v_1&v_3\end{vmatrix}+\vec{k}\begin{vmatrix}u_1&u_2\\v_1&v_2\end{vmatrix}$$

We can use the following formula, too.

Theorem 231 (vector product formula).
For vectors $\vec{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ their vector product $\vec{u}\times\vec{v}$ can be evaluated as follows:

$$\vec{u}\times\vec{v}=\begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix}$$

[3] which is easier, but not required by the IB

Main applications of a vector product

FACTS, REMARKS, OBSERVATIONS.

(1) Finding an equation $\alpha x + \beta y + \gamma z = \delta$ of the plane passing through given points A, B and C.

Vector normal to the plane is $\vec{n} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \vec{AB} \times \vec{AC}$

(2) Evaluating area of triangle ABC.

$Area_{ABC} = \frac{1}{2}|\vec{AB} \times \vec{AC}|$

29.3 PROPERTIES OF SCALAR AND VECTOR PRODUCT

PROPERTIES OF A SCALAR AND OF A VECTOR PRODUCT.

name	***scalar product*** *(dot product)*	***vector product*** *(cross product)*
notation	$\vec{u} \cdot \vec{v}$	$\vec{u} \times \vec{v}$
angle	$\vec{u} \cdot \vec{v} = \lvert\vec{u}\rvert\lvert\vec{v}\rvert \cos\theta$	$\lvert\vec{u} \times \vec{v}\rvert = \lvert\vec{u}\rvert\lvert\vec{v}\rvert \sin\theta$
zero value	$\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$	$\vec{u} \times \vec{v} = 0 \Leftrightarrow \vec{u} \parallel \vec{v}$
	or $\vec{u} = 0$ *or* $\vec{v} = 0$	
commutativity	*YES* $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$	*NO* $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
associativity	*NO* *(why?)*	*NO* $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$
distributivity over addition	*YES* $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$	*YES* $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
		$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{i} = -\vec{k}$ $\vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{j} = -\vec{i}$ $\vec{k} \times \vec{i} = \vec{j}, \quad \vec{i} \times \vec{k} = -\vec{j}$

QUESTIONS 29.2.

Q3. Given $A = (-3, 5, 1)$, $B = (-1, 9, 5)$, $C = (-5, 14, -5)$

(i) Find $\vec{AB} \times \vec{AC}$.

Hence:

(ii) Find the equation of a plane passing through A, B and C.

(iii) Calculate the area of triangle ABC.

Q4. Given $A = (4, -3, 2)$, $B = (-6, 2, -8)$, $C = (7, -5, -4)$

(i) Find $\vec{AB} \times \vec{AC}$.

Hence:

(ii) Find the equation of a plane passing through A, B and C.

(iii) Calculate the area of triangle ABC.

Q5. Given $A = (1, -2, -2)$, $B = (-5, 8, 2)$, $C = (10, -17, -8)$

(i) Find $\vec{AB} \times \vec{AC}$.

Hence:

(ii) Find the equation of a plane passing through A, B and C.

(iii) Calculate the area of triangle ABC.

Q6. Given $A = (-2, 3, -2)$, $B = (-5, 4, 2)$, $C = (0, 2, 3)$

(i) Find D such that $ABCD$ is a parallelogram.

(ii) Find $\vec{AB} \times \vec{AD}$.

(iii) Hence find the area of $ABCD$.

Q7. Given $A = (-1, 9, 5)$, $B = (-5, 6, 2)$, $C = (-2, 4, 3)$

(i) Find D such that $ABCD$ is a parallelogram.

(ii) Find $\vec{AB} \times \vec{AD}$.

(iii) Hence find the area of $ABCD$.

Q8. Given the lines $\vec{r} = 2\vec{i} - 3\vec{j} - \vec{k} + \lambda(\vec{i} + 2\vec{j} - 3\vec{k})$ and $\vec{r} = 2\vec{i} - 3\vec{j} - \vec{k} + \mu(-2\vec{i} + \vec{j} + 2\vec{k})$ find the Cartesian equation of a plane that contains both lines.

Q9. Given the lines $\vec{r} = -\vec{i} + 2\vec{k} + \lambda(2\vec{i} - \vec{j} - \vec{k})$ and $\vec{r} = -\vec{i} + 2\vec{k} + \mu(3\vec{i} + 2\vec{j} + 4\vec{k})$ find the Cartesian equation of a plane that contains both lines.

Q10. Given the lines: $\frac{x-2}{3} = \frac{3-y}{2} = \frac{z+1}{4}$ and $\frac{2x-4}{3} = y - 3 = \frac{2z+2}{4}$ find the Cartesian equation of a plane that contains both lines.

Q11. Given $A = (1, -2, -2)$, $B = (-3, 2, 1)$, $C = (5, -4, -3)$, $D = (8, 8, -15)$:

(i) Find the equation of a plane π passing through A, B and C.

(ii) Write down a vector normal to π.

(iii) Write down the equation of the line L that passes through C and is perpendicular to π.

(iv) Verify that D lies on L.

(v) Find the area of the triangle ABC.

(vi) Find the volume of the pyramid $ABCD$.

Q12. Given $A = (5, 0, 2)$, $B = (2, 3, -1)$, $C = (1, 2, -6)$, $D = (4, -1, -3)$.

(i) Show that the quadrilateral $ABCD$ is a rhombus.

(ii) Find the equation of a plane π passing through A, B, C and D.

(iii) Write down a vector normal to π.

(iv) Find M, the point of intersection of diagonals of $ABCD$.

(v) Write down the equation of the line L that passes through M and is perpendicular to π in a vector or parametric form.

Let P be the point on L with parameter t.

(vi) Express $\vec{PA}$ in terms of t.

(vii) Find the value of t such that $|\vec{PA}| = 7$.

(viii) Hence find the volume of the pyramid $ABCDP$.

Q13. Let $\vec{u}$ and $\vec{v}$ be any vectors.

(1) Show that $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

(2) Show that $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$.

(3) Given $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$ find $\vec{u} \cdot \vec{v}$.

(4) Given that $|\vec{u}| = |\vec{v}|$ find the angle between $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$.

(5) Show that $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$.

(6) Show that $|\vec{u}| \leq |\vec{v}| + |\vec{u} - \vec{v}|$.

(7) Show that $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$.

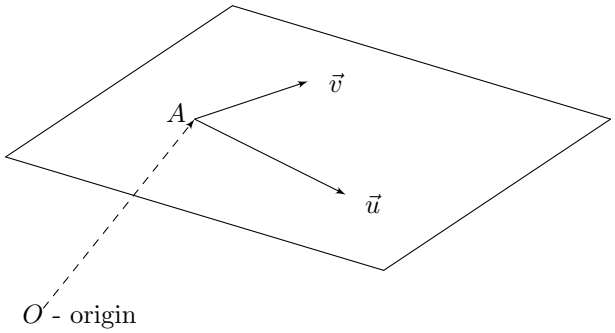
Q14. Find the intersection of $x - 4y - 5z = 6$ and $\frac{x-3}{2} = \frac{y+2}{3} = \frac{1-z}{2}$.

Q15. Find the equation of the plane containing line $\frac{x-2}{2} = 1 - y = \frac{2z+6}{3}$ and point $(-1, 2, 0)$.

Q16. Find the equation of the plane parallel to y-axis and passing through points $(2, 3, -5)$ and $(-1, 3, -2)$.

Q17. Evaluate the acute angle between plane $5x - 3y + 4z = 11$ and y-axis. Find the intersection of the plane and y-axis.

29.4 Alternate forms of equations of planes



Definition 232 (vector equation of a plane).
Let $\vec{u}$ and $\vec{v}$ be two non-parallel vectors that spread the plane π, i.e. each of them is parallel to π. Let $\vec{a}$ be a position vector of a point in the plane. Then the ***vector equation*** *of π is*

$$\vec{r} = \vec{a} + \lambda\vec{u} + \mu\vec{v}$$

where $\lambda, \mu \in \mathbb{R}$.

Theorem 233 (normal vector equation of a plane).
Let $\vec{n}$ be a vector normal to the plane π. Let $\vec{a}$ be a position vector of a point in the plane. Then the equation

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{a}$$

describes the plane π. The equation is called ***normal vector equation of a plane****.*

Proof. Let the Cartesian equation of the plane π be $Ax + By + Cz = D$. Then vector $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is normal to π, so let $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$.
The Cartesian equation of π can be transformed:

$$Ax + By + Cz = D \Leftrightarrow \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = D$$

To get the value of D one should substitute $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with position vector of any point on the plane, let it be $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

The above equation may be rewritten in the form:

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{a}$$

□

Questions 29.3.

Q18. Express the following equations of planes in Cartesian form.

(1) $\vec{r} = 5\vec{i} - 2\vec{j} + 4\vec{k} + \lambda(\vec{i} - \vec{j} + 2\vec{k}) + \mu(3\vec{i} + 2\vec{j})$

(2) $\vec{r} = -3\vec{i} + 2\vec{k} + \lambda(2\vec{i} - \vec{j} - \vec{k}) + \mu(-\vec{i} + 2\vec{j} + \vec{k})$

(3) $\begin{cases} x = 2 + \lambda - 2\mu \\ y = -1 - 2\lambda + \mu \\ z = 3 + 2\lambda - \mu \end{cases}$

Q19. Express the following equations of planes in vector form.

(1) $2x - 4y + z = 5$

(2) $5x - 3y + 6z + 2 = 0$

(3) $x + 2y = 3$

(4) $z = 5$

(5) $\vec{r} \cdot (2\vec{i} + 3\vec{j} - 5\vec{k}) = 5$

Q20. Express the following equations of planes in normal vector form.

(1) $2x - 4y + z = 5$

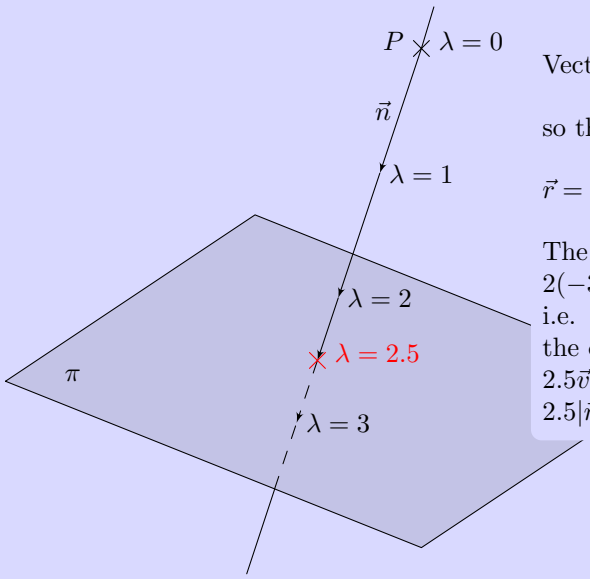
(2) $5x - 3y + 6z + 2 = 0$

(3) $\vec{r} = -\vec{i} + \vec{j} + 2\vec{k} + \lambda(\vec{i} - 3\vec{j} - \vec{k}) + \mu(-\vec{i} + \vec{j} + 2\vec{k})$

29.5 3D distances

Distance between a point and a plane

Example 234 (distance between a point and a plane).
Find the distance between point $P(-3, 5, -2)$ and plane π: $2x - y + 3z = 19$.



Vector $\vec{n}$ normal to π is $\vec{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$
so the line through P normal to π is
$\vec{r} = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.
The line intersects π when:
$2(-3 + 2\lambda) - (5 - \lambda) + 3(-2 + 3\lambda) = 19$,
i.e. $\lambda = 2.5$ which means that
the distance between P and π is shown by
$2.5\vec{v}$, so it is equal to:
$2.5|\vec{n}| = 2.5\sqrt{2^2 + 1^2 + 3^2} = \frac{5}{2}\sqrt{14}$

Questions 29.4.

Q21. Plane π is given by $2x - 5y + z - 3 = 0$. Point A is $(-4, 9, -4)$.

(i) Write down a vector normal to π.

(ii) Write down a vector equation of line l that passes through A and is perpendicular to π.

(iii) Find the intersection of l and π.

(iv) Hence calculate the distance of A from π.

Q22. Calculate the distance between point $(\frac{19}{2}, -5, \frac{33}{2})$ and the plane $3x - 2y + 7z + 1 = 0$.

Q23. Calculate the distance between the point $(-5, 7, 2)$ and the plane $x = 3$.

Q24. Calculate the distance between the point $(0, 0, 7)$ and the plane $x + 3z - 1 = 0$.

Distance between two (parallel) planes

Q25. Planes π_1 and π_2 are given by the equations $x + 3y - 2z = 14$ and $x + 3y - 2z + 14 = 0$ respectively.

(i) Find A - any point on π_1.
(ii) Calculate the distance between A and π_2.
(iii) Write down the distance between the planes.

Q26. Calculate the distance between the planes: $7x - y + z = 34$ and $7x - y + z + 119 = 0$.

Q27. Calculate the distance between the planes: $12x - 3y + 4z + 10 = 0$ and $12x - 3y + 4z = 159$.

Q28. Calculate the distance between the planes: $3x - 4y + 5z = 8$ and $4x - 5y + 3z + 2 = 0$.

Distance between a point and a line

Q29. (1st method) Consider point $A(1, 1, -1)$ and line $l : \frac{x}{3} = \frac{7-y}{5} = \frac{2z+7}{4}$.

(i) Write down an equation of l in parametric form with a parameter t.
(ii) Write down the position vector of B - a point on l in terms of t.
(iii) Find $\vec{AB}$.
(iv) What is the scalar product of $\vec{AB}$ and the direction vector of the line when they are perpendicular? Find B such that $\vec{AB}$ is perpendicular to the line.
(v) Calculate the distance between A and l.

Q29. (2nd method) Consider point $A(1, 1, -1)$ and line $l : \frac{x}{3} = \frac{7-y}{5} = \frac{2z+7}{4}$..

(i) Write down $\vec{n}$ - a direction vector of l.
(ii) Find the equation of π - plane normal to $\vec{n}$ and passing through A.
(iii) Find B - point of intersection of π and l.
(iv) Hence find the distance between A and l.

Q29. (3rd method) Consider point $A(1, 1, -1)$ and line $l : \frac{x}{3} = \frac{7-y}{5} = \frac{2z+7}{4}$.

(i) Write down $\vec{n}$ - a direction vector of l.
(ii) Write down B - any point on l.

Let Q be the projection of A on l.

(iii) Find $\vec{BQ}$, the projection of $\vec{BA}$ on l.[4]
(iv) Considering magnitudes of vectors $\vec{BQ}$ and $\vec{BA}$ find the distance between A and Q.
(v) Write down the distance between A and l.

Q30. Find the distance between point $(1, \frac{1}{2}, -2)$ and line $\frac{x+2}{7} = \frac{y}{2} = \frac{1-x}{3}$.

Q31. Find the distance between point $(3, 2, 1)$ and line $\begin{cases} x = 4 \\ \frac{y+2}{3} = \frac{z+2}{-4} \end{cases}$.

Distance between parallel lines

Q32. Lines l_1 and l_2 are given by the equations $\frac{1-x}{2} = \frac{y+1}{3} = -z - 10$ and $\frac{2-x}{2} = \frac{y-5}{3} = 6 - z$ respectively.

(i) Write l_2 in parametric form.
(ii) Find A - any point on l_1.
(iii) Calculate the distance between A and l_2.
(iv) Write down the distance between the lines.

Q33. Find the distance between the lines $\frac{x-1}{4} = \frac{3-2y}{4} = \frac{z+2}{3}$ and $\frac{x+2}{4} = \frac{3-y}{2} = \frac{z-3}{3}$.

Q34. Find the distance between the lines $x = y - 2 = \frac{1-x}{2}$ and $x + 6 = y - 2 = \frac{z-1}{-2}$.

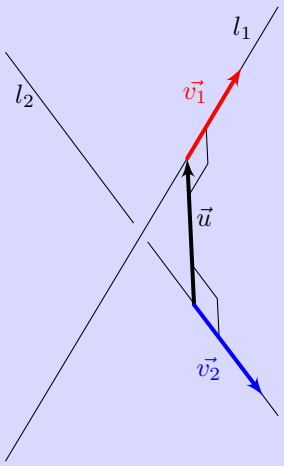
Q35. Find the distance between the lines $\begin{cases} \frac{x+1}{3} = \frac{y+1}{-4} \\ z = -1 \end{cases}$ and $\begin{cases} \frac{x-6}{3} = \frac{-2-y}{4} \\ z = 1 \end{cases}$.

[4] The projection of $\vec{u}$ on $\vec{v}$ is given by $\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$. Why?

Distance between skew lines

EXAMPLE 235 (distance between lines).

Find the distance between lines l_1 given by $\vec{r} = \begin{pmatrix}2\\4\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\-2\\1\end{pmatrix}$ and l_2 given by $\vec{r} = \begin{pmatrix}-1\\2\\14\end{pmatrix} + \mu\begin{pmatrix}-1\\2\\2\end{pmatrix}$.



Let $\vec{v_1}$ be a direction vector of l_1, $\vec{v_1} = \begin{pmatrix}4\\-2\\1\end{pmatrix}$

and $\vec{v_2}$ be a direction vector of l_2, $\vec{v_2} = \begin{pmatrix}-1\\2\\2\end{pmatrix}$.

Let $\vec{u}$ be a vector connecting a point on l_1 with a point on l_2:

Then $\vec{u} = \begin{pmatrix}2+4\lambda\\4-2\lambda\\3+\lambda\end{pmatrix} - \begin{pmatrix}-1-\mu\\2+2\mu\\14+2\mu\end{pmatrix} = \begin{pmatrix}3+4\lambda+\mu\\2-2\lambda-2\mu\\-11+\lambda-2\mu\end{pmatrix}$

Vector $\vec{u}$ shows the shortest distance when:

$\vec{u} \perp \vec{v_1}$ and $\vec{u} \perp \vec{v_2}$

i.e. $\vec{u} \cdot \vec{v_1} = 0$ and $\vec{u} \cdot \vec{v_2} = 0$

$$\vec{u} \cdot \vec{v_1} = \begin{pmatrix}3+4\lambda+\mu\\2-2\lambda-2\mu\\-11+\lambda-2\mu\end{pmatrix} \cdot \begin{pmatrix}4\\-2\\1\end{pmatrix} = 0$$

$$\vec{u} \cdot \vec{v_2} = \begin{pmatrix}3+4\lambda+\mu\\2-2\lambda-2\mu\\-11+\lambda-2\mu\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\2\end{pmatrix} = 0$$

$$\Leftrightarrow \begin{cases}7\lambda+2\mu=1\\-2\lambda-3\mu=7\end{cases} \Leftrightarrow \begin{cases}\lambda=1\\\mu=-3\end{cases}$$

Finally $\vec{u} = \begin{pmatrix}3+4\times1+(-3)\\2-2\times1-2(-3)\\-11+1-2(-3)\end{pmatrix} = \begin{pmatrix}4\\6\\-4\end{pmatrix} = 2\begin{pmatrix}2\\3\\-2\end{pmatrix}$,

so the distance is $2\sqrt{2^2+3^2+(-2)^2} = 2\sqrt{17}$.

QUESTIONS 29.5.

Q36. (1st method) Lines l_1 and l_2 are given by the equations $\frac{x+2}{3} = \frac{y-8}{3} = \frac{z-3}{4}$ and $x+4 = \frac{y+7}{2} = \frac{z-2}{3}$ respectively.

(i) Write down an equation of l_1 in parametric form with a parameter t.

(ii) Write down an equation of l_2 in parametric form with a parameter s.

(iii) Write down the position vector of A - a point on l_1 in terms of t.

(iv) Write down the position vector of B - a point on l_2 in terms of s.

(v) Find $\vec{AB}$ in terms of t and s.

(vi) Form and solve the system of equations for which $\vec{AB}$ is perpendicular to the lines l_1 and l_2 (i.e. their direction vectors).

(vii) Find the distance between the lines.

Q36. (2nd method) Lines l_1 and l_2 are given by the equations $\frac{x+2}{3} = \frac{y-8}{3} = \frac{z-3}{4}$ and $x+4 = \frac{y+7}{2} = \frac{z-2}{3}$ respectively.

(i) Write down $\vec{u}$ - a direction vector of l_1.

(ii) Write down $\vec{v}$ - a direction vector of l_2.

(iii) Find A - any point on l_1.

(iv) Find B - any point on l_2.

(v) Find $\vec{n}$ - a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

(vi) Find the equation of π - the plane that passes through A and is perpendicular to $\vec{n}$.

(vii) Does π contain l_1? (yes - why?)

(viii) Does l_2 intersect ? (no - why?)

(ix) Find the distance between B and π.

(x) Write down the distance between l_1 and l_2.

Q36. (3rd method) Lines l_1 and l_2 are given by the equations $\frac{x+2}{3} = \frac{y-8}{3} = \frac{z-3}{4}$ and $x+4 = \frac{y+7}{2} = \frac{z-2}{3}$ respectively.

(i) Write down $\vec{u}$ - a direction vector of l_1.

(ii) Write down $\vec{v}$ - a direction vector of l_2.

(iii) Find A - any point on l_1.

(iv) Find B - any point on l_2.

(v) Find $\vec{n}$ - a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

Let P and Q be the points on l_1 and l_2 respectively such that $\vec{PQ}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. $\vec{PQ}$ is then the projection of $\vec{AB}$ on $\vec{n}$. (Why?)

(vi) Find $\vec{PQ}$.[4]

(vii) Hence find the distance between l_1 and l_2.

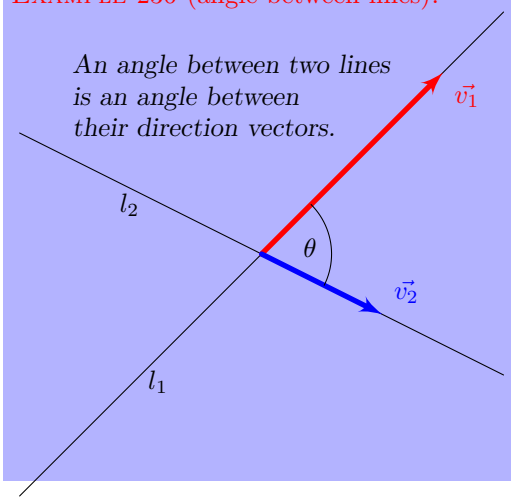
Q37. Find the distance between the lines: $\frac{x-10}{2} = -y-6 = \frac{z-12}{7}$ and $\begin{cases} \frac{3-x}{3} = \frac{y-3}{2} \\ z = 1 \end{cases}$.

Q38. Find the distance between the lines: $4-x = \frac{y+6}{2} = z-9$ and $x-1 = \frac{y+3}{-2} = -z$.

29.6 Angles between lines, planes...

Angle between two lines

Example 236 (angle between lines).



To find the angle θ between the lines

$$l_1: \quad \vec{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \quad \text{and } l_2: \quad \vec{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$$

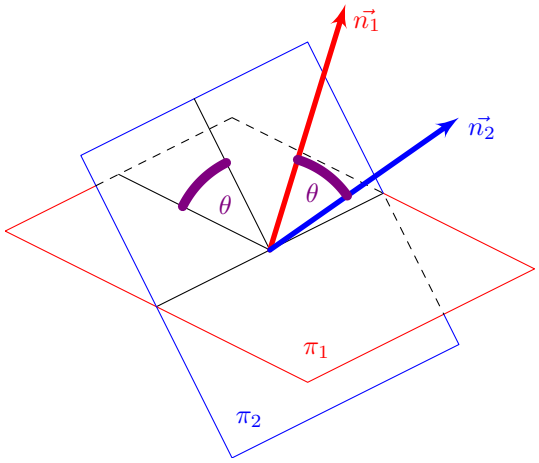
we will consider their direction vectors

$$\vec{v_1} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \text{ and } \vec{v_2} = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}.$$

$$\cos\theta = \frac{\vec{v_1} \cdot \vec{v_2}}{|\vec{v_1}||\vec{v_2}|} = \frac{(4)(-2)+(1)(3)+(-2)(3)}{\sqrt{4^2+1^2+2^2}\sqrt{2^2+3^2+3^2}} = \frac{-11}{\sqrt{21}\sqrt{22}}$$

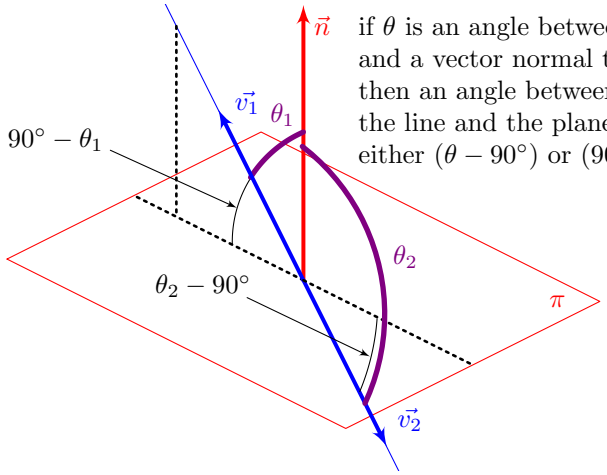
Hence $\theta \approx 121°$. If we are looking for the acute angle between the lines, the answer is $180° - 121° \approx 59.2°$.

Angle between two planes



an angle between
planes π_1 and π_2
is equal to
an angle between
their normal vectors
$\vec{n_1}$ and $\vec{n_2}$

Angle between line and plane



if θ is an angle between the line and a vector normal to the plane then an angle between the line and the plane is either $(\theta - 90°)$ or $(90° - \theta)$

EXAMPLE 237 (angle between a line and a plane).

To find the angle between the plane $3x - y + 2z + 5 = 0$ *and the line* $\vec{r} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ *we will consider the direction vector of a line* $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ *and the vector normal to the plane* $\vec{n} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

The angle θ *between* $\vec{v}$ *and* $\vec{n}$ *is such that:*

$$\cos\theta = \frac{(-2)(3) + (1)(-1) + (4)(2)}{\sqrt{2^2 + 1^2 + 4^2}\sqrt{3^2 + 1^2 + 2^2}} = \frac{1}{\sqrt{21}\sqrt{14}}$$

Hence $\theta \approx 86.7°$ *and the angle between the plane and the line is* $90° - \theta \approx 3.34°$.

QUESTIONS 29.6.

Q39. Find the angle between lines: $4 - x = \frac{y+6}{2} = z - 9$ and $x - 1 = \frac{y+3}{-2} = -z$. Give your answer to the nearest degree.

Q40. Find the acute angle between lines $\frac{x-10}{2} = -y - 6 = \frac{z-12}{7}$ and $\begin{cases} \frac{3-x}{3} = \frac{y-3}{2} \\ z = 1 \end{cases}$. Give your answer to the nearest degree.

Q41. Find the acute angle between line $\frac{x-10}{2} = -y - 6 = \frac{z-12}{7}$ and plane $4x - 5y + 3z + 2 = 0$. Give your answer to the nearest degree.

Q42. Find the angle between line $\begin{cases} x = 4 \\ \frac{y+2}{3} = \frac{z+2}{-4} \end{cases}$ and plane $x + 3y - 2z + 14 = 0$. Give your answer to the nearest degree.

Q43. Find the acute angle between the planes $4x - 5y + 3z + 2 = 0$ and $x + 3y - 2z + 14 = 0$.

Q44. Find the acute angle between the planes $3x - 7y + z - 3 = 0$ and $4y - 3z + 5 = 0$.

29.6 Summary of key points

For points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$:
their **position vectors** are:

$$\vec{a} = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} = x_A\vec{i} + y_A\vec{j} + z_A\vec{k} \text{ and}$$

$$\vec{b} = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix} = x_B\vec{i} + y_B\vec{j} + z_B\vec{k}$$

where $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then

$$\vec{AB} = \vec{AO} + \vec{OB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$$

where O denotes the origin $(0, 0, 0)$.

The midpoint of a line segment AB:

$$M = \left(\tfrac{x_A + x_B}{2}, \tfrac{y_A + y_B}{2}, \tfrac{z_A + z_B}{2}\right)$$

For a vector $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ its **magnitude** is

$$|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$$

Hence the distance between A and B is

$$|AB| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

A **unit vector** is a vector $\vec{u}$ whose magnitude is 1, $|\vec{u}| = 1$.
A unit vector in the direction of $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$.

For vectors $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ their
scalar product (or *dot product*) is defined:

(1) $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$,
where θ is the angle between $\vec{u}$ and $\vec{v}$.

or (2) $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Definitions (1) and (2) are equivalent.

Hence for the angle θ between vectors $\vec{u}$ and $\vec{v}$:

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}||\vec{v}|}$$

Note:

$$\vec{u} \perp \vec{v} \quad \Rightarrow \vec{u} \cdot \vec{v} = 0$$

$$\vec{u} \parallel \vec{v} \quad \Rightarrow |\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}|$$

A **matrix** (pl. matrices) is a rectangular array of numbers, e.g.

$$\begin{pmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \end{pmatrix}$$

is an example of 2×3 matrix.

A matrix with n rows and m columns is said to be of *order* $n \times m$.
When $n = m$ a matrix is said to be of order n.

Square $(n \times n)$ matrix I such that for any other $n \times n$ matrix A:

$$AI = IA = A$$

is called **the identity matrix**.

For 2×2 matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For 3×3 matrices:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Square matrix B is called the **inverse matrix** of a square matrix A if and only if

$$AB = BA = I$$

(where I is **the identity matrix**).

Notation:

$$B = A^{-1}, \text{ so}$$
$$AA^{-1} = A^{-1}A = I$$

Determinant of a square 2×2 matrix

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Notation:

$$\det A = |A|$$
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\det A = 0 \Leftrightarrow A^{-1}$ does **not** exist

Matrix A such that $\det A = 0$ is called **singular** / noninvertible.

The result holds for **all square matrices** of any order.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \Rightarrow$$
$$\Rightarrow \det A = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \Rightarrow \det A = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\det(AB) = (\det A)(\det B)$$

$$\det A^{-1} = \frac{1}{\det A}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$
OR $\vec{r} = \vec{a} + \lambda\vec{v}$

VECTOR EQUATION

$$\begin{cases} x = a_1 + \lambda v_1 \\ y = a_2 + \lambda v_2 \\ z = a_3 + \lambda v_3 \end{cases}$$

PARAMETRIC EQUATION

$$\frac{x-a_1}{v_1} = \frac{y-a_2}{v_2} = \frac{z-a_3}{v_3}$$

CARTESIAN EQUATION

$$\begin{cases} \frac{x-a_1}{v_1} = \frac{z-a_3}{v_3} \\ y = a_2 \end{cases}$$

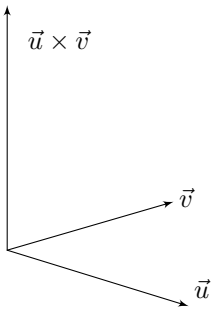
Cartesian equation when one of the coordinates of $\vec{v}$ is 0, e.g. $\vec{v} = \begin{pmatrix} v_1 \\ 0 \\ v_3 \end{pmatrix}$

Cartesian equation of a plane π: $Ax + By + Cz + D = 0$

vector normal to π: $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \qquad \vec{n} \perp \pi$

The **vector product** (cross product) of two 3-dimensional vectors $\vec{u}$ and $\vec{v}$ is the third vector denoted as $\vec{u} \times \vec{v}$ such that:

1. $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$
2. $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$
3. orientation of $\vec{u} \times \vec{v}$ is determined by the so-called **right-hand rule** (if a thumb points direction of $\vec{u}$, an index finger - direction of $\vec{v}$ then the middle finger points the direction of $\vec{u} \times \vec{v}$)



For $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

For $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$:

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

	scalar product (dot product) $\vec{u}\cdot\vec{v}$	**vector product** (cross product) $\vec{u}\times\vec{v}$
	$\vec{u}\cdot\vec{v}=\lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos\theta$	$\lvert\vec{u}\times\vec{v}\rvert=\lvert\vec{u}\rvert\lvert\vec{v}\rvert\sin\theta$
	$\vec{u}\cdot\vec{v}=0\Leftrightarrow\vec{u}\perp\vec{v}$	$\vec{u}\times\vec{v}=0\Leftrightarrow$ **1.** $\vec{u}\parallel\vec{v}$ **or** **2.** $\vec{u}=0$ or $\vec{v}=0$
commutativity	YES $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$	NO $\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$
associativity	NO (why?)	NO $(\vec{u}\times\vec{v})\times\vec{w}\neq\vec{u}\times(\vec{v}\times\vec{w})$
distributivity over addition	YES $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$	YES $\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}$
		$\vec{i}\times\vec{j}=\vec{k},\quad\vec{j}\times\vec{i}=-\vec{k}$ $\vec{j}\times\vec{k}=\vec{i},\quad\vec{k}\times\vec{j}=-\vec{i}$ $\vec{k}\times\vec{i}=\vec{j},\quad\vec{i}\times\vec{k}=-\vec{j}$

	scalar product (dot product) $\vec{u}\cdot\vec{v}$	**vector product** (cross product) $\vec{u}\times\vec{v}$
	$\vec{u}\cdot\vec{v}=\lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos\theta$	$\lvert\vec{u}\times\vec{v}\rvert=\lvert\vec{u}\rvert\lvert\vec{v}\rvert\sin\theta$
	$\vec{u}\cdot\vec{v}=0\Leftrightarrow\vec{u}\perp\vec{v}$	$\vec{u}\times\vec{v}=0\Leftrightarrow$ **1.** $\vec{u}\parallel\vec{v}$ **or** **2.** $\vec{u}=0$ or $\vec{v}=0$
commutativity	YES $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$	NO $\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$
associativity	NO (why?)	NO $(\vec{u}\times\vec{v})\times\vec{w}\neq\vec{u}\times(\vec{v}\times\vec{w})$
distributivity over addition	YES $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$	YES $\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}$
		$\vec{i}\times\vec{j}=\vec{k},\quad\vec{j}\times\vec{i}=-\vec{k}$ $\vec{j}\times\vec{k}=\vec{i},\quad\vec{k}\times\vec{j}=-\vec{i}$ $\vec{k}\times\vec{i}=\vec{j},\quad\vec{i}\times\vec{k}=-\vec{j}$

Vector equation of a plane

$$\vec{r}=\vec{a}+\lambda\vec{u}+\mu\vec{v}$$

where $\vec{u}$ and $\vec{v}$ are non-parallel vectors that spread the plane (and $\lambda,\mu\in\mathbb{R}$).

Normal vector equation of a plane

$$\vec{n}\cdot\vec{r}=\vec{n}\cdot\vec{a}$$

where $\vec{n}$ is vector normal to the plane and
$\vec{a}$ - position vector of a point on the plane.

($\vec{r}$ as in case of vector equation of a line stands for vector $\begin{pmatrix}x\\y\\z\end{pmatrix}$)

Part VIII

COMPLEX NUMBERS

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Chapter 30

Introduction to complex numbers

30.1 Polar coordinates

Definition 238 (polar coordinates).
For a point with cartesian coordinates (x, y) its polar coordinates (r, θ) are defined by to the following rules.

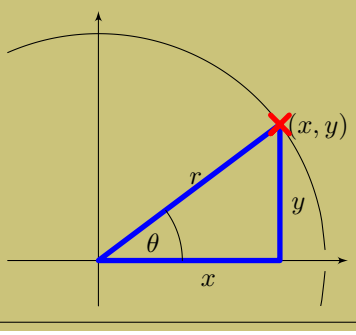
$(x, y) \longrightarrow (r, \theta)$

$x = r\cos\theta$

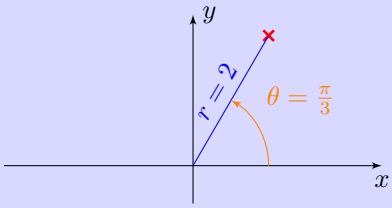
$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x}$



Example 239 (polar to cartesian coordinates).
Express in cartesian coordinates the point given in polar coordinates as $(r, \theta) = (2, \frac{\pi}{3})$.



polar → cartesian

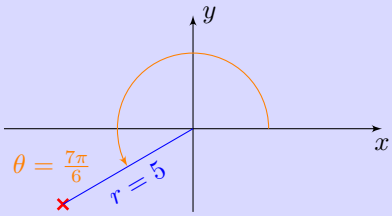
$r = 2$

$\theta = \frac{\pi}{3}$

$x = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$

$y = 2\sin\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

Example 240 (polar to cartesian coordinates).
Express in cartesian coordinates the point given in polar coordinates as $(r, \theta) = (5, \frac{7\pi}{6})$.



polar → cartesian

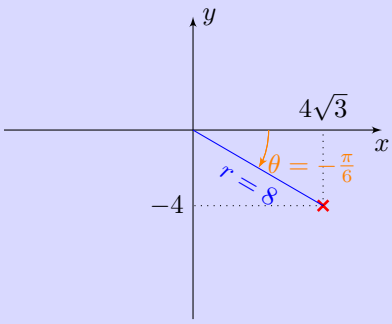
$r = 5$

$\theta = \frac{7\pi}{6}$

$x = 5\cos\frac{7\pi}{6} = -\frac{5}{2}\sqrt{3}$

$y = 5\sin\frac{7\pi}{6} = -\frac{5}{2}$

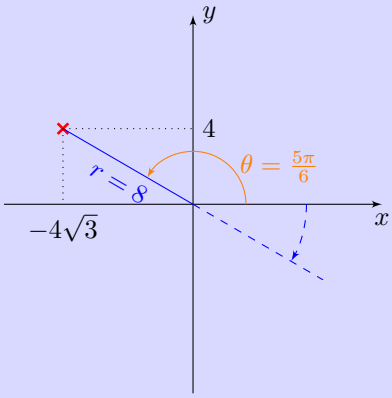
Example 241 (cartesian to polar coordinates).
Express in polar coordinates the point given in cartesian coordinates as $(x, y) = (4\sqrt{3}, -4)$.



$y = -4$

$= \sqrt{16 + 16 \times 3} =$
$= 8$
$\tan\theta = \frac{-4}{4\sqrt{3}} =$
$= -\frac{1}{\sqrt{3}}$
$\theta = \arctan\left(-\frac{1}{\sqrt{3}}\right)$
$= -\frac{\pi}{6}$

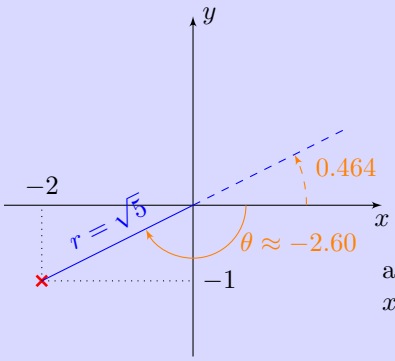
Example 242 (cartesian to polar coordinates).
Express in polar coordinates the point given in cartesian coordinates as $(x, y) = (-4\sqrt{3}, 4)$.



cartesian	→	polar
$x = -4\sqrt{3}$		$r = 8$ (as above)
$y = 4$		$\tan\theta = -\frac{1}{\sqrt{3}}$ (as above)

$\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$,
but $x < 0 \Rightarrow \theta \neq -\frac{\pi}{6}$;

Example 243 (cartesian to polar coordinates).
Express in polar coordinates the point given in cartesian coordinates as $(x, y) = (-2, -1)$.



cartesian	→	polar
$x = -2$		$r = \sqrt{5}$
$y = -1$		$\tan\theta = \frac{1}{2}$

$\arctan\left(\frac{1}{2}\right) \approx 0.464$
$x < 0 \Rightarrow \theta \approx 0.464 - \pi \approx$
≈ -2.60

Questions 30.1.

Q1. Change the cartesian coordinates of the points below into polar form (r, θ), where $r \geq 0$, $\theta \in]-\pi, \pi]$. Where possible, give the answer in exact form.

(1) $(2, 2\sqrt{3})$
(2) $(3, \sqrt{3})$
(3) $(-3, 0)$
(4) $(0, -2)$
(5) $(-5, 5)$
(6) $(-2\sqrt{3}, -6)$
(7) $(4\sqrt{3}, -4)$
(8) $(3, -3)$
(9) $(-2.5, 2.5)$
(10) $(-9, 3\sqrt{3})$
(11) $(\frac{10}{3}, 0)$
(12) $(-8, -8)$
(13) $(-\sqrt{3}, 3)$
(14) $(-6\sqrt{2}, -2\sqrt{6})$
(15) $(0, \sqrt{5})$
(16) $(3, 4)$
(17) $(6, 1)$
(18) $(-8, 3)$
(19) $(4\sqrt{2}, -\sqrt{6})$
(20) $(-5, -7)$
(21) $(-0.5, 6)$
(22) $(7, -0.34)$
(23) $(-5.23, -2.35)$
(24) $(-0.212, 0.703)$
(25) $(-12, 5)$
(26) $(4.23, -11)$
(27) $(-15, -8)$

Q2. Change the polar coordinates of the points below into cartesian form.

(1) $(6, \frac{\pi}{3})$
(2) $(4\sqrt{2}, \frac{\pi}{4})$
(3) $(4, \frac{\pi}{6})$
(4) $(6, \frac{5\pi}{6})$
(5) $(2, -\frac{2\pi}{3})$
(6) $(2.4, \frac{3\pi}{4})$
(7) $(3, -\frac{\pi}{3})$
(8) $(\frac{9\sqrt{2}}{5}, -\frac{3\pi}{4})$
(9) $(2\sqrt{3}, \frac{2\pi}{3})$
(10) $(2\sqrt{3}, -\frac{\pi}{6})$
(11) $(\frac{3}{\sqrt{2}}, -\frac{\pi}{4})$
(12) $(12, -\frac{5\pi}{6})$
(13) $(3, 3)$
(14) $(4, 1)$
(15) $(2.5, -2)$
(16) $(1, -2.5)$
(17) $(8, 0.46)$
(18) $(\pi, 1)$
(19) $(12, -0.74)$
(20) $(9.3, -2.1)$

30.2 Introducing complex numbers

The complex numbers begin with a positive answer to the question "*Do all quadratic equations have solutions?*"

Definition 244 (imaginary unit i).
The imaginary number i is defined by the property that its square is -1

$$i^2 = -1$$

Example 245 (complex solutions).
Let us consider the equation

$$x^2 + 1 = 0$$

that has no real solutions. Then

$$x^2 = -1$$

or

$$x = \pm\sqrt{-1}$$
$$x = \pm i$$

Clearly, both $x = i$ and $x = -i$ are the solutions of the above equation, as:

$$i^2 + 1 = i \times i + 1 = i^2 + 1 = -1 + 1 = 0$$
$$(-i)^2 + 1 = (-i)(-i) + 1 = i \times i + 1 = i^2 + 1 = -1 + 1 = 0$$

Note that the expression $x^2 + 1$ can be transformed into a factorised form that shows both roots:

$$x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - i)(x + i)$$

Questions 30.2.

Q3. Show by a counterexample that the rule

$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$

does not work for complex numbers.

Q4. Show by a counterexample that the rule

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

does not work for complex numbers.

Notation.
The rules shown in the above questions will still work if we avoid wrting square roots of negative numbers. Thus instead of ie. "$\sqrt{-5}$" we should write "$i\sqrt{5}$".

Questions 30.3.

Q5. Find the complex solutions of the following equations.

(1) $x^2 + 9 = 0$
(2) $x^2 - 2x + 5 = 0$
(3) $x^2 + 2x + 10 = 0$
(4) $x^2 - 2x + 3 = 0$
(5) $x^2 + 6x + 11 = 0$
(6) $x^2 - x + 7 = 0$
(7) $x^2 + 2x + 7 = 0$
(8) $9x^2 + 16 = 0$
(9) $4x^2 + 12 = 0$
(10) $2x^2 + 2x + 3 = 0$
(11) $3x^2 - 2x + 7 = 0$
(12) $2x^2 + 2x + 1 = 0$
(13) $3x^2 - 2x + 1 = 0$
(14) $x^3 + x^2 + 4x + 4 = 0$
(15) $2x^3 - 6x^2 + x - 3 = 0$
(16) $x^4 + 5x^2 + 4 = 0$
(17) $x^4 - 1 = 0$
(18) $x^3 + 1 = 0$

30.3 Algebra of complex numbers

Definition 246 (complex number).
z is a complex number if $z = a + ib$, where
$a, b \in \mathbb{R}$,
i is the imaginary unit.

Questions 30.4.

Q6. Given $z = 2 + 3i$, $v = 3 - i$, $w = -1 + 2i$ evaluate the following.
Express the answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

(1) zv
(2) vw
(3) zw
(4) $v(z + w)$
(5) $(z + v)(w - v)$
(6) $(v - z)^2$

Q7. Evaluate.

(1) i^3
(2) i^4
(3) i^5
(4) i^6
(5) i^7
(6) i^8
(7) $(1 + i)^2$
(8) $(1 + i)^3$
(9) $(1 + i)^4$
(10) $(1 + i)^8$
(11) $(1 + i)^9$
(12) $i^{11} + i^9 + i^7 + i^5$

Q8. Consider the function f defined by:

$$f(x) = x^3 + ax^2 + bx + 5$$

Find the real values of a and b for which $f(2 + i) = 2 + i$.

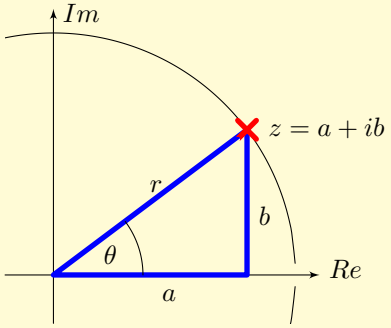
NOTATION & TERMINOLOGY FOR COMPLEX NUMBERS.

$$i^2 = -1$$

$z = a + ib$, *where* $a, b \in \mathbb{R}$ *is a* ***complex number****.*

$z \in \mathbb{C}$

a	*is the* ***real part*** *of* z.	$a = Re(z)$
b	*is the* ***imaginary part*** *of* z.	$b = Im(z)$
$z^* = \overline{z}$	*is the* ***conjugate*** *of* z.	$z^* = a - ib$
$\lvert z\rvert$	*is the* ***modulus*** *of* z.	$\lvert z\rvert = r = \sqrt{a^2 + b^2}$
$Arg(z)$	*is the* ***argument*** *of* z.	$\arg z = \theta$ *such that* $a = r\cos\theta$, $b = r\sin\theta$, $(\tan\theta = \frac{b}{a})$



$z = a + ib$ ***cartesian*** *form of* z

$= r\cos\theta + ri\sin\theta$

$= r(\cos\theta + i\sin\theta)$ ***polar*** *form of* z

$= rcis\theta$

30.4 CONJUGATE NUMBERS. POLAR FORM OF A COMPLEX NUMBER

QUESTIONS 30.5.

Q9. Complete the table.

z	$\lvert z\rvert$	z^*	zz^*
$3 - 4i$			
$12i - 5$			
$2\sqrt{3} + 6i$			
$4 - 4i$			
$-1 + 3i$			

THEOREM 247 (product of complex numbers).
For any complex number z*:*

$$zz^* = |z|^2$$

COROLLARY 248 (product of complex numbers).
For a complex number with modulus equal 1 ($|z| = 1$)*:*

$$zz^* = 1$$

QUESTIONS 30.6.

Q10. Given $z = 2 + 3i$, $v = 3 - i$, $w = -1 + 2i$ evaluate the following.
Express the answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

(i) $\frac{z}{v}$ (ii) $\frac{v}{w}$ (iii) $\frac{w}{z}$ (iv) $\frac{v}{z}$ (v) $\frac{w}{v}$ (vi) $\frac{z}{w}$

Q11. Complete the table.

z in cartesian form	z in polar form	z^* in cartesian form	z^* in polar form
$4 + 4i$			
$-\sqrt{3} + 3i$			
$\sqrt{3} - i$			
	$2cis\frac{5\pi}{6}$		
	$4cis(-\frac{2\pi}{3})$		
	$3cis(-\frac{\pi}{2})$		

THEOREM 249 (modulus and argument of conjugate).
For any complex number z:

1. $|z^*| = |z|$,

2. $\arg z^* = \arg z$.

QUESTIONS 30.7.

Q12. For a given pair of numbers z and v:

(i) find zv,
(ii) express z, v and zv in polar form.

(1) $z = 2\sqrt{3} + 2i$, $v = 3i$
(2) $z = 6 + 2i\sqrt{3}$, $v = -1 + i\sqrt{3}$
(3) $z = -5 + 5i$, $v = -2i$
(4) $z = -3 - i\sqrt{3}$, $v = -2 + 2i\sqrt{3}$
(5) $z = -2 - 2i$, $v = -3 - 3i$
(6) $z = 3 + 3i$, $v = -1 + i$
(7) $z = -1 - i\sqrt{3}$, $v = 1 - i\sqrt{3}$
(8) $z = 1 + i$, $v = \sqrt{3} - i$

THEOREM 250 (modulus and argument of a product of complex numbers).
For any two complex numbers z and v:

1. $|zv| = |z||v|$

2. $\arg(zv) = \arg z + \arg v \ (\pm 2\pi)$

QUESTIONS 30.8.

Q13. For a given pair of numbers z and v:

(i) find $\frac{z}{v}$,
(ii) express z, v and $\frac{z}{v}$ in polar form.

(1) $z = 2\sqrt{3} + 2i$, $v = 3i$
(2) $z = 6 + 2i\sqrt{3}$, $v = -1 + i\sqrt{3}$
(3) $z = -5 + 5i$, $v = -2i$
(4) $z = -3 - i\sqrt{3}$, $v = -2 + 2i\sqrt{3}$
(5) $z = -2 - 2i$, $v = -3 - 3i$
(6) $z = 3 + 3i$, $v = -1 + i$
(7) $z = -1 - i\sqrt{3}$, $v = 1 - i\sqrt{3}$
(8) $z = 1 + i$, $v = \sqrt{3} - i$

THEOREM 251 (modulus and argument of a quatioent of complex numbers).
For any two complex numbers z and v, $v \neq 0$:

1. $\left|\frac{z}{v}\right| = \frac{|z|}{|v|}$

2. $\arg\left(\frac{z}{v}\right) = \arg z - \arg v \ (\pm 2\pi)$

COROLLARY 252 (product and quotient of complex numbers in polar form).
Let $z = r(\cos\alpha + i\sin\alpha)$ and $v = t(\cos\beta + i\sin\beta)$. Then:

1. $zv = rt\,(\cos(\alpha+\beta) + i\sin(\alpha+\beta))$

2. *if $v \neq 0$ then* $\frac{z}{v} = \frac{r}{t}\,(\cos(\alpha-\beta) + i\sin(\alpha-\beta))$

Chapter 31

De Moivre's theorem

31.1 Powers of complex numbers

Questions 31.1.

Q1. For a given complex number z:

(i) find z^2 and z^3,

(ii) express z, z^2 and z^3 in polar form.

(1) $z = 2\sqrt{3} + 2i$
(2) $z = 3i$
(3) $z = 6 + 2i\sqrt{3}$
(4) $z = -1 + i\sqrt{3}$
(5) $z = -5 + 5i$
(6) $z = -2i$
(7) $z = -3 - i\sqrt{3}$
(8) $z = -2 + 2i\sqrt{3}$
(9) $z = -2 - 2i$
(10) $z = 3 + 3i$
(11) $z = -1 + i$
(12) $z = -1 - i\sqrt{3}$
(13) $z = 1 - i\sqrt{3}$
(14) $z = \sqrt{3} - i$

Q2. For a given complex number z:

(i) find z^4,

(ii) express z^4 in polar form.

(1) $z = \sqrt{3} + i$
(2) $z = 2i$
(3) $z = -1 + i\sqrt{3}$
(4) $z = -5 + 5i$
(5) $z = -3i$
(6) $z = -3 - i\sqrt{3}$
(7) $z = -2 - 2i$
(8) $z = 1 + i$
(9) $z = -1 + i$
(10) $z = -1 - i\sqrt{3}$
(11) $z = 1 - i\sqrt{3}$
(12) $z = \sqrt{3} - i$

Theorem 253 (power of a complex number).
Let $z = r(\cos\theta + i\sin\theta)$. Then for $n \in \mathbb{N}$:

$$z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Questions 31.2.

Q3. Sketch in separate Argand diagrams sets of complex numbers described with the following equations / inequalities.

(1) $|z| = 2$
(2) $|z| = 5$
(3) $\arg z = \frac{\pi}{3}$
(4) $\arg z = -\frac{5\pi}{6}$
(5) $|z - 2| = 2$
(6) $|z + 3i| = 2$
(7) $|z - 2 + 3i| = 2$
(8) $\arg(z - 2) = \frac{\pi}{3}$
(9) $\arg(z - i) = -\pi$
(10) $|z| = |z - 2|$
(11) $|z + 2| = |z - 4i|$
(12) $|z - 3 + 5i| = |z + 5 - i|$
(13) $|z| \leq 3$ and $0 \leq \arg z \leq \frac{\pi}{4}$
(14) $2 \leq |z| \leq 4$ and $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$
(15) $\arg z^2 = \frac{\pi}{3}$
(16) $\arg(z + 1)^2 = \frac{\pi}{3}$
(17) $|z| = |2z - 6|$
(18) $|z + 1| = |2z - 2|$

Q4. Find all possible real values of a and b such that $(a + ib)^2 = 3 + 4i$.
Hence state the possible square roots of $3 + 4i$.

Q5. Find all possible real values of a and b such that $(a + ib)^2 = 8 - 6i$.
Hence state the possible square roots of $8 - 6i$.

Q6. z is a complex number such that $|z| \leq 3$ and $\frac{\pi}{3} \leq \arg z < \frac{\pi}{2}$.
Describe clearly the range of modulus and argument of each of the following numbers.

(1) $3z$
(2) $\frac{z}{2}$
(3) $-z$
(4) z^2
(5) z^3
(6) $\sqrt{z}$

Q7. z is a complex number such that $1 < |z| \leq 2$ and $-\frac{\pi}{3} \leq \arg z < \frac{\pi}{2}$.
Describe clearly the range of modulus and argument of each of the following numbers.

(1) $3z$
(2) $\frac{z}{2}$
(3) $-z$
(4) z^2
(5) z^3
(6) $\sqrt{z}$
(7) $\frac{1}{z}$
(8) $2z^2$

Q8. z is a complex number such that $1 \leq |z|$ and $-\frac{\pi}{2} \leq \arg z < -\frac{\pi}{6}$.
Describe clearly the range of modulus and argument of each of the following numbers.

(1) $3z$
(2) $\frac{z}{2}$
(3) $-z$
(4) z^2
(5) z^3
(6) $\sqrt{z}$
(7) $\frac{1}{z}$
(8) $2z^2$

Q9. Consider the functions:

$$f(x) = \cos x + i \sin x \text{ and}$$
$$g(x) = e^{-ix} f(x).$$

(i) Show that $\frac{df}{dx}(x) = if(x)$.
(ii) Show that $\frac{dg}{dx}(x) = 0$ for all values of x.
(iii) Consider $g(0)$ and deduce that $g(x) = 1$ for all values of x.
(iv) Hence show that $e^{ix} = \cos x + i \sin x$.

Theorem 254 (Euler's formula).
A complex number with modulus r and argument θ can be written in the form:

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

31.2 De Moivre's theorem and its generalization

Theorem 255 (The original de Moivre's theorem).
Let $r \geq 0, r, \theta \in \mathbb{R}$. Let $n \in \mathbb{Z}$.
Then

$$(rcis\theta)^n = r^n cis(n\theta)$$

or

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

or

$$(re^{i\theta})^n = r^n e^{in\theta}$$

Note, that all above express exactly the same statement but in different notation.

Generalization

As seen in e.g. question 4 above, non-integer powers give multiple answers. Let us try to use the de Moivre's theorem in two different cases:

1. $r = 1, n = \frac{1}{2}, \theta = 0$

$$(\cos 0 + i\sin 0)^{\frac{1}{2}} = \cos(\tfrac{1}{2} \times 0) + i\sin(\tfrac{1}{2} \times 0)$$
$$(1 + i \times 0)^{\frac{1}{2}} = \cos 0 + i\sin 0$$
$$1^{\frac{1}{2}} = 1$$

2. $r = 1, n = \dfrac{1}{2}, \theta = 2\pi$

$$(\cos 2\pi + i\sin 2\pi)^{\frac{1}{2}} = \cos(\tfrac{1}{2} \times 2\pi) + i\sin(\tfrac{1}{2} \times 2\pi)$$
$$(1 + i \times 0)^{\frac{1}{2}} = \cos\pi + i\sin\pi$$
$$1^{\frac{1}{2}} = -1$$

Theorem 256 (generalization of the de Moivre's theorem).
*For **non-integer** values of n the expression*

$$r^n cis(n\theta) \qquad = r^n(\cos(n\theta) + i\sin(n\theta)) \qquad = r^n e^{in\theta}$$

gives one of the possible multiple values of

$$(rcis\theta)^n \qquad = (r(\cos\theta + i\sin\theta))^n \qquad = (re^{i\theta})^n.$$

Questions 31.3.

Q10. (i) Solve $r^3 = 8$ in real numbers.

(ii) Find three distinct values of θ such that

$$3\theta = \frac{\pi}{2} + 2k\pi, \text{ where } k \in \mathbb{Z}.$$

(two values of θ are to be considered distinct if the difference between them is not a multiple of 2π) Is it possible to find more distinct values of θ? Why?

(iii) Hence, expressing z as $rcis\theta$ and $8i$ in polar form, find the three solutions of the equation

$$z^3 = 8i$$

Give the answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

Q11. Solve the following equations. Give the answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

(i) $z^2 = 1$ (ii) $z^3 = 1$ (iii) $z^4 = 1$ (iv) $z^6 = 1$

Q12. Solve the following equations. Give the answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

(i) $z^3 = 8$
(ii) $z^3 = -27$
(iii) $z^4 = -16$
(iv) $z^3 = -54\sqrt{2}$
(v) $z^4 = 8i(i + \sqrt{3})$
(vi) $i(z^4 + 2) - 2\sqrt{3} = 0$
(vii) $\frac{z^3\sqrt{3}+3i}{i} = 12$

Q13. Considering numbers $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $v = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and $\frac{z}{v}$ in both cartesian and polar form find the exact values of $\cos\frac{\pi}{12}$, $\sin\frac{\pi}{12}$ and $\tan\frac{\pi}{12}$.

Q14. Choosing appropriate complex numbers find the exact values of sine, cosine and tangent of $\frac{5\pi}{12}$ and $\frac{7\pi}{12}$.

Q15. Numbers $1, w_1, w_2, w_3, , w_8$ are the ninth roots of unity (i.e. $w_i^9 = 1$ for $i = 1, \ldots, 8$) such that $0 < \arg w_1 < \arg w_2 < \cdots < \arg w_8 < 2\pi$. Find the smallest possible natural numbers a, b, c, d such that:

(1) $w_2^a = w_1$
(2) $w_2^b = w_3$
(3) $(\sqrt{2}w_4)^c = 4\sqrt{2}w_d$

Q16. A number $3 + i\sqrt{3}$ is a solution of $z^3 = a + ib$.

(1) Find the real values of a and b.
(2) Find the other solutions of the equation in cartesian form.
(3) Hence solve $(iz + 2)^3 = 3\sqrt{3}i$

Q17. Find the modulus and the argument (from $-\pi$ to π) of the following complex numbers.

(1) $5e^{\frac{i\pi}{2}}$
(2) $3e^{2i}$
(3) e^{2+2i}
(4) $4e^{2-i}$
(5) $\pi e^{2-\pi i}$
(6) $\sqrt{3e^{6i}}$
(7) 2^i
(8) 3^{i+1}
(9) 5^{2-3i}

Chapter 32

Conjugate roots

32.1 Fundamental theorem of algebra

Theorem 257 (Fundamental theorem of algebra).
Every non-constant polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

with complex coefficients $(a_0, a_1, \ldots, a_n \in \mathbb{C})$ *has a root.*

Corollary 258 (to the Fundamental theorem of algebra).
Every non-constant polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

with complex coefficients $(a_0, a_1, \ldots, a_n \in \mathbb{C})$
has a linear factor $(z - \alpha)$, $\alpha \in \mathbb{C}$.

Corollary 259 (to the Fundamental theorem of algebra).
Every non-constant polynomial
with complex coefficients $(a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0)$
can be expressed as a product of its n *linear factors:*

$$P(z) = a_n(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)$$

Questions 32.1.

Q1. Solve the equations over the complex numbers

(1) $x^4 + 4x^2 + 4 = 0$
(2) $x^3 + x^2 + x + 1 = 0$
(3) $x^3 - 5x^2 + 7x + 13 = 0$
(4) $x^3 - 2x^2 - 2x - 3 = 0$

32.2 Conjugate Root theorem

Theorem 260 (Conjugate Root theorem).
A complex number w ***is a root*** *of a polynomial*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

with real coefficients $(a_0, a_1, \ldots, a_n \in \mathbb{R})$
if and only if w^* ***is a root***.

Questions 32.2.

Q2. Find a polynomial of the smallest possible degree and integer coefficients whose roots are (among others):

(1) $2i$
(2) $1 - 2i$
(3) $2i, 3i$
(4) $2 - i, -1 + 2i$

Q3. Solve the equation $x^4 - 4x^3 + 14x^2 - 36x + 45 = 0$ in complex numbers knowing that $i + 2$ is one of its solutions.

Q4. Solve in complex numbers equation $x^4 + 4x^3 + 34x^2 + 60x + 125 = 0$ knowing that $2i - 1$ is one of the solutions.

Q5. Solve the equation $x^4 + 4x^3 + 4x^2 + 48 = 0$ over the complex numbers knowing that $\sqrt{3}i - 3$ is one of the solutions.

Q6. Solve the equation $x^4 - 5x^3 + 11x^2 - 10x + 6 = 0$ over the complex numbers knowing that $2 + i\sqrt{2}$ is one of its solutions.

Q7. Solve the equation $4x^4 + 28x^3 + 81x^2 + 82x + 65 = 0$ over the complex numbers knowing that $-3 - 2i$ is one of its solutions.

Q8. Solve the equation $x^6 - 5x^4 + 19x^2 + 25 = 0$ over the complex numbers.

Q9. Solve the equation $x^4 - 2x^3 + 3x^2 - 2x + 2 = 0$ knowing that one of its complex solutions has no real part.

Q10. Complex solutions of the equation $x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$ have positive integer real parts and integer imaginary parts. In one of the solutions real part is twice greater than the imaginary part, while in the other the imaginary part is twice greater than the real part. Find all solutions.

Q11. Solve the equation $z^4 - 2z^3 + 12z^2 - 18z + 27 = 0$ over the complex numbers knowing that one of its complex solutions has no real part.

Q12. Let $z = \cos\theta + i\sin\theta$.

(i) Show that $z + \frac{1}{z} = 2\cos\theta$.
(ii) Deduce that $z^n + \frac{1}{z^n} = 2\cos n\theta$.
(iii) Find the binomial expansion of $(z + \frac{1}{z})^4$.
(iv) Hence express $\cos^4\theta$ in terms of $\cos 2\theta$ and $\cos 4\theta$.
(v) Hence, evaluate $\int_0^{\frac{\pi}{4}} \cos^4 x dx$.

Q13. Let $z = \cos\theta + i\sin\theta$.

(i) Show that $z - \frac{1}{z} = 2i\sin\theta$.
(ii) Deduce that $z^n - \frac{1}{z^n} = 2i\sin n\theta$.
(iii) Find the binomial expansion of $(z - \frac{1}{z})^5$.
(iv) Hence express $\sin^5\theta$ in terms of $\sin\theta$, $\sin 3\theta$ and $\sin 5\theta$.
(v) Hence, evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x dx$.

Q14. Let ω be a non-real complex number such that $\omega^3 = 1$. Show that:

(i) $\omega^2 + \omega + 1 = 0$
(ii) $\omega^4 + \omega^5 = -1$

32.3 Summary of key points

$$zz^* = |z|^2$$

$$|z| = |z^*|$$

$$\arg z^* = -\arg z$$

$$|zv| = |z||v| \qquad \arg(zv) = \arg z + \arg v \quad (\pm 2\pi)$$

$$\left|\frac{z}{v}\right| = \frac{|z|}{|v|} \qquad \arg\left(\frac{z}{v}\right) = \arg z - \arg v \quad (\pm 2\pi)$$

$$(rcis\alpha)(tcis\beta) = rtcis(\alpha + \beta)$$

$$\frac{rcis\alpha}{tcis\beta} = \frac{r}{t}cis(\alpha - \beta)$$

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$(rcis\theta)^n = r^n cis(n\theta) = r^n e^{in\theta}$$

Part IX

STATISTICS

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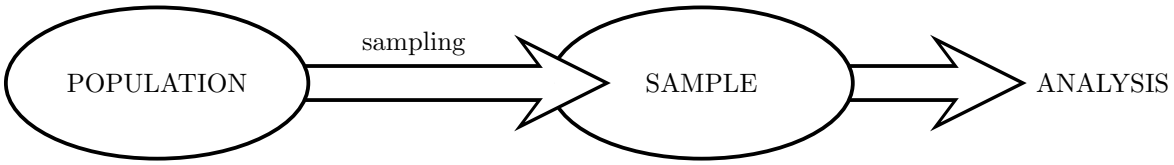
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Chapter 33

Population, sample, types of data

33.1 Population, sample, random sample



33.2 Types of data

Numerical data are **quantative**.
(e.g. height or weight).
Categorical data are **qualitative**.
(e.g. colour).
Discrete data are **counted**.
(e.g. number of people, number of cars, shoe size)
Continuous data are **measured**.
(e.g. height, weight, time, speed)

Chapter 34

Presentation of data

34.1 Frequency tables and diagrams

Example 261 (frequency).
Heights of students

The following numbers show the heights of a group of 50 students.

158	156	143	160	140	195	168	156	140	171
172	141	183	165	173	175	173	151	167	171
198	182	196	164	149	162	179	166	184	175
165	156	159	185	176	182	148	168	168	153
151	158	167	160	189	166	148	147	173	177

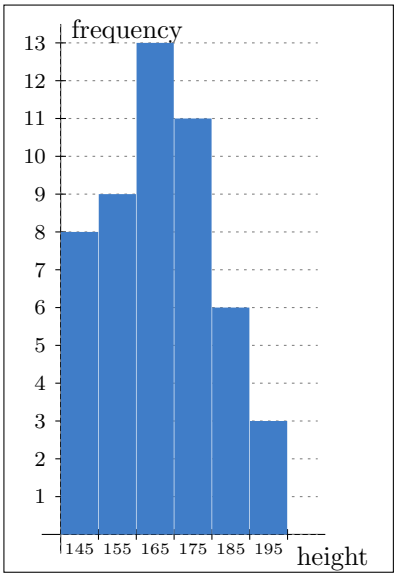
The numbers alone do not give us any information about the heights of the students. It is hard to see tendencies among them as there are simply too many of them. This is why we are going to group them.
Fill in the following table.

heights	tally	frequency
140-149	√√√√√√√√	8
150-159		
160-169		
170-179		
180-189		
190-199		

To display raw data in a graphical form that is easier to interpret we use different forms of frequency diagrams. Commonly used frequency diagrams are

- *bar charts*,
- *pie charts*,
- *scatter diagrams*
- and ***histograms***.

34.2 Histogram



The data can be shown in a form of a histogram. Histograms are usually used to represent data varying continuously. As the classes (140-149, 150-159 etc.) are of equal width, the height of each bar is to represent frequency. If the classes were of different widths, the area of bars would be proportional to frequency. However, in the IB it is not going to be the case.

The horizontal axis represents height - the **continuous** variable. The first class represents students with heights from 140cm to 149cm. However, the values might have been rounded to the nearest whole centimetre, thus anyone with the height in the interval $[139.5, 149.5[$ should have been placed in this class. Therefore to be precise the divisions between the classes should be labelled in the diagram as 139.5, 149.5 etc.

Similarly, the labels below the bars should be, technically speaking, 144.5, 154.5 etc.

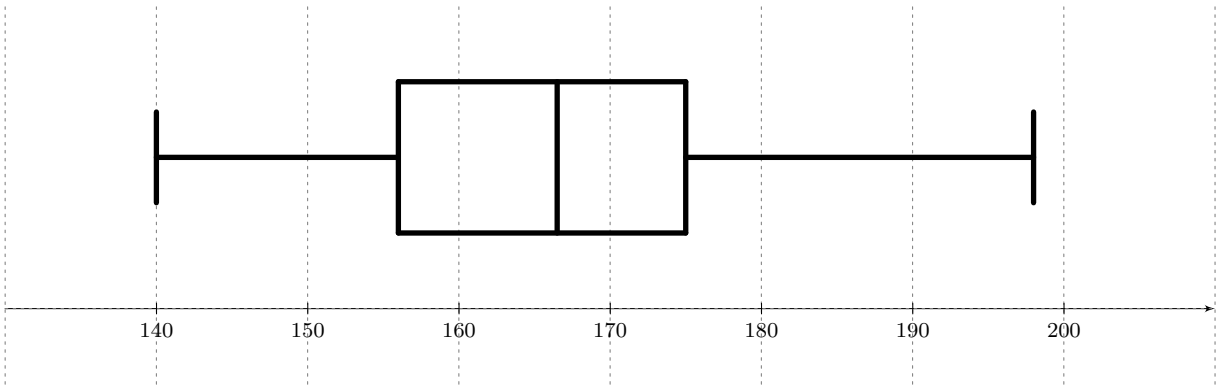
GDC: **Use STAT and STATPLOT options on your GDC to create the above histogram**.

34.3 Box and whisker plot

From the data above you can find:

minimum height	(Q_0)	$140cm$	
lower quartile	(Q_1)	$156cm$	*If you cannot remember what the*
median height	(Q_2)	$166.5cm$	*quartiles are you can read more in*
upper quartile	(Q_3)	$175cm$	*Section 35.3.*
maximum height	(Q_4)	$198cm$	

Box and whisker plot (box plot) is the simplest diagram showing all 5 quartiles.



It shows the **skewness** of the data. The set considered is almost symmetrical (or slightly positively skewed). Skewness, however, is not in the concern of the IB course.

Questions 34.1.

Q1. Consider the following set of numbers (heights of students).
Complete the table of frequencies below. Find the mean height.

152	153	157	163	163	165
152	153	160	163	163	168
152	157	160	163	165	168
152	157	160	163	165	168

height (h)	frequency (f)	$h \times f$
152	4	608
153	2	
157		
160		

Q2. For the data from question **1**:

(i) Write down the mode. .....

(ii) Write down the minimum height (Q_0). .....

(iii) Write down the maximum height (Q_4). .....

(iv) Find the range of the data. .....

(v) Find the median height (Q_2). .....

(vi) Find the lower quartile (Q_1). .....

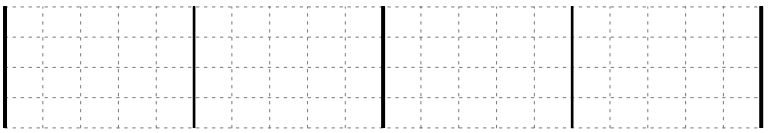
(vii) Find the upper quartile (Q_3). .....

(viii) Find the interquartile range (I.Q.R.) .....

Copy the results to the table below:

Q_0	Q_1	Q_2	Q_3	Q_4

Sketch the box plot.



Q3. For the given numbers

61	62	64	65	67
68	69	70	72	72
76	79	80	88	90

find:

(i) the median

(ii) the lower quartile

(iii) the upper quartile.

(iv) Sketch the box plot.



Q4. A hundred children were asked how many sandwiches they ate in a lunch break. The table below shows the results of the survey.

number of sandwiches	number of children (frequency)	cumulative frequency
0	6	6
1	14	20
2	48	68
3	17	
4	9	
5	6	

(i) Complete the table.

(ii) Find the median and the quartiles.

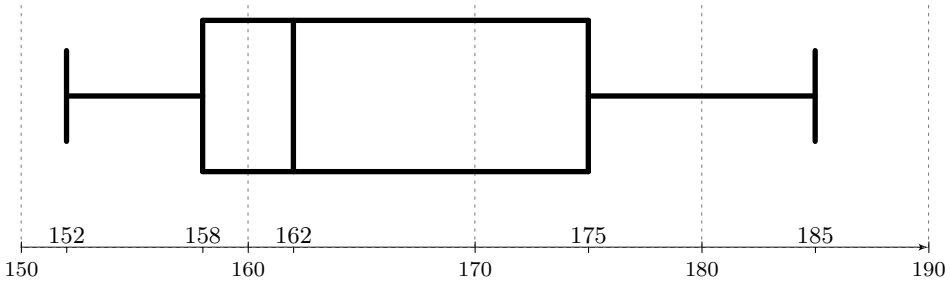
(iii) Draw a box plot.



(iv) Looking at the diagram above state whether the following statements are true or false.

- At least 50% of children ate not more than 2 sandwiches. .....
- At least 50% of children ate more than 2 sandwiches. .....
- A child in top 25% ate more than 3 sandwiches. .....
- A child in the bottom 25% ate less than 2 sandwiches. .....
- A child in middle 50% ate 2 or 3 sandwiches. .....

Q5. The box plot below shows the spread of heights of a group of students.



Looking at the diagram above state whether the following statements are true or false.

- At least 50% of the students are higher than $162cm$. .....
- Exactly 50% of the students are higher than $162cm$. .....
- At least 50% of the students are shorter than $162cm$. .....
- There are more students in a group of 25% highest than in a group of 25% shortest students. .....

Chapter 35

Measures of tendency, measures of spread

35.1 Mean

Definition 262 (mean).
The mean *of numbers $x_1, x_2, \ldots, x_n$ is*

$$\mu = \frac{\sum_{i=1}^{n} x_i}{n}$$

If the numbers $x_1, x_2, \ldots, x_s$ have frequencies $f_1, f_2, \ldots, f_s$ respectively then the mean is

$$\mu = \frac{\sum_{i=1}^{s} x_i f_i}{n}$$

where $n = \sum_{i=1}^{s} f_i$.

Notation.
The formulae above are often written in simplified way:

$$\mu = \frac{\sum x}{n} \quad \text{and} \quad \mu = \frac{\sum xf}{n}$$

When the data is grouped in classes (see example 261 above) the midpoints of the classes are taken as values of x. In the above case the mean is therefore calculated as follows:

$$\mu = \frac{144.5 \times 8 + 154.5 \times 9 + 164.5 \times 13 + 174.5 \times 11 + 184.5 \times 6 + 194.5 \times 3}{8 + 9 + 13 + 11 + 6 + 3}$$

Notation.
The Greek letter μ is used to describe the mean of the whole population considered. However, when we calculate the mean of the sample, we use the symbol $\bar{x}$. It must be noted that it is very often impossible to calculate μ and then $\bar{x}$ serves as the estimate of μ. $\bar{x}$ is sometimes called an ***unbiased estimate*** *of the population mean.*

35.2 Mode

Definition 263 (mode).
The mode *is the most common value.*

Sometimes we say that the set of data is bi-modal or that there is no mode for the given set. In case of grouped data when the real values are not known, a **modal class** can be considered.

35.3 Median and quartiles, percentiles

Definition 264 (median).
The median *is the value in the middle when the data are arranged in an ascending order.*
Let n be the size of the data set. Then $\frac{n+1}{2}$ shows the position of the median.

- *when n is odd then the formula $\frac{n+1}{2}$ gives the position of the value in the middle of the data set*
- *when n is even then the formula $\frac{n+1}{2}$ gives the outcome between two positions of the two values in the data set whose mean is considered to be the meadian.*

Example 265 (median).
Find the median weight of the group of women:

51	52	63	57	59	61	65	68	54	61	61	49	51	62	62

Firstly, the weight have to be arranged in ascending order:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
49	51	51	52	54	57	59	61	61	61	62	62	63	65	68

Median weight is the one in the middle, i.e. the one with the same number of "results" below and above.
Its position is given by $\frac{n+1}{2} = \frac{16}{2} = 8$.

1	2	3	4	5	6	7	**8**	9	10	11	12	13	14	15
49	51	51	52	54	57	59	**61**	61	61	62	62	63	65	68

The median weight is 61 (kg).

Example 266 (median).
Find the meadian height of the group of men:

175	*190*	*180*	*165*	*177*	*188*	*172*	*182*

The heights need to be arranged in ascending order:

1	2	3	4	5	6	7	8
165	172	175	177	180	182	188	190

Median height's position is given by the formula $\frac{n+1}{2} = \frac{9}{2} = 4.5$.
It means that the median is the mean of heights on positions 4 and 5.

1	2	3	**4**	**5**	6	7	8
165	172	175	**177**	**180**	182	188	190

Hence, the medina is $\frac{177+180}{2} = 178.5$ (cm).

Notation.
Median is sometimes called the second quartile (and denoted as Q_2).

Definition 267 (quartiles).
*The first or the **lower quartile** (Q_1) is the result in the quarter from the smallest, while the third or the **upper quartile** (Q_3) is the one in the quarter from the highest.*
More precisely:
Q_1 is the median of the set of values smaller than the median of the original data set.
Q_3 is the median of the set of values larger than the median of the original data set.

Example 268 (quartiles).
Find the lower and the upper quartile for the following set o weights (in kg):

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
49	51	51	52	54	57	59	61	61	61	62	62	63	65	68

			Q_1				Q_2				Q_3			
1	2	3	**4**	5	6	7	8	9	10	11	**12**	13	14	15
49	51	51	**52**	54	57	59	61	61	61	62	**62**	63	65	68

Hence $Q_1 = 52$, $Q_3 = 62$.

FACTS, REMARKS, OBSERVATIONS.
Let n be the number of data in the set considered (in our case $n = 15$).
When the outcomes are numbered from smallest to highest, the quartiles are to be found with use if the following simple formulae showing their position in the row.

quartile	*small n*	*large n, e.g. $n > 100$*	*example above*
Q_1	$\frac{n+1}{4}$	$\frac{n}{4}$	$\frac{15+1}{4} = 4$
Q_2	$\frac{2(n+1)}{4}$	$\frac{2n}{4}$	$2 \times 4 = 8$
Q_3	$\frac{3(n+1)}{4}$	$\frac{3n}{4}$	$3 \times 4 = 12$

DEFINITION 269 (percentiles).
For large sets of data percentiles *are used. The k^{th} percentile is a value in $k\%$ from the bottom of the data arranged in ascending order.*
For the data set of size n the position of k^{th} percentile is therefore given by $\frac{kn}{100}$.

PROPERTIES OF MEAN, MEDIAN, MODE, QUARTILES AND PERCENTILES.

- *The lower quartile is 25^{th} percentile, the upper quartile is 75^{th} percentile.*
- *Sometimes the smallest and largest values are denoted by Q_0 and Q_4 respectively.*
- *Mean, mode and median are used to describe central tendency of a given set of data. They are different types of an average.*
- *For symmetrical data the mean is equal to the median (and to the mode if the mode exists).*

35.4 Range and interquartile range

DEFINITION 270 (range of data).
The range of quantitative data is the difference between highest and smallest value ($Q_4 - Q_0$).

DEFINITION 271 (interquartile range).
The interquartile range is difference between upper and lower quartiles ($I.Q.R. = Q_3 - Q_1$).

EXAMPLE 272 (range and interquartile range).
Consider the set of weights from example 265:

49	51	51	52	54	57	59	61	61	61	62	62	63	65	68

*The **range** of the data is $68 - 49 = 19$ (kg).*
The interquartile range is $I.Q.R. = 62 - 52 = 10$ (kg).

35.5 Variance, standard deviation

Both **variance** and **standard deviation** indicate how the data is spread from the mean. The larger are the indicators, the greater is the spread. The example below shows two sets of data (heights of small groups of students) with the same mean of 162 centimetres.

A	159	160	161	161	162	162	163	164	164	164
B	157	158	159	161	162	162	163	164	166	168

$\mu_A = \mu_B = 162$

To investigate, how much the data differ from the mean we will evaluate the differences of each height from the mean and then the squares of the differences.

h	$\lvert h-\mu_A\rvert$	$(h-\mu_A)^2$
159	3	9
160	2	4
161	1	1
161	1	1
162	0	0
162	0	0
163	1	1
164	2	4
164	2	4
164	2	4

h	$\lvert h-\mu_B\rvert$	$(h-\mu_B)^2$
157	5	25
158	4	16
159	3	9
161	1	1
162	0	0
162	0	0
163	1	1
164	2	4
166	4	16
168	6	36

Summing up the squares and dividing the sum by number of students we get the **variance** of the heights (denoted as σ^2 or s_n^2):

- For set **A**: $\sigma^2 = \frac{\sum(h-\mu_A)^2}{10} = 2.8$
- For set **B**: $\sigma^2 = \frac{\sum(h-\mu_B)^2}{10} = 10.8$

Standard deviation (σ or s_n) is defined simply as a square root of vari-ance.

$$\sigma = \sqrt{\sigma^2}$$

In the case considered:

$$\sigma_A \approx 1.67cm \text{ and } \sigma_B \approx 3.29cm$$

DEFINITION 273 (variance, standard deviation).
Let x be the data with frequencies f such that $\sum f = n$. Let the mean of the data be μ.
Then the variance σ^2 *is given by:*

$$\sigma^2 = \frac{\sum f(x-\mu)^2}{n} - \mu^2$$

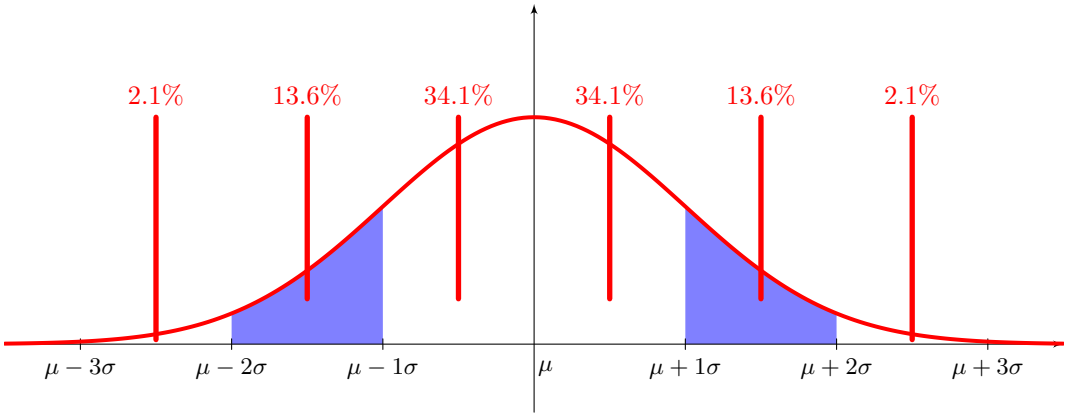
The standard deviation σ is simply:

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum f(x-\mu)^2}{n} - \mu^2}$$

PROPERTIES OF VARIANCE AND STANDARD DEVIATION.
From the so-called Central Limit Theorem (which is not in the concern of the IB course) we may deduct that under certain conditions if the amount of data is sufficiently large then they are normally distributed (you can read more about normal distribution in Chapter 44).
In practice it means that for an individual randomly taken from the population the probability that he is within:

- *one standard deviation from the mean is approximately* 68%,
- *two standard deviation from the mean is approximately* 95%,
- *three standard deviations from the mean exceeds* 99%.

In the following sections we will extend our knowledge to be able to get more precise results.



Range, interquartile range, variance and standard deviation are used to describe spread of data.

NOTATION.
The notation σ or s_n (for standard deviation) and σ^2 or s_n^2 (for variance) are used to describe the standard deviation and variance of the whole population considered. However, when we calculate the measures of the sample, we use the symbols s_{n-1} and s_{n-1}^2 respectively. They are called ***unbiased estimates of standard deviation / variance****. It must be noted that it is very often impossible to calculate σ / s_n and therefore s_{n-1} serves as the estimate of population standard deviation.*

35.6 Calculating variance

THEOREM 274 (alternative formula for the variance).
The formula for the variance $\sigma^2 = \frac{\sum(x-\mu)^2}{n}$ can be rearranged to

$$\sigma^2 = \frac{\sum x^2}{n} - \mu^2$$

Sometimes the formulae are expressed with use of frequencies, in a more formal way:

$$\sigma^2 = \frac{\sum_{i=1}^{s} f_i(x_i-\mu)^2}{n} = \frac{\sum_{i=1}^{s} f_i x_i^2}{n} - \mu^2, \text{ where } n = \sum_{i=1}^{s} f_i$$

Note. To calculate **unbiased estimate of population variance** s_{n-1}^2 we replace n with $n-1$ in the first version of formulae for variance, i.e.

$$s_{n-1}^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$$

or

$$s_{n-1}^2 = \frac{\sum_{i=1}^{n} x_i^2}{n-1} - \frac{n}{n-1}\bar{x}^2$$

Note, that μ was replaced with $\bar{x}$. (Why? See section 35.1.)

QUESTIONS 35.1.

Q1. For the set of numbers shown below find

(i) the mean .....

(ii) the range .....

165 187 174 169 170 182 168 173 166 171

Q2. In the following set of data:

164 ? 172 164 164 ? 172 166 172 174

2 numbers are missing.
The mode is 164 while the mean is 169.
Find the two missing numbers.

Q3. Find the mean and then the standard deviation of each of the 2 sets of numbers given below.

SET A
28
22
24
19
30
24

SET B
25
23
26
22
27
24

Q4. Consider the following 2 sets of data. Both describe the weights in grams of 8 packs of sugar bought in two different shops (A & B).

A	990	975	950	1050	925	1055	1045	970
B	1015	995	970	1010	1015	985	980	990

(i) Show that the mean weight of packs from both shops is $995g$.

(ii) Find standard deviation for both.

Q5. It has been surveyed that the mean life time of a certain type of a light bulb is 453 hours ($\mu = 453$) with standard deviation 28 hours ($\sigma = 28$). Fill in the table below.

$\mu - 2\sigma$	$\mu - \sigma$	μ	$\mu + \sigma$	$\mu + 2\sigma$
		453		

A light bulb of that type is chosen at random. What is the chance that:

(i) the life time of the light bulb will be between 425 and 481 hours? .....

(ii) the life time of the light bulb will be between 397 and 509 hours? .....

(iii) the light bulb is going to live longer than 509 hours? .....

(iv) the light bulb is going to live shorter than 425 hours? .....

(v) the light bulb is going to live longer than 397 hours? .....

(vi) the light bulb is going to live shorter than 509 hours? .....

Q6. The length of certain species of fish is known to has mean of 28cm ($\mu = 28$) with standard deviation 4 cm ($\sigma = 4$). Fill in the table below.

$\mu - 2\sigma$	$\mu - \sigma$	μ	$\mu + \sigma$	$\mu + 2\sigma$
		28		

A fisherman has caught a fish of this species. What is the chance that:

(i) the fishs length will be between 24 and 32 cm? .....

(ii) the fishs length will be between 20 and 36 cm? .....

(iii) the fish is going to be longer than 32 cm? .....

(iv) the fish is going to be shorter than 20 cm? .....

(v) the fish is going to be longer than 24 cm? .....

(vi) the fish is going to be shorter than 36 cm? .....

Chapter 36

Cumulative frequency

Questions 36.1.

Q1. Consider the following set of numbers (heights of students).
Complete the table of **grouped** frequencies below. Find the **estimate** of the mean height.

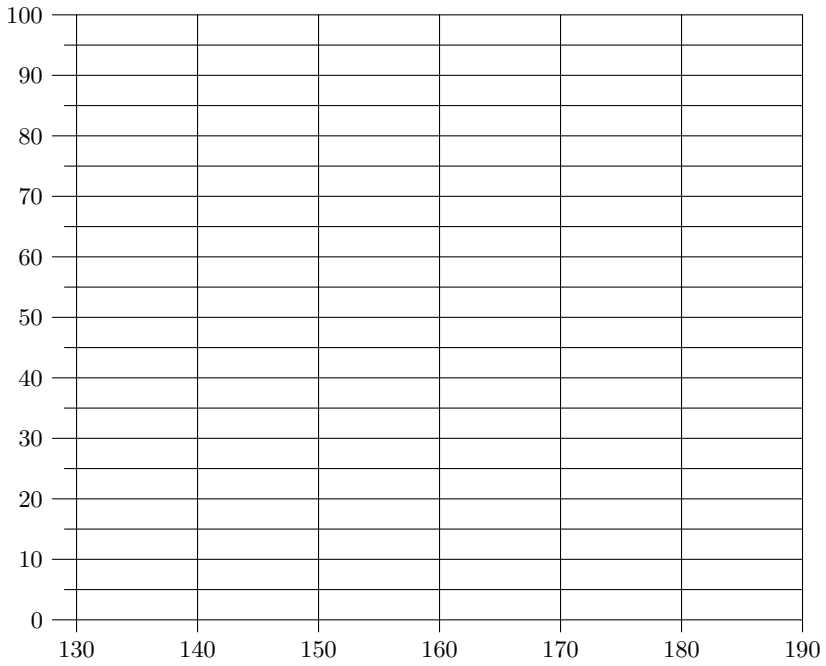
139	148	153	156	159	161	164	167	171	176
140	148	153	156	159	162	164	168	171	176
141	149	153	157	159	162	165	168	172	177
142	149	154	157	160	162	165	168	172	178
143	150	154	157	160	162	165	169	173	179
144	150	154	158	160	163	166	169	173	180
145	151	155	158	160	163	166	169	174	181
146	151	155	158	161	163	166	170	174	182
147	152	155	158	161	163	167	170	175	183
147	152	156	159	161	164	167	170	175	184

height (h)	frequency (f)	estimated height (x)	xf
$130 < h \leq 140$	2	135	270
$140 < h \leq 150$	14	145	

Q2. For the data from question **1** complete the cumulative frequency table below.

height	frequency	cumulative frequency
$h \leq 140$	2	2
$h \leq 150$	14	16

Q3. Using cumulative frequencies from question **2** sketch the **cumulative frequency curve**.



Using the graph:

(i) Find the median.

.....

(ii) Find the lower quartile (Q_1).

.....

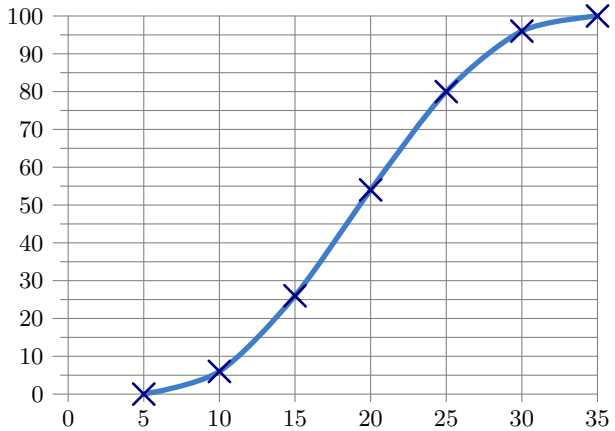
(iii) Find the upper quartile (Q_3).

.....

(iv) Find the interquartile range.

.....

Q4. The following **cumulative frequency diagram** shows heights of 100 plants in cm.



height of plant	cumulative frequency
5	0
10	6
15	26
20	54
25	80
30	96
35	100

(i) Find the median height. .....

(ii) Find the quartiles of the data. .....

(iii) Find the interquartile range (I.Q.R.) .....

(iv) Find the values of a and b such that 80% of plants have heights larger than a cm but smaller than b cm. .....

Part X

PROBABILITY

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Chapter 37

Counting principles

$$\begin{array}{ccccccccc}
 & & & \binom{0}{0} & & & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & 1 & & & & & \\
 & & 1 & & 1 & & & & \\
 & 1 & & 2 & & 1 & & & \\
1 & & 3 & & 3 & & 1 & & \\
 & 4 & & 6 & & 4 & & 1 &
\end{array}$$

PASCAL'S TRIANGLE

$\binom{n}{r} = \binom{n}{n-r}$	(symmetry of Pascal's triangle)
$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$	(each number is a sum of the two above it)
$\binom{n}{0} = \binom{n}{n} = 1$	
$\binom{n}{1} = \binom{n}{n-1} = n$	
$\binom{n}{2} = \binom{n}{n-2} = \frac{n\times(n-1)}{2}$	
$\binom{n}{3} = \binom{n}{n-3} = \frac{n\times(n-1)\times(n-2)}{1\times2\times3}$	etc.

THEOREM 275 (counting prinicples).

1. *If one of n events can occur on each of k trials, the total number of possible outcomes is*

$$n^k.$$

2. *If on the 1^{st} trial there are n_1 possible events, on the 2^{nd} trial there are n_2 possible events, and so on, i.e. on every k^{th} trial there are n_k possible events then the total number of possible outcomes after m trials is*

$$n_1 \times n_2 \times \cdots \times n_m.$$

3. *The total number of ways that n different objects can be arranged in a row is*

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

NOTATION.
For simplification we use the ***factorial*** *notation. By definition, n* ***factorial*** *is:*

$$n! = \begin{cases} 1 & \text{for } n = 0 \\ n \times (n-1)! & \text{for } n > 0, n \in \mathbb{N} \end{cases}$$

We say that there are $n!$ permutations of n objects.

4. *The total number of ways (permutations) that r objects chosen from a set of n different objects can be arranged in a row is*

$$n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}.$$

NOTATION.
$\frac{n!}{(n-r)!}$ *is denoted by ${}^nP_r$, which is read as "$n-p-r$".*

5. *The number of ways of selecting r objects out of n different objects is*

$$\frac{n!}{r!(n-r)!}.$$

NOTATION.
$\frac{n!}{r!(n-r)!}$ *is denoted by ${}^nC_r$ or by $\binom{n}{r}$.*
Both ${}^nC_r$ and $\binom{n}{r}$ are read as "$n-c-r$" or "n−choose−r".

6. *The total number of permutations of n objects of which n_1 are identical, n_2 are identical, ..., n_k are identical is*

$$\frac{n!}{(n_1!)(n_2!)\cdots(n_k!)}.$$

QUESTIONS 37.1.

Q1. Evaluate without a calculator:

(1) $\frac{45!}{44!}$ (2) $\frac{26!}{24!}$ (3) $\frac{41!}{39!}$ (4) $\frac{10!}{7!}$ (5) $\frac{12!}{9!}$ (6) $\frac{26!}{23!4!}$ (7) $\frac{9!}{6!3!}$ (8) $\frac{14!}{10!4!}$

Q2. Find n such that $\frac{n!-(n-1)!}{(n-2)!} = 9$.

Q3. Solve the following:

(1) ${}^nP_2 = 12$ (2) ${}^5P_r = 60$ (3) ${}^nC_2 = 15$ (4) ${}^nC_3 = 20$

Q4. Show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$.

Q5. Show that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+2}{k+1} - \binom{n}{k+1} - \binom{n}{k}$.

Q6. (i) Use the binomial theorem to show that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

(ii) What is the number of all subsets of an n-element set?

Q7. Five books are to be placed on five shelves. In how many ways can this be done if:

(1) there must be at least one book on each shelf?

(2) There are no restrictions on number of books on a shelf?

Q8. Six marbles are to be placed in 3 boxes. In how many ways can this be done if:

(1) there are no additional restrictions?

(2) there must be the same number of marbles in each box.

(3) no box is to be left empty?

Q9. Paul plays a car-racing game on his computer. To start the game he has to choose one of four available types of his car, one of five available colours and one of three engines.

(1) How many different cars can he get?

To finish the game Paul has to be successful on six different missions. He can attempt them in any order.

(2) In how many different ways can he choose the order of missions?

Q10. How many different 5-digit numbers can be formed using 1, 2, 3, 4, 5, 6 and 7 if:

(1) repetitions are allowed?

(2) no digit is repeated?

(3) repetitions are allowed and the number is odd?

(4) no digit is repeated and the number is odd?

Q11. How many different 5-digit numbers can be formed using 0, 1, 2, 3, 4, 5 and 6 if:

(1) repetitions are allowed?

(2) no digit is repeated?

(3) repetitions are allowed and the number is even?

(4) no digit is repeated and the number is even?

Q12. In how many ways can 6 people be seated in a row if the tallest and the shortest are to sit together?

Q13. In how many ways can 6 people be seated in a row if the tallest and the shortest must not sit together?

Q14. In how many ways can 6 people be seated in a row if the tallest and the shortest must not sit at either end?

Q15. In how many ways can 4 men and 3 women be seated if:

(1) all men and all women are to be seated together?

(2) all men are to be seated together?

(3) between every two men there must be at least one woman?

Q16. In how many ways can a committee of 5 people be chosen from a group of 6 women and 7 men if:

(1) there are no additional restrictions?

(2) there should be at least one woman in the committee?

(3) there should be at least one woman and one man in the committee?

(4) the number of women and men should not differ by more than one?

Q17. In how many ways can 8 people be seated around a circular table if:

(1) there are no additional restrictions?

(2) the tallest and the shortest must sit together?

(3) the tallest and the shortest must not sit together?

Q18. In how many ways can 3 men and 3 women be seated around a circular table if:

(1) there are no additional restrictions?

(2) between every two men there must be at least one woman?

(3) all women must sit together?

Q19. In how many different ways can the letters of the word PIRATE be arranged?

Q20. In how many different ways can the letters of the word ALABAMA be arranged?

Q21. In how many different ways can the letters of the word MISSISSIPPI be arranged?

Q22. A collection of books consists of 3 poetry books, 4 fictional books and 5 science-fiction books. In how many ways 4 books can be chosen if:

(1) there are no additional restrictions?

(2) the selection must contain 2 science-fiction books?

(3) the selection must contain at least 2 science-fiction books?

Q23. In how many ways 2 poetry books, 4 fictional books and 3 science-fiction books can be arranged on a shelf if:

(1) there are no additional restrictions?

(2) poetry books cannot be placed together?

(3) the books of each type are to be kept together?

Q24. There are 9 people standing in a tram: Adam, Basil and Cindy are kids, Dorothy and Eunice are pregnant women while Feekou, Greg, Horace and Ike are male adults. In how many ways can they sit if there are 5 places in a row available and:

(1) there are no restrictions?

(2) both pregnant women must be seated?

(3) only adults are to sit?

(4) both pregnant women and no kids must be seated?

(5) both pregnant women and no male adults must be seated?

In how many ways can they sit if there are 4 places in a row available and:

(6) there are no restrictions?

(7) both pregnant women must be seated?

(8) only adults are to sit?

(9) both pregnant women and no kids must be seated?

(10) both pregnant women and no male adults must be seated?

The following questions are about playing cards and card games.

They are usually played with a standard deck of 52 cards. From highest to lowest, the cards are ranked A (ace), K (king), Q (queen), J (jack), 10, 9, 8, 7, 6, 5, 4, 3, 2. Suits are ranked Spades (♠), Hearts (♡), Diamonds (♢) and Clubs (♣). There are therefore 13 cards in each of the suites.

Q25. The list below shows all possible hands in the game of poker. They are shown in descending order.

- **Straight flush** - consecutive cards in the same suit, (e.g. 4♡, 5♡, 6♡, 7♡, 8♡), aces can be treated both as lowest and as highest cards;
- **Four of a Kind** - four cards of the same height and any fifth card (e.g. J♠, J♡, J♢, J♣, 5♢);
- **Full House** - three cards of the same height + two cards of the same height (e.g. Q♠, Q♢, Q♣, 10♡, 10♢);
- **Flush** - all cards of the same suit; they are not to be consecutive (e.g. 3♢, 6♢, 10♢, J♢, Q♢);
- **Straight** - consecutive cards of various suits (e.g. 6♠, 7♣, 8♠, 9♢, 10♣), aces can be treated both as lowest and as highest cards;
- **Three of a Kind** - three cards of the same height + two additional cards of two different heights (e.g. J♠, J♡, J♢, A♣, 5♢);

- **Two Pair** - two pairs of cards of the same heights; the fifth of different height (e.g. K♠, K♡, 10♢, 10♣, Q♢);
- **One Pair** - one pair of cards of the same height; all others of different heights (e.g. 9♠, 9♢, A♣, J♡, 5♢);
- **High Cards** - no two cards of the same height; cards of various suits (e.g. K♠, J♢, 8♣, 4♡, 3♢).

(1) How many different hands a player can get?

(2) How many different hands of each type a player can get?

Q26. In the game of bridge each of four players gets 13 cards. How many different hands a player can get if the hand is to contain:

(1) all aces?

(2) all aces and all kings?

(3) all face cards?

(4) at least one ace?

(5) at most two face cards?

(6) at least two aces and one king?

Q27. Consider the following sets:

- $A = \{a, b, c, d\}$,
- $B = \{1, 2, 3, 4, 5\}$,
- F_{AA} - set of all functions from A into A,
- F_{BB} - set of all functions from B into B,
- F_{AB} - set of all functions from A into B,
- F_{BA} - set of all functions from B into A,
- I - set of all injections,
- S - set of all surjections.

Find:

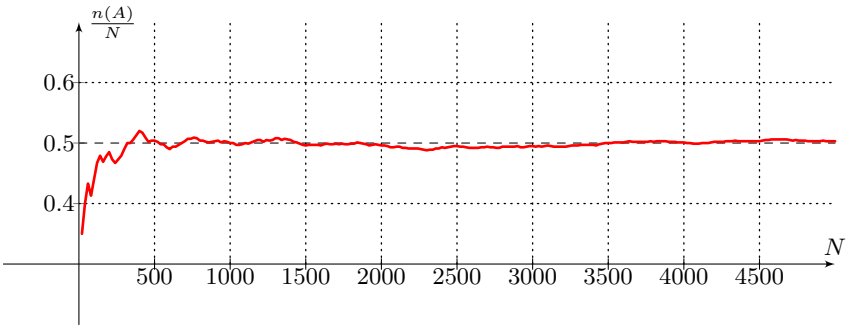
(i) $n(F_{AA})$
(ii) $n(F_{BB})$
(iii) $n(F_{AB})$
(iv) $n(F_{BA})$
(v) $n(F_{AA} \cap I)$
(vi) $n(F_{AA} \cap S)$
(vii) $n(F_{AB} \cap I)$
(viii) $n(F_{AB} \cap S)$
(ix) $n(F_{BA} \cap I)$
(x) $n(F_{BA} \cap S)$

Chapter 38

Introduction to probability

38.1 Concept of probability

When talking of probability we think of a limit of *relative frequency*.
Let an experiment be repeated N times in a way such that each repetition (each trial) is **identical** and **independent**. Let *event* A be one of the possible outcomes of the experiment. Let then $n(A)$ be the number of times event A occurred in N trials. By *relative frequency* we understand the ratio $\frac{n(A)}{N}$.
Then, when N tends to infinity, the ratio $\frac{n(A)}{N}$ approaches$P(A)$, the probability of event A.
The diagram below shows the results of an experiment in which a coin was tossed $N = 5000$ times. Event A denotes a *HEAD* appearing uppermost.



As we can see

$$\lim_{N\to+\infty} \frac{n(A)}{N} = 0.5 = P(A)$$

38.2 Formal approach

Set of all possible **outcomes** of an experiment is called a **sample space** and is usually denoted by U. An **event** is a subset of a sample space. For a finite sample U space where all outcomes are equally likely probability of an event A is

$$P(A) = \frac{n(A)}{n(U)}$$

Example 276 (coin tossed).
A fair coin is tossed.

When a fair coin is tossed there are two possible equally likely outcomes:
H - a head appears uppermost OR
T - a tail appears uppermost.
Then $U = \{\mathbf{H}, \mathbf{T}\}$ and $P(\mathbf{H}) = P(\mathbf{T}) = \frac{1}{2}$.

EXAMPLE 277 (die rolled).
A fair six-faced die is rolled.

When a fair die is rolled there are six possible and equally likely outcomes.
$U = \{1, 2, 3, 4, 5, 6\}$ and $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$

38.3 Axioms of probability

1. For any event A:	$0 \leq P(A) \leq 1$
2. $P(\emptyset) = 0$	(impossible event)
3. $P(U) = 1$	(necessary event)
4. if	$A, B \subset U$ and $A \cap B = \emptyset$
then	$P(A \cup B) = P(A) + P(B)$

Note. If $A \cap B = \emptyset$ then A and B are called **mutually exclusive**.

Note. For $A' = U - A$, $A \cap A' = \emptyset$ and thus $P(A') = 1 - P(A)$.

38.4 Exemplary simple probability problems

1. Two fair dice are rolled and sum of the results seen uppermost noted. Use a lattice diagram to show possible results. What is the probability that the sum is equal 7?

2. A card is drawn from a standard pack of 52 cards. What is the probability that the card is:

(i) an ace?
(ii) not an ace?
(iii) a king or a queen?
(iv) a spade?
(v) red?
(vi) a ten or a diamond?

3. An experiment.

(i) Toss a coin and write down whether a head or tail appeared uppermost. Repeat the experiment 10 000 times. Comment on the results.
(ii) Toss two coins 10 000 times. Each time write down the number of heads seen uppermost. Comment on the results.
(iii) Repeat the above for three coins.
(iv) Repeat the above for four coins.

38.5 Counting principles

Questions 38.1.

Q1. Two men and two women are randomly seated in four chairs standing in a row. What is the probability that:

(i) men will occupy two neighbouring chairs?
(ii) neither men nor women will be separated?
(iii) no one will have a person of the same gender as a neighbour?

Q2. An urn contains 3 white balls and 2 black balls. What is the probability that two balls chosen at random are white?

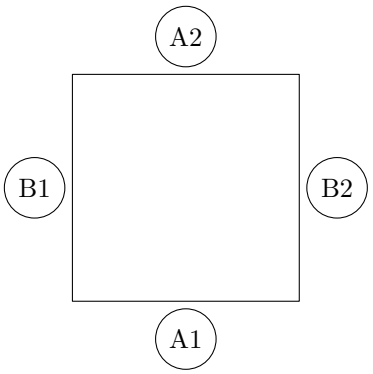
Q3. A committee of two people was chosen from the group of three men and four women. What is the probability that:

(i) no woman is chosen to the committee?

(ii) one woman is chosen to the committee?

(iii) at least one man is chosen to the committee?

Q4. In the game of bridge four players play in two teams with partners sitting opposite each other (in the diagram team A consists of players A1 and A2, while team B consists of players B1 and B2).
In one of the games players A1 and A2 had 8 spades together. What is the probability that:



(i) one of the players from team B has 3 spades while the other one has 2 spades?

(ii) one of the players from team B has 4 spades while the other one has 1 spade?

(iii) one of the players from team B has all 5 spades while the other one has no spades?

In the other game players A1 and A2 had 9 hearts together.
What is the probability that:

(iv) both players from team B have 2 hearts?

(v) one of the players from team B has 3 hearts while the other one has 1 heart?

(vi) one of the players from team B has all 4 hearts while the other one has no hearts?

In another game players A1 and A2 had 10 diamonds together. The three missing were K♢, 9♢ and 7♢.
What is the probability that:

(vii) one of the players from team B has all 3 diamonds while the other one has none?

(viii) one of the players from team B has the king of diamonds while the other one has both 9♢ and 7♢?

(ix) one of the players from team B has K♢ and one other card, while the other one has the third missing card?

Chapter 39

Conditional probability

39.1 Combined events

Questions 39.1.

Q1. Use Venn diagram to verify whether:

(i) $P(A \setminus B) = P(A \setminus (A \cap B))$
(ii) $P((A \cup B)') = P(A' \cap B')$
(iii) $P((A \cap B')') = P(A' \cup B)$
(iv) $P(A \cap B') = P(A) - P(A \cap B)$

Q2. The events A and B are such that $P(A) = 0.83, P(B) = 0.72, P(A \cup B) = 0.94$.
Find:

(i) $P(A \cap B)$ (ii) $P(B \cap A')$ (iii) $P(A \cap B')$ (iv) $P(A' \cup B')$

Q3. The events A and B are such that $P(A) = 0.8, P(B) = 0.7, P(A \cap B) = 0.6$. Find:

(i) $P(A \cup B)$ (ii) $P(A')$ (iii) $P(A \setminus B)$ (iv) $P(A \cup B')$

Q4. The events A and B are such that $P(A') = 0.4, P(B) = 0.65, P(A \cap B) = 0.4$. Find:

(i) $P(A \cup B)$ (iii) $P((A \cap B)')$ (v) $P(B \cup A')$
(ii) $P(B \cap A')$ (iv) $P(A \setminus B)$ (vi) $P(A' \cup B')$

Q5. The events A and B are such that $P(A) = 0.65, P(A \cap B) = 0.2, P(A \cup B) = 0.8$. Find:

(i) $P(B)$ (ii) $P(B \setminus A)$ (iii) $P(A \cup B')$

Q6. The events A and B are such that $P(A) = 0.55, P(A \setminus B) = 0.2, P(B') = 0.3$. Find:

(i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(B \setminus A)$

Q7. The events A and B are such that $P(A) = p, P(B) = 2p^2$ and $P(A \cup B) = p + p^2$. Find:

(i) $P(A \cap B)$ (ii) $P(A \cap B')$ (iii) $P(A \cup B')$

Q8. A letter is chosen at random from the letters of the word ALABAMA.

(i) Find the probability that B was selected.
(ii) Find the probability that A was chosen.
(iii) Find the probability that a consonant is chosen.

Q9. A letter is chosen at random from the letters of Latin alphabet (consisting of 26 letters).

(i) Find the probability that the letter comes from the word ALABAMA.
(ii) Find the probability that the letter is a vowel or comes from the word ALABAMA.

Q10. Among 25 students in a class 19 learn English, 12 learn German and 3 learn none of the two languages. Find the probability that a randomly chosen student

(i) learns both languages;
(ii) learns only one of the two languages.

Q11. Two fair dice, one red and one yellow, are rolled.

(i) Draw a lattice diagram.
(ii) Find the probability that two sixes are obtained.
(iii) Find the probability that the numbers obtained on both dice are the same.
(iv) Find the probability that the number on the yellow die is larger.
(v) Find the probability that the sum of both results is 7.
(vi) Find the probability that the sum of 6 or an even number on the red die is obtained.

Q12. A fair coin is tossed three times.

(i) Draw a tree diagram and hence write down the sample space.
(ii) Find the probability that exactly one head is seen uppermost.
(iii) Find the probability that exactly two heads are seen uppermost.
(iv) Find the probability that at least two heads are seen uppermost.
(v) Find the probability that exactly two heads in succession are seen uppermost.

39.2 Conditional probability

Suppose the die is rolled but you do not know the result.
The sample space is $U = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{4, 6\}$.

- If you were to bet on A or on A' what would you choose?

An obvious answer is that it makes no sense to bet on A, as:

$$P(A) = \frac{2}{6} = \frac{1}{3} < \frac{1}{2}.$$

Suppose that someone tells you that event $B = \{2, 4, 6\}$ occurred.

- If you were to bet on A or on A' given that B occurred, what would you choose?

We have to realize that the sample space is narrowed. The events we are to compare are:

$$A \cap B = \{4, 6\} \text{ and } A' \cap B = \{2\}.$$

Obviously, this time it does make sense to bet on A, as our chances are:

$$\frac{n(A \cap B)}{n(B)} = \frac{2}{3} > \frac{1}{2}.$$

The above probability is called **conditional probability** of event A when B is given:

$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{n(A \cap B) \div n(U)}{n(B) \div n(U)} = \frac{P(A \cap B)}{P(B)}.$$

Conditional probability of event A when B is known

(*probability of A given B*) is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

39.3 Independent events

Event A is **independent** of event B if and only if

$$P(A|B) = P(A)$$

i.e. when knowledge of B occurring does not influence the probability of A.
By definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

or

$$P(A \cap B) = P(A)P(B)$$

This shows that "A independent of B" is equivalent to "B independent of A".

Events A and B are independent if and only if

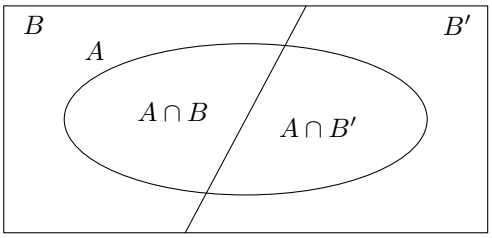
$$P(A \cap B) = P(A)P(B)$$

39.4 The law of total probability

(The law of total probability for two events - B and B'.)

The diagram illustrates the fact that

$$P(A) = P(A \cap B) + P(A \cap B').$$



When we combine the result with the definition of conditional probability
i.e.

$$P(A \cap B) = P(A|B)P(B)$$

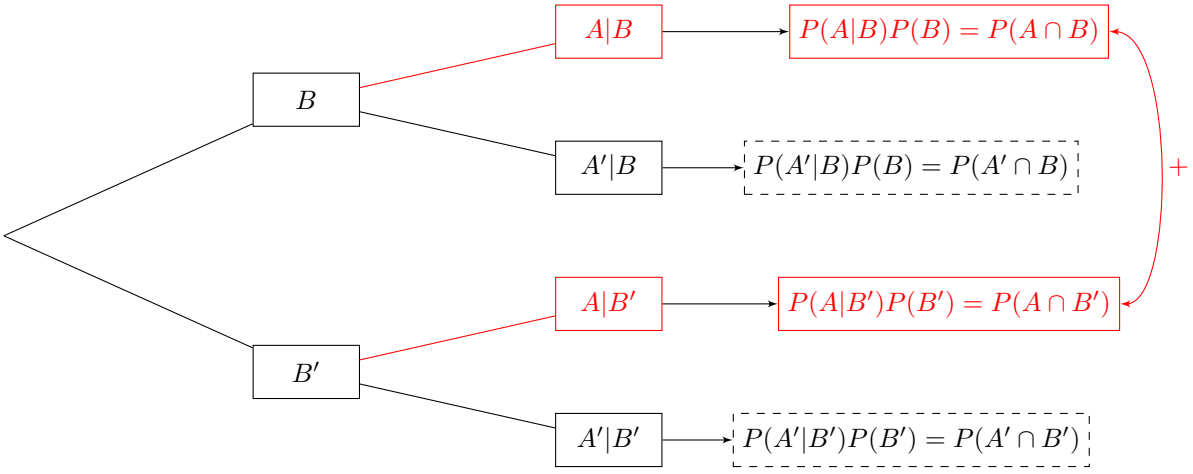
and $P(A \cap B') = P(A|B')P(B')$ we get

the law of total probability for two events:

$$P(A) = P(A|B)P(B) + P(A|B')P(B')$$

Similarly we can get a version for n events expressing the sample space U as a union of n mutually exclusive sets:
$U = B_1 \cup B_2 \cup ... \cup B_n$, $B_k \cap B_i = \emptyset$ for $k \neq i$
instead of $U = B \cup B'$, $B \cap B' = \emptyset$.

The result may be illustrated on a tree diagram.



39.5 Bayes' theorem

The tree diagram above shows only conditional probability when B or B' is given. When we want to reverse the situation, we use the definition of conditional probability:

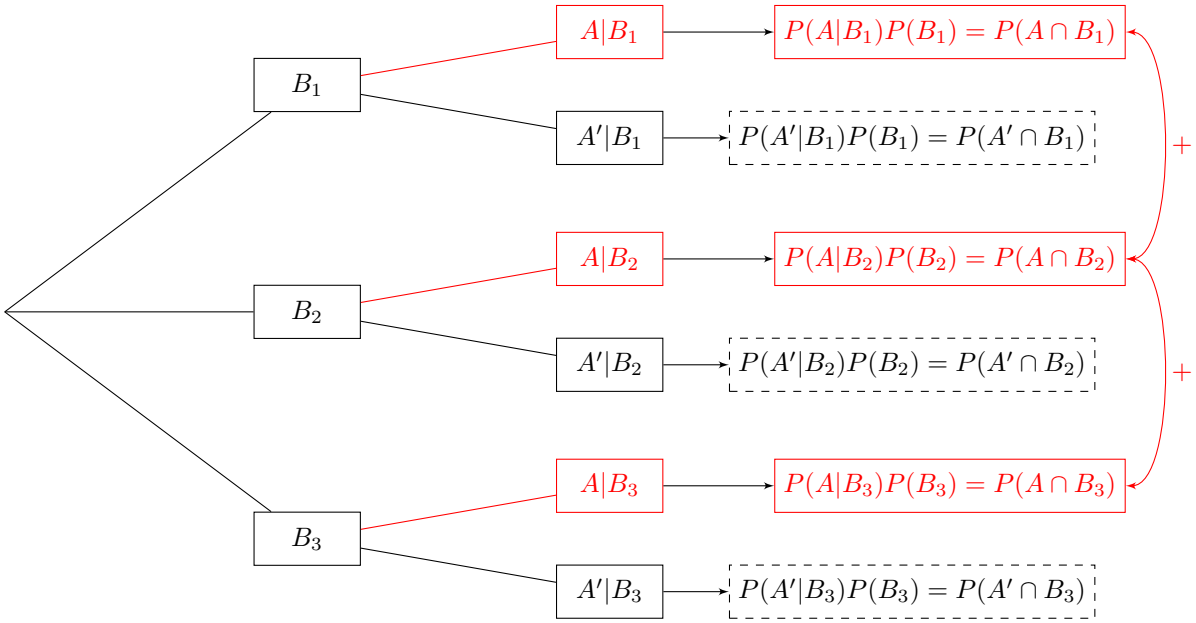
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

and then using the law of total probability we get the so-called

Bayes' theorem for 2 events

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|B')P(B')}$$

(You do not have to memorize the theorem. It is sufficient to understand that you can "read" it from the tree diagram. Simply divide probability in one of the branches by the sum of this one and one other; in the diagram above the first by sum of first and third.)
The theorem can be applied to more than two events. The tree diagram below illustrates the Bayes' theorem for three events.



Bayes' theorem for 3 events

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

Questions 39.2.

Q13. Let A and B be events such that $P(A) = 0.7$, $P(B) = 0.3$ and $P(A \cup B) = 0.8$. Find the probability of:

(i) $P(A \cap B)$
(ii) $P(A|B)$
(iii) $P(B|A)$
(iv) $P(A|B')$
(v) are A and B independent?

Q14. Let A and B be events such that $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.3$. Find the probability of:

(i) $P(A \cup B)$
(ii) $P(A|B)$
(iii) $P(B|A)$
(iv) $P(B|A')$
(v) are A and B independent?

Q15. Let A and B be events such that $P(A) = 0.4$, $P(B) = 0.75$ and $P(A \cup B) = 0.85$. Find the probability of:

(i) $P(A \cap B)$ (ii) $P(A|B)$ (iii) $P(B|A)$ (iv) $P(A'|B')$ (v) are A and B independent?

Q16. Let A and B be events such that $P(A) = 0.65$, $P(B) = 0.35$ and $P(A \cap B) = 0.3$. Find the probability of:

(i) $P(A \cup B)$ (ii) $P(A|B)$ (iii) $P(B|A)$ (iv) $P(B'|A)$ (v) are A and B independent?

Q17. Let A and B be events such that $P(A) = 0.6$, $P(B) = 0.2$ and $P(A \cup B) = 0.8$. Find the probability of:

(i) $P(A \cap B)$ (ii) $P(A|B)$ (iii) $P(B|A)$ (iv) $P(B'|A')$

Q18. Let A and B be independent events such that $P(A) = 0.8$ and $P(B) = 0.45$. Find the probability of:

(i) $P(A \cup B)$ (ii) $P(A'|B)$ (iii) $P(B|A)$ (iv) $P(B'|A')$

Q19. Let A and B be independent events such that $P(A) = 0.48$ and $P(A \cup B) = 0.74$. Find the probability of:

(i) $P(A \cap B)$ (ii) $P(B)$ (iii) $P(A'|B)$ (iv) $P(B'|A')$

Q20. A box contains 3 black tokens and 7 red tokens. Two tokens are selected at random and in succession from the box.

(i) Draw a tree diagram that represents the situation.
(ii) Find the probability that both tokens are red.
(iii) Find the probability that the first token chosen is red.
(iv) Find the probability that only the first token chosen red.
(v) Find the probability that at least one token chosen is red.
(vi) Find the probability that only one token is red.
(vii) If the first token is black, find the probability that the second is red.
(viii) If the first token is red, find the probability that the second is red.
(ix) Find the probability that the second token chosen is red.
(x) If only one token is red, find the probability that it is the first chosen.

Q21. There are four coins placed in a bag: one fair coin, two double-headed and one double-tailed. A coin is selected at random from the bag and then tossed.

(i) Draw a tree diagram that represents the situation.
(ii) Find the probability that the coin tossed lands with a head shown uppermost.
(iii) If the coin shows a head uppermost, find the probability that it is a fair coin.

Q22. A box contains 4 red balls and 8 white balls. A ball is randomly drawn out of the box, its colour is noted and the ball is then replaced. Then another ball is drawn.

(i) Find the probability that both balls are white.
(ii) Find the probability that the two balls are of different colours.

Q23. A box contains 4 red balls and 5 white balls. A ball is randomly drawn out of the box, its colour is noted and the ball is then replaced together with additional ball of the same colour. Then another ball is drawn.

(i) Find the probability that both balls are white.
(ii) Find the probability that the two balls are of different colours.

Q24. Mugs are packed in boxes of 10. There are three different colours of mugs: red, blue and yellow. In a box considered there are 5 red and 2 blue mugs. Two mugs are selected at random and in succession from the box. The first one is not replaced before the second is selected.

(i) Find the probability that the two mugs are of different colours.
(ii) Find the probability that the second mug is blue.

(iii) Given that the second mug is blue, find the probability that the first one is red.

(iv) Given that the second mug is red, find the probability that the first one is blue.

Q25. Two fair coins are tossed in succession.

(i) Find the probability that the first coin shows a head.

(ii) Find the probability that the second coin shows a head.

(iii) Find the probability that both coins show heads.

(iv) Find the probability that the 1st coin shows a head given that both show heads.

(v) Find the probability that the 1st coin shows a head given there was at least one head on both coins.

Q26. Two fair six-sided dice (red and yellow) are rolled.

(i) Find the probability that the sum shown on the two dice is 8,

(ii) Given that the sum is 8, find the probability that the red die shows 6.

(iii) Given that the sum is 8, find the probability that the one of the dies shows 6.

(iv) Given that the sum is 8, find the probability that the faces are different.

(v) Given that the faces are different, find the probability that the sum is 8.

Q27. 0.2% of a population is infected with a virus. The test diagnoses the virus correctly in 99% of cases when a person tested is infected and in 95% of cases when a person tested is healthy.

(i) Given that the test is positive, find the probability that the person is infected.

(ii) Given that the test is positive, find the probability that the person is healthy.

(iii) Given that the test is negative, find the probability that the person is infected.

Q28. Show that the events A and B are independent whenever events A' and B' are independent.

Q29. A short test in mathematics consists of one question with 5 answers given. Only one of them is correct. Adam has absolutely no knowledge on the topic of the test, while Borat knows that one of the answers is wrong. They decided to choose the answer at random hoping they guess the correct one.

(i) Find the probability that they both answer the question correctly.

(ii) Find the probability that only one of them answers the question correctly.

(iii) Given that only one of them answered the question correctly, find the probability that it was Borat.

Q30. The probability that Adam passes a biology test is 80%. The probability that Borat passes the test is 50%. However, because of the regrettable test cooperation syndrome, the probability that Borat passes the test when Adam passes is 60%. What is the probability that:

(i) both pass the test?

(ii) Borat passes the test given Adam fails?

(iii) Adam passes given that Borat fails?

Q31. A fair coin is tossed three times in succession. What is the probability of observing

(i) three heads?

(ii) exactly two heads?

(iii) at least one head?

(iv) exactly two heads given that at least one head was observed?

Q32. A coin is weighted in such a way that the odds in favour of heads are 2:1. The coin is tossed three times in succession. What is the probability of observing

(i) three heads?

(ii) exactly two heads?

(iii) at least one head?

(iv) exactly two heads given that at least one head was observed?

Q33. In the school there are 60% of girls and 40% of boys out of which 25% and 75% are taller than $175cm$. Find the probability that a randomly chosen student:

(i) is taller than $175cm$,

(ii) is taller than $175cm$ given that the student is a girl,

(iii) is a girl, given that the student is taller than $175cm$,

(iv) is a boy, given that the student is shorter than $175cm$.

Q34. Two factories (A and B) produce light bulbs for one store. They produce 30% and 70% of the amount respectively. It turns out that respectively 2% and 5% of the output is defective. Find the probability that a randomly selected light bulb:

(i) is defective,

(ii) is from factory A given it is defective,

(iii) is from factory B given it is defective.

Q35. An urn labelled A contains 7 cards numbered 1 to 7. An urn labelled B contains 5 cards numbered 1 to 5. Two cards are selected at random, one from each urn. Find the probability that:

(i) the numbers on both cards are even,

(ii) the product of the two numbers is even,

(iii) the sum of the two numbers is even,

(iv) the card from B has an even number, given that the product is even,

(v) both cards are even, given that the product is even.

Q36. An urn labelled A contains 7 cards numbered 1 to 7. An urn labelled B contains 5 cards numbered 1 to 5. An urn labelled C contains 3 cards numbered 1 to 3. A fair six-sided die is rolled. When a number 1 appears, a card is drawn from urn A, when 2 or 3 appear, a card is drawn from urn B. Otherwise a card is drawn from urn C. Find the probability that

(i) the number on a card drawn is 3,

(ii) the number on a card drawn is 4,

(iii) the number on a card drawn is even,

(iv) a card was drawn from urn A given that the number on the card is even,

(v) a card was drawn from urn C given that the number on the card is odd.

Q37. Ann goes to school five days a week. Twice a week she takes a bus, twice a week she bikes and once a week she walks. When she goes by bus she is late on 5% of cases. When she bikes she is late on 10% of cases while when she walks she is late on 15% of cases. Find the probability that on a randomly chosen day

(i) Ann is late,

(ii) she took a bus given she is late,

(iii) she walked to school given she is on time.

Questions 39.3.

Q1. (*Monty Hall problem*)
Suppose you're on a game show, and you're given the choice of three doors: Behind one door there is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Q2. A natural number a is chosen at random from the first one hundred positive natural numbers. What is the probability that $a + \frac{10000}{a} > 500$?

Q3. A ball is drawn from a bag which contains 4 balls each of which is equal likely to be a black or a white. If the colour of the drawn ball is white then what is the probability that the bag contained 3 white and 1 black ball?

Q4. Eight people entered the elevator on the ground floor of a 20-floor building (this includes the ground floor). Suppose each of them, independently and with equal probability, can leave the cabin at any of the other nineteen floors. Find out the probability of all eight people leaving at different floors.

Q5. A fair die is tossed twice in succession. Let A denote the first score and B the second score. Consider the quadratic equation $x^2 + Ax + B = 0$. Find the probability that

(i) the equation has 2 distinct roots.

(ii) the equation has a double root.

Q6. There are 6 blue, 7 brown, 8 black, and 5 gray pairs of socks in the drawer. The drawer is placed in a room so dark that the colours are indistinguishable. How many socks must a man pull out to be certain of having a matched pair?

Q7. There are five possible answers to each question in a multiple-choice test. A student thinks he knows the answer for 75% of the questions and for the other 25% he guesses at random. When he thinks he knows the answer, he is right 80% of the time. Find the probability of getting a randomly chosen question right.

Q8. Let $f(x) = (x + 3)(2x - 1)(x - 4)$. Let $S = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. Two different numbers a and b are drawn at random from set S. Find the probability that

(i) $f(a)f(b)$ is positive

(ii) $f(a)f(b)$ is negative

(iii) $f(a)f(b) = 0$

Chapter 40

Discrete random variables

When a fair coin is tossed three times the sample space is as follows:

$$U = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$$

where H and T denote a head or a tail seen uppermost respectively. If our concern is about the number of heads seen uppermost, certain numerical values can be assigned to all the outcomes:

number of heads	outcomes
0	TTT
1	TTH, THT, HTT
2	THH, HTH, HHT
3	HHH

The numbers are **random**, i.e. they are not known before the experiment is conducted.
The variable taking these values is called a **random variable**.
The random variable can be **discrete** or **continuous** depending on kind of data from which the values result.
The concept of a random variable can be shown with use of our example, where X denotes the number of heads.

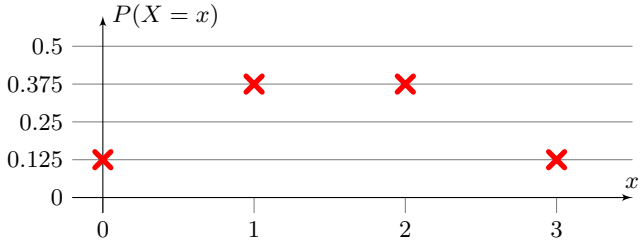
sample space *outcomes*	→	**random variable** *numerical values* X *(usually capital letters)*	→	**probability** *that* X *takes certain value/-s* $P(X = x)$
TTT		0		$P(X = 0) = \frac{1}{8}$
TTH, THT, HTT		1		$P(X = 1) = \frac{3}{8}$
THH, HTH, HHT		2		$P(X = 2) = \frac{3}{8}$
HHH		3		$P(X = 3) = \frac{1}{8}$

This **distribution** of probability can be shown in a variety of forms:

1. TABLE

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

2. GRAPH



3. FUNCTION $P(X=x) = \binom{3}{x}(\frac{1}{8}), x \in \{0,1,2,3\}$

40.1 Discrete random variables

Properties of a discrete random variable

1. $0 \leq P(X=x_i) \leq 1$ for all possible values of x_i
2. $\sum P(X=x_i) = 1$, i.e. sum of all probabilities is 1

Sometimes it is convenient to introduce a **cumulative distribution function** $F : \mathbb{R} \longmapsto [0,1]$ defined as follows:

$$F(x) = P(X \leq x)$$

Properties of a cumulative distribution function of a discrete random variable X

1. $0 \leq F(x) \leq 1$ for all possible values of x
2. F is non-decreasing
3. when $x \to -\infty$, $F(x) \to 0$
4. when $x \to +\infty$, $F(x) \to 1$
5. $P(a < X \leq b) = F(b) - F(a)$ for $a \leq b$

Questions 40.1.

Q1. Find the value of k for which the random variable X describes a probability distribution.

x	1	2	3	4	5
$P(X=x)$	0.2	0.1	k	0.15	0.1

Q2. Find the value of k for which the random variable X describes a probability distribution.

x	1	2	4	8	16
$P(X=x)$	0.25	0.2	k	0.15	$3k$

Q3. The random variable Y has the following probability distribution.

y	-1	0	2	3
$P(Y=y)$	k^2	$1-k$	k^2	$2-4k$

(i) Find the value of k.
(ii) Find $P(Y > 2)$.
(iii) Find $P(Y \leq \frac{1}{2})$.
(iv) Find $F(2)$.
(v) Find $F(1.5)$.

Q4. There are 5 TV-sets in the store. Three of them are defective. A customer buys two TV-sets. Let D denote the number of defective sets he buys.

(i) List the possible values of D.

(ii) Construct the probability table.

(iii) Find $P(D = 2|D \geq 1)$.

Q5. A fair die is rolled twice. Let S describe the sum of the results seen.

(i) Find $P(S > 6)$.

(ii) Find the probability that $S > 6$ given that S is even.

(iii) Find the probability that S is even given that $S > 6$.

Q6. Two fair six-sided dice have faces with numbers: 1, 1, 2, 2, 3 and 3. The dice are rolled. Let X denote the sum of numbers seen uppermost.

(i) Find the probability distribution of X.

(ii) Find $P(X \leq 4|X \geq 3)$.

(iii) Find the probability that $X > 3$ given that the two dice show different numbers.

(iv) Find the probability that the dice show the same number and hence the probability that the number on one chosen die is larger.

Q7. A coin is weighted in such a way that the odds heads: tails are 3:2. The coin is tossed twice. Let X denote the number of heads seen uppermost.

(i) Draw a tree diagram that shows the situation and construct the probability table for X.

(ii) Find $P(X \leq 1)$.

(iii) Find $P(X = 2|X > 0)$.

The coin is tossed repeatedly till tail appears uppermost. Let Y denote the number of tosses.

(iv) Write down

(a) $P(Y = 1)$

(b) $P(Y = 2)$

(c) $P(Y = 3)$

(v) Find $P(Y = 7)$.

(vi) Find $P(Y > 2)$.

(vii) Find $P(2 \leq Y \leq 4)$.

(viii) Find $P(Y \leq 4|Y > 2)$.

(ix) Find n, the minimum number of tosses such that the probability that a tail appears at least once in n tosses is larger than 0.99.

Q8. A fair coin is tossed three times.

(i) Draw a tree diagram that represents the situation.

(ii) Find the probability distribution of T - the number of heads.

(iii) Find $P(T \geq 2)$.

(iv) Find $P(T < 3|T \geq 2)$.

Q9. A random variable X has the following probability distribution:

x	-1	0	1	2
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$

Let a random variable Y be defined as $Y = X^2 - X$.

(i) Find a probability distribution of Y.

(ii) Find $P(Y \leq 0)$.

Q10. There are 4 cats ~~in a bag~~ on the roof of which 2 are black and the other 2 white. Two cats are randomly chosen to play a major part in "Puss in boots". Let B denote the number of black cats chosen.

(i) Construct a probability table for B.

(ii) Find the probability that the cats chosen are of different colours.

(iii) Given that the cats chosen are of the same colour, find the probability that they are both white.

Q11. There are 6 donuts in a bag of which 2 are filled with strawberry jam and the other 4 with cherry jam. A girl chooses two donuts at random. Let X denote the number of donuts chosen that contain strawberry jam. Construct a probability table for X.

Q12. There are 9 girls 11 boys in a class. The committee of 5 is to be chosen at random. Let X denote the number of girls chosen for the committee.

(i) Construct a probability table for X.

(ii) Find $P(X > 0)$.

(iii) Find $P(X = 3|X \geq 2)$.

Q13. There are 20 players in a handball team. Of them 3 are left-handed. Five players are chosen at random from the team. Let L denote the number of left-handed players chosen.

(i) Construct a probability table for L.

(ii) Find $P(L \geq 1)$.

(iii) Find $P(L = 2|L \geq 1)$.

Q14. The random variable X has a probability distribution defined by:

$$P(X = x) = \frac{x}{k}, x \in \{1, 2, 3, 4\}.$$

(i) Find the value of k.

(ii) Find:

(a) $P(X > 2)$ (b) $P(X > 2|X > 1)$

Q15. The random variable X has a probability distribution defined by:

$$P(X = x) = k(x - 1), x \in \{1, 2, 4, 8, 16\}$$

(i) Find the value of k.

(ii) Find:

(a) $P(X > 3)$ (b) $P(X > 3|X < 10)$

Q16. Let X be a discrete random variable that has non-zero probabilities for $x \in \{1, 2, 3\}$. Let F defined $F(x) = \frac{x^2}{9}$ for $x \in \{1, 2, 3\}$ be the cumulative distribution function for X.

(i) Fill in the table of values of F.

x	0	0.5	1	1.5	2	2.5	3	5
$F(x)$								

(ii) Fill in the distribution table of X.

x	0	0.5	1	1.5	2	2.5	3	5
$P(X = x)$								

40.2 Expected value of a discrete random variable

For a discrete random variable X its **expected value** is defined:

$$E(X) = \sum x_i P(X = x_i)$$

where sum is calculated for all possible values of x_i.

To find other measures of central tendency:

mode - most common / probable value

median - value with half probabilities below and half above the median value, i.e. m is the median if and only if $P(X < a) < 0.5$ for $a < m$ **and** $P(X < a) > 0.5$ for $a > m$

40.3 The meaning of expected value

Example 278 (expected value for a die rolled).
Show that when a fair die is rolled and X denotes the number of dots seen on a face uppermost then its expected value $E(X) = 3.5$.

Example 279 (expected value for 3 coins tossed).
Show that when three fair coins are tossed and X denote the number of heads seen uppermost then $E(X) = 1.5$.

Does it mean that we should expect to see 3.5 dots on a face of a die (**Example 278**) or 1.5 heads in 3 tosses (**Example 279**)? (*Don't answer. No, it does not.*)
It should be understood as a **mean** result if the experiment was conducted many times, i.e.

- if a fair die was rolled 1000 times the total number of dots should be approximately 3500 and the mean number 3.5;
- if three fair coins were tossed 1000 times the total number of heads should be approximately 1500 and the mean number 1.5.

40.4 Properties of expected value of a discrete random variable

1. $E(a) = a$, where a is constant
2. $E(aX) = aE(X)$, where a is a constant.
3. $E(g(X)) = \sum g(x_i) \times P(X = x_i)$
4. $E(aX + b) = \sum(ax_i + b) \times P(X = x_i) = aE(X) + b$, where a and b are constants.
Note: **(4)** is a special case of **(3)**; $g(X) = aX + b$

NOTE: $E(X^2) \neq (E(X))^2$
$E(g(X)) \neq g(E(X))$, $(g(X) = aX + b$ is an exception)

While expected value is to be associated with mean, the other measures of central tendency should be understood in a simple intuitive way:
mode - most common value; value with highest probability;
median - value with half probabilities below and half above it.

40.5 Variance of a discrete random variable

For a discrete random variable X its **variance** is defined:
$$Var(X) = E((X - \mu)^2) = \sum(x_i - \mu)^2 P(X = x_i)$$
where sum is calculated for all possible values of x_i.
For calculations:
$$Var(X) = E(X^2) - (E(X))^2 = E(X^2) - \mu^2$$
$$= \left(\sum x_i^2 P(X = x_i)\right) - \mu^2$$

Notation:
$$Var(X) = \sigma^2$$
Standard deviation is obviously defined as:
$$Sd(X) = \sigma = \sqrt{Var(X)}$$

40.6 Properties of variance of a discrete random variable

1. $Var(a) = 0$, where a is constant
2. $Var(aX + b) = a^2 Var(X)$, where a and b are constant.
3. $Var(g(X)) = \sum(g(x_i))^2 \times P(X = x_i) - (\sum g(x_i) \times P(X = x_i))^2$ where a and b are constants.

Questions 40.2.

Q17. A discrete random variable X has a probability distribution as given.

x	1	2	3	4	5
$P(X = x)$	0.2	0.1	0.45	0.15	0.1

(i) Find $E(X)$.
(ii) Find $VarX$.

Q18. A discrete random variable X has a probability distribution as given.

x	1	2	4	8	16
$P(X = x)$	0.25	0.2	0.3	0.15	0.1

Find:

(i) $E(X)$
(ii) $VarX$
(iii) $E(3 - 2X)$
(iv) $Sd(3 - 2X)$
(v) $E(X - 4)^2$
(vi) $Var(X - 4)^2$

Q19. The random variable Y has the following probability distribution.

y	-1	0	2	4
$P(Y = y)$	0.2	0.5	0.25	0.05

Find:

(i) $E(Y)$
(ii) $VarY$
(iii) $E(2Y + 1)$
(iv) $Var(2Y + 1)$
(v) $E(Y^2)$
(vi) $E(3 - Y)$
(vii) $Var(3 - Y)$
(viii) $E(\frac{1}{Y-1})$
(ix) $Var(\frac{1}{Y-1})$

Q20. Among 4 cats there are 2 black and 2 white cats. Two cats are randomly chosen to play a major part in "Puss in boots". Let B denote the number of black cats chosen. Find the mean and variance of B.

Q21. There are 6 donuts in a bag of which 2 are filled with strawberry jam and the other 4 with cherry jam. A girl chooses two donuts at random. Let X denote the number of donuts chosen that contain strawberry jam. Find the mean and variance of X.

Q22. There are 9 girls 11 boys in a class. The committee of 5 is to be chosen at random. Let X denote the number of girls chosen for the committee. Find the mean and variance of X.

Q23. The random variable X has a probability distribution defined by:

$$P(X = x) = \frac{x}{k}, x \in \{1, 2, 3, 4\}.$$

(i) Find the value of k.
(ii) Find the mean and the variance of the distribution.

Q24. The random variable X has a probability distribution defined by:

$$P(X = x) = k(x - 1), x \in \{1, 2, 4, 8, 16\}$$

(i) Find the value of k.

(ii) Find the mean and the variance of the distribution.

Q25. Two fair dice are rolled. Let X denote the sum of results on both dice. Find the mean and standard deviation of X.

Q26. A fair die is rolled. Find the mean of X - a square of the result seen on the die.

Q27. Three fair coins are tossed. Let X denote the number of heads seen uppermost. Find the mean and standard deviation of X.

Q28. A box contains 3 black tokens and 7 red tokens. Two tokens are selected at random and in succession from the box. Let X denote the number of black tokens chosen. Find the mean and variance of X.

Q29. There are four coins placed in a bag: two fair coins, one double-headed and one double-tailed. A coin is selected at random from the bag and then tossed twice. Let X denote the number of heads observed. Find the mean and variance of X.

Q30. A box contains 4 red balls and 8 white balls. A ball is randomly drawn out of the box, its colour is noted and the ball is then replaced. Then another ball is drawn. Let X denote the number of red balls drawn. Find the mean and variance of X.

Q31. A box contains 4 red balls and 5 white balls. A ball is randomly drawn out of the box, its colour is noted and the ball is then replaced together with additional ball of the same colour. Then another ball is drawn. Let X denote the number of red balls drawn. Find the mean and variance of X.

Q32. In a game a fair die is rolled twice. If 3 or 6 are seen twice, the player wins \$10.
If 3 or 6 are seen once, the player wins \$2. If any other number but 3 or 6 are seen twice, the player looses \$6. Is the game fair?

Q33. In a game a biased coin (odds heads : tails are 3:2) is tossed twice. If two tails are seen, the player wins \$10. If two heads are seen uppermost, the player looses \$8. Should the player loose or win when one head and one tail are seen so that the game is fair? How much should he win / loose?

Q34. To enter a game a player pays the fee of $\$a$. In the game the player throws a dart at a circular dartboard that consists of three regions made by three concentric circles with radii $10cm$, $20cm$ and $30cm$ respectively. The largest circle makes the whole dartboard. The player wins \$7, \$5 and \$1 when he hits the smallest, middle and the largest region respectively. Assuming that the player misses the dartboard with probability $\frac{1}{2}$ and the probability of hitting each of the regions is proportional to its area, find the player's expected winnings. What should be the entrance fee a for the game to be fair?

Q35. Given that $E(X) = 1.5$ and $E(X^2) = 3.25$ find:

(i) $Var(X)$ (ii) $Var(2X + 3)$ (iii) $E((X - 2)^2)$

Q36. Given that $E(X) = 1.3$ and $Var(X) = 0.99$ find:

(i) $E(X^2)$ (ii) $Var(2 - 3X)$ (iii) $E((X - 1)(X + 1))$

Chapter 41

Binomial distribution

The Binomial distribution describes experiments consisting of a **fixed** number of **independent** and **identical** trials, each having **two possible out-comes** called a *success* and a *failure*. Such trials are called Bernoulli trials. In a Bernoulli trial probability of a success (same for every trial) is usually denoted with p and probability of failure with $q = 1 - p$. We say that a random variable X follows a binomial distribution with n trials and with probability of success p, which is denoted by $X \sim B(n, p)$. Then for $x \in \{0, 1, 2, ..., n\}$:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

41.1 Expected value, variance, standard deviation

For $X \sim B(n, p)$:

$$E(X) = np$$

$$Var(X) = npq = np(1-p)$$

$$Sd(X) = \sqrt{npq} = \sqrt{np(1-p)}$$

Questions 41.1.

Q1. A box contains 10 balls, 3 of which are red and the others - white.
A ball is selected at random and then replaced in the box. The process is carried out 6 times.

(i) What is the probability that no red balls will be chosen?

(ii) What is the probability that all chosen balls will be red?

(iii) What is the probability exactly 2 red balls will be chosen?

(iv) What is the probability at least 2 red balls will be chosen?

(v) What is the probability that 2 to 5 red balls will be chosen?

(vi) What is the expected number of the red balls chosen?

(vii) What is the standard deviation of the number of the red balls chosen?

Q2. A box contains 10 balls, 3 of which are red and the others - white.
A ball is selected at random and then replaced in the box. The process is carried out 60 times.

(i) What is the probability that no red balls will be chosen?

(ii) What is the probability that all chosen balls will be red?

(iii) What is the probability exactly 30 red balls will be chosen?

(iv) What is the probability at least 20 red balls will be chosen?

(v) What is the probability that 15 to 35 red balls will be chosen?

(vi) What is the expected number of the red balls chosen?

(vii) What is the standard deviation of the number of the red balls chosen?

Q3. At an election 35% of voters supported the Support Our Uncle Party. 10 randomly chosen voters were interviewed.

(i) What is the probability that all of them supported SOUP?

(ii) What is the probability that half of them supported SOUP?

(iii) What is the probability that at least half of them supported SOUP?

Q4. A test consists of 20 questions. There are 4 answers to each question. A student needs to score more than 40% of points to pass a test. A student will sit the test but his knowledge is sufficient to try to guess all the answers only.

(i) What is the probability that he will answer all questions correctly?

(ii) What is the probability that he will answer correctly exactly 10 questions?

(iii) What is the probability that he will pass the test?

(iv) What is the probability that he will answer all questions incorrectly?

(v) What is the expected number of questions answered correctly?

(vi) What is the expected number of questions answered incorrectly?

Q5. A manufacturer finds that during the production 10% of the items produced contain flaws.
A sample of 12 items are selected for inspection.

(i) What is the probability that they will be all flawless?

(ii) What is the probability that they will be all defective?

(iii) What is the probability that at least three of them will be defective?

A sample of 200 items is then selected.

(iv) What is the expected number of defective items in the sample?

Q6. Let $X \sim B(6, \frac{2}{3})$. Find:

(i) $P(X = 4)$

(ii) $P(X \leq 4)$

(iii) $P(X > 4)$

(iv) $P(X < 4)$

(v) $P(X = 3|X \leq 4)$

(vi) $P(X \leq 3|X \leq 4)$

(vii) $P(X \leq 4|X > 2)$

(viii) $P(X < 4|X > 1)$

Q7. Six-faced fair die with faces numbered from 1 to 6 is rolled

(i) Write down the probability that a multiple of 3 appears uppermost.

(ii) If the die is rolled 8 times, what is the probability that a multiple of 3 appears exactly twice?

(iii) If the die is rolled 8 times, what is the probability that a multiple of 3 appears at least once?

A die is rolled a number of times until a multiple of 3 appears.

(iv) Find the probability that a multiple of 3 appeared for the 1^{st} time in the 5^{th} roll.

A die is rolled a number of times.

(v) How many times the die must be rolled so that the probability of seeing the multiple of 3 is greater than 90%?

Q8. Knowing that $X \sim B(n, p)$, where $E(X) = 6$ and $Var(X) = 4.2$ find:

(i) the values of n and p,
(ii) $P(X > 20)$
(iii) $P(X \leq 3)$
(iv) $P(X < 5)$
(v) $P(X > 8)$
(vi) $P(X \geq 7)$
(vii) $P(X \leq 5|X > 3)$
(viii) $P(X < 5|X \geq 3)$

Q9. The probability that Robin, the archer, hits the target is 0.75. (As over a long period of time he has been successful on 75% of his attempts.) He takes part in a competition in which he participates in 2 rounds, having 4 attempts in each round.

(i) Find the probability that Robin is successful on all 8 attempts.
(ii) Find the probability that Robin is successful on 4 attempts.
(iii) Find the probability that Robin is successful on at least 4 attempts.
(iv) Find the probability that Robin manages exactly 1 perfect round.
(i.e. he is successful on all attempts in this round and he is successful on not all attempts in the other round.)

The new competition consists of 5 rounds, 4 attempts in each round.

(v) What is the probability that Robin manages all perfect rounds?
(vi) What is the probability that Robin manages exactly 2 perfect rounds?
(vii) What is the probability that Robin manages at least 2 perfect rounds?

Chapter 42

Poisson distribution

The Poisson distribution is a **discrete** probability distribution that describes the probability of a given number of events occurring in a fixed interval of time and/or space given the events occur with a known average rate and independently of the time / space since the last event.
The average rate per unit of time / space is usually denoted with μ, m or λ and X following the Poisson distribution with **mean** m is denoted with $X \sim P_0(m)$. Then for $x \in \{0, 1, 2, 3, ...\}$:

$$P(X = x) = \frac{m^x e^{-m}}{x!}$$

42.1 Expected value, variance, standard deviation

For a random variable $X \sim P_0(m)$:

$$E(X) = m, \quad Var(X) = m, \quad Sd(X) = \sqrt{m}$$

42.2 Examples of variables that may follow Poisson distribution

1. Number of answered telephone calls in a 5-hour period
2. Number of cars passing certain point on a road
3. Number of flaws in a board
4. Number of typing mistakes in 10 pages
5. Number of people in a street between 3pm and 4pm
6. Number of rotten apples in a large quantity

Questions 42.1.

Q1. Let $X \sim P_0(3.2)$. Find the following probabilities.

(i) $P(X = 3)$
(ii) $P(X \leq 3)$
(iii) $P(X < 3)$
(iv) $P(X > 0)$
(v) $P(X \geq 2)$
(vi) $P(3 < X < 8)$
(vii) $P(3 \leq X \leq 8)$
(viii) $P(X = 3|X > 2)$
(ix) $P(X \leq 5|X \geq 3)$
(x) the mode of X
(xi) the median of X
(xii) the variance of X

Q2. Let $X \sim P_0(m)$. Given that $P(X = 5) = 6.68 \times 10^{-2}$ find:

(i) the value of m
(ii) $P(X = 2)$
(iii) $P(X \leq 5)$
(iv) $P(X \geq 3)$
(v) $P(2 \leq X \leq 5)$
(vi) $P(X > 2|X \geq 1)$
(vii) the median of X
(viii) the mode of X
(ix) the st.deviation of X

Q3. Let $X \sim P_0(\mu)$. Given that $P(X \leq 2) = 0.21$ find:

(i) the value of μ
(ii) $P(X = 4)$
(iii) $P(X > 5)$
(iv) $P(X < 4)$
(v) $P(2 \leq X < 7)$
(vi) $P(X > 4|X \geq 4)$

Q4. Let $X \sim P_0(\mu)$. Given that $P(X \geq 2) = 0.769$ find:

(i) the value of μ
(ii) $P(X = 3)$
(iii) $P(X \geq 4)$
(iv) $P(X < 6)$
(v) $P(4 \leq X < 8)$
(vi) $P(X \geq 8|X \geq 4)$

Q5. The number of typing mistakes made by Alexandra occur at an average rate of 7 every 10 pages. Assuming that the number follows Poisson distribution find the probability that there will be:

(i) no mistakes in a 10-pages long section,
(ii) no mistakes in a 15-pages long section,
(iii) no mistakes in a randomly chosen page,
(iv) at most 2 mistakes in a randomly chosen page,
(v) at least two mistakes in a randomly chosen page?

A reviser chose a page and found a mistake in it. What is the probability that:

(vi) he will find no more mistakes in the page,
(vii) he will find more than 2 more mistakes in the page?

A page is considered acceptable when no more than 2 mistakes are found. If the page is acceptable what is the probability that:

(viii) it contains no mistakes,
(ix) it contains 2 mistakes,
(x) it is not free of mistakes?

Q6. The number of flaws in wooden board appears at a rate of 2 per 5 square metres. Find the probability that:

(i) there will be no flaws in $1m^2$ of board,
(ii) there will be flaws in $2m^2$ of board,
(iii) there will be only one flaw in $1m^2$ of board given there will be some flaws.

Q7. Cars have been observed to enter a highway at a given gate at a rate of 280 per hour. Find the probability that:

(i) no cars pass the gate in 5 minutes,
(ii) at least 20 cars pass the gate in 15 minutes,
(iii) not more than 150 cars pass the gate in half an hour given that at least 100 cars pass the gate at that time.

Q8. Accidents occur at a crossroad at a rate of 1 every 5 days.

(i) Find the probability that there will be no accidents in a week.
(ii) Find the probability that there will be more than 3 accidents in two weeks,
(iii) If an accident occurred in a certain week, find the probability that there will be less than 3 accidents this week.

(iv) If an accident occurred in a certain week, find the probability that there will be no more accidents this week.

Q9. An office receives phone calls at a rate of 50 per hour. Find the probability that:

(i) there will be no calls in a minute,

(ii) there will be at least two calls in a minute,

(iii) there will be more than 30 calls in a half an hour,

(iv) there will be more than 30 calls in a half an hour given there will be less than 40 calls.

Q10. A company imports large quantities of oranges. Approximately 2% of them are rotten. A box of 25 oranges in selected at random for inspection. Assuming that the number of rotten oranges follows Poisson distribution[1], what is the probability that:

(i) the box contains rotten oranges? (*i.e. at least one rotten orange*)

(ii) the box contains more than 2 rotten oranges?

Twenty such boxes are chosen for sale. Find the probability that:

(iii) they are free of rotten oranges,

(iv) only one of them contains rotten oranges,

(v) at least 10 boxes contain rotten oranges,

(vi) at most 2 boxes contain rotten oranges.

Q11. The number of flaws in glass sheets follows Poisson distribution with a mean of 0.8 flaw per square metre. 50cm by 75cm glass sheets are packed in packages of 8. If there is at least one flaw in more than one sheet from the package, the package is returned to the manufacturer.

(i) Find the probability that there are no flaws in a single sheet.

(ii) Find the probability that there are no flaws in all sheets from a single package.

(iii) Find the probability that a package inspected is sent back to the manufacturer.

(iv) Given that the package was returned to the manufacturer, find the probability that only two sheets contained flaws.

Q12. The number of orders of pizzas in a Pizza Hat restaurant follows Poisson distribution. On average, 92.4 pizzas are ordered every day. Let X denote the number of pizzas ordered in a 5-day week. Let $E(X) = \mu$ and $Var(X) = \sigma^2$. Find the following probabilities:

(i) $P(X > 500)$

(ii) $P(X < 400)$

(iii) $P(400 < X < 500)$

(iv) $P(X > 400|X < 500)$

(v) $P(X < 500|X > 400)$

(vi) $P(400 < X < 500|X > 400)$

(vii) $P(X > \mu + \sigma)$

(viii) $P(\mu - \sigma < X < \mu + \sigma)$

(ix) $P(X > \mu + \sigma|X > \mu)$

Q13. The number of cars stopping at two petrol stations (A and B) follow Poisson distribution at a rate of 12 cars per 10 and 15 minutes respectively. Find the probability that:

(i) at least 50 cars will stop at station A between 1pm and 2pm,

(ii) more than 50 cars will stop at station B between 1pm and 2pm,

(iii) at most 2 cars will stop at station A in 5 minutes period given that at least one car stops there at that time,

(iv) at least 20 cars will stop at station A in a half an hour period given that at least 10 cars stop there at that time,

(v) no cars stop at the two stations in a given 10-minutes period,

(vi) one car stops at the two stations together in a 10-minutes period,

(vii) the number of cars that stop at A exceeds that of B given that not more than one car stops at B, all in a 10-minutes period.

[1] Can the number be modelled by binomial distribution?

Q14. The number of fish that Ian catches in a lake follows Poisson distribution. The probability that he catches more than 1 fish in 5 hours is 0.594.

(i) What is the mean number of fish that Ian catches per hour?

(ii) What is the probability that he catches more than one fish in 2 hours?

(iii) How many **complete** hours should he fish so that the probability that he catches more than one fish exceeds 75%?

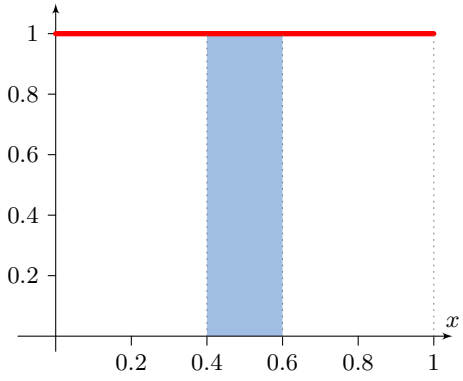
Chapter 43

Continuous random variables

When dealing with continuous data we have to change our methods. We cannot **calculate** any more, we have to start **measuring**. For example, when considering time taken by the kids to travel to school, we cannot state *what is the number of kids who travel exactly 17.5 minutes.* Neither can we say *what is the probability that randomly chosen kid travelled to school exactly 17.5 minutes.* We can think of e.g. time interval *between 15 and 20 minutes.*
Let's start with an example.
Let X denote the decimal part of a second of time measured on a *very accurate* stopwatch. In theory, X can take any value from $[0, 1[$. However, there is no real chance to stop a stopwatch **exactly** at $X = 0.5$ (similarly there is no real chance to stop it at exactly $X = 0.10825592$).
We can, however, think of probability of X being between 0.4 and 0.6.
Assuming that X is **uniformly distributed** over interval $[0, 1[$ (i.e. each interval of the same length is equally likely) we can state that:

$$P(0.4 \leq X < 0.6) = \frac{0.6 - 0.4}{\text{length of the whole interval}} = \frac{0.2}{1}$$

Note, that the inequality $0.4 \leq X < 0.6$ can be changed - each of the signs " $<$ " and " $\leq$ " can be replaced with the other one. The above remark that "there is no real chance to stop a stopwatch **exactly** at $X = 0.5$" means simply that $P(X = 0.5) = 0$. More generally, for any continuous random variable X and any real number a: $P(X = a) = 0$.



In the diagram we can see that the probability discussed above, $P(0.4 \leq X < 0.6)$, can be shown as the area under the graph of $y = 1$ between $x = 0.4$ and $x = 0.6$, as the area between 0 and 1 is equal 1.
A function like $y = 1$ in this example is called a **probability density function**.
We say that $y = f(x)$, a *non-negative* real function is a probability density function of a random variable X when for $a \leq b$

$$P(a < X < b) = \int_a^b f(x)dx$$

43.1 Properties of a probability density function of a continuous random variable

Let $f : \mathbb{R} \to \mathbb{R}$ be a probability density function of a continuous random variable X. Then:

1. the domain of f is $\mathbb{R}$,
2. $f(x) \geq 0$ for all $x \in \mathbb{R}$,
3. $\int_{-\infty}^{+\infty} f(x)dx = 1$.

Note, that in the example above, the density function $f : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, 1[\\ 0 & \text{otherwise} \end{cases}$$

It shows that in case of $f : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f(x) \geq 0 & \text{for } x \in [a, b] \quad (\text{OR } x \in [a, b[\text{ or } x \in]a, b[\text{ or } x \in]a, b]) \\ f(x) = 0 & \text{otherwise} \end{cases}$$

the 3^{rd} property changes into $\int_a^b f(x)dx = 1$.

Similarly to a discrete case, it is sometimes convenient to introduce a **cumulative distribution function** $F : \mathbb{R} \longmapsto [0, 1]$ defined as follows:

$$F(x) = P(X \leq x)$$

Properties of a cumulative distribution function of a continuous random variable X

1. $0 \leq F(x) \leq 1$ for all values of x
2. F is non-decreasing
3. when $x \to -\infty$, $F(x) \to 0$
4. when $x \to +\infty$, $F(x) \to 1$
5. $P(a < X \leq b) = F(b) - F(a) = \int_a^b f(x)dx$ for $a \leq b$
6. $f(x) = \frac{d}{dx}F(x)$

43.2 Expected value of a continuous random variable

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

43.3 Properties of expected value of a continuous random variable

1.	$E(a) = a$, where a is constant
2.	$E(aX) = aE(X)$, where a is a constant.
3.	$E(aX + b) = aE(X) + b$, where a and b are constants.

NOTE: $E(X^2) \neq (E(X))^2$
$E(g(X)) \neq g(E(X))$, $\quad (g(X) = aX + b$ is an exception)

To find other measures of central tendency:

mode - most common / probable value - this should not be treated literally;
mode is shown by the maximum of f
(as the *neighborhoods* of it are more probable than any other intervals)

median - value with half probabilities below and half above the median value, i.e.
m is the median if and only if $\int_{-\infty}^{m} f(x)dx = \frac{1}{2} \Leftrightarrow \int_{m}^{+\infty} f(x)dx = \frac{1}{2}$
or equivalently $F(m) = \frac{1}{2}$

variance - $Var(X) = E((X-\mu)^2) = \int_{-\infty}^{+\infty}(x-\mu)^2 f(x)dx$

$= E(X^2) - (E(X))^2 = \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2$

Questions 43.1.

Q1. Consider the function

$$f(x) = \begin{cases} a & \text{for } -0.5 \le x \le 2.5 \\ 0 & \text{otherwise,} \end{cases}$$

where a is a positive constant. Let X be a continuous random variable defined by a probability density function f.

(i) Show that $f(x) \ge 0$ for all x where f is defined.

(ii) Evaluate $\int_{-\infty}^{+\infty} f(x)dx$ in terms of a.

(iii) Find the value of a.

(iv) Sketch the graph of f.

(v) Find the probability $P(-0.5 \le X \le 1.5)$.

(vi) Find the probability $P(-0.5 < X < 1.5)$.

(vii) Find the mode of X.

(viii) Find the median of X.

(ix) Find the mean of X.

(x) Find the variance of X.

(xi) Find $F(0)$ and $F(1)$.

(xii) Find the expression for $F(x)$.

Q2. Consider the function

$$f(x) = \begin{cases} \frac{4-x}{b} & \text{for } 1 \le x \le 3 \\ 0 & \text{otherwise,} \end{cases}$$

where b is a positive constant. Let X be a continuous random variable defined by a probability density function f.

(i) Show that $f(x) \ge 0$ for all x where f is defined.

(ii) Evaluate $\int_{-\infty}^{+\infty} f(x)dx$ in terms of b.

(iii) Find the value of b.

(iv) Sketch the graph of f.

(v) Find $P(X = 2)$.

(vi) Find the mode of X.

(vii) Find the median of X.

(viii) Find the mean of X.

(ix) Find the variance of X.

(x) Find the expression for $F(x)$.

Q3. Consider the function

$$f(x) = \begin{cases} \frac{1}{1+x} & \text{for } 0 \leq x \leq c \\ 0 & \text{otherwise,} \end{cases}$$

where c is a positive constant. Let X be a continuous random variable defined by a probability density function f.

(i) Show that $f(x) \geq 0$ for all x where f is defined.

(ii) Evaluate $\int_{-\infty}^{+\infty} f(x)dx$ in terms of c.

(iii) Find the value of c.

(iv) Sketch the graph of f.

(v) Find $P(X < 1)$.

(vi) Find the mode of X.

(vii) Find the median of X.

(viii) Find the mean of X.

(ix) Find the variance of X.

(x) Find the expression for $F(x)$.

Q4. Consider the function

$$f(x) = \begin{cases} kx(3-x) & \text{for } 1 \leq x \leq 3 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a positive constant. Let X be a continuous random variable defined by a probability density function f.

(i) Show that $f(x) \geq 0$ for all x where f is defined.

(ii) Evaluate $\int_{-\infty}^{+\infty} f(x)dx$ in terms of k.

(iii) Find the value of k.

(iv) Sketch the graph of f.

(v) Find the probability $P(X < 2)$.

(vi) Find the mode of X.

(vii) Find the median of X.

(viii) Find the mean of X.

(ix) Find the variance of X.

Q5. Consider the function

$$f(x) = \begin{cases} se^x & \text{for } 0 \leq x \leq \ln 3 \\ 0 & \text{otherwise,} \end{cases}$$

where s is a positive constant. Let X be a continuous random variable defined by a probability density function f.

(i) Show that $f(x) \geq 0$ for all x where f is defined.

(ii) Evaluate $\int_{-\infty}^{+\infty} f(x)dx$ in terms of s.

(iii) Find the value of s.

(iv) Sketch the graph of f.

(v) Find the probability $P(X < 2)$.

(vi) Find the probability $P(\ln 2 < X < \ln 3)$.

(vii) Find the mode of X.

(viii) Find the median of X.

(ix) Find the mean of X.

(x) Find the variance of X.

Q6. Consider the function

$$f(x) = \begin{cases} t\cos(\frac{x}{2}) & \text{for } -\frac{\pi}{3} \leq x \leq \frac{\pi}{3} \\ 0 & \text{otherwise,} \end{cases}$$

where t is a positive constant. Let X be a continuous random variable defined by a probability density function f.

(i) Show that $f(x) \geq 0$ for all x where f is defined.

(ii) Evaluate $\int_{-\infty}^{+\infty} f(x)dx$ in terms of t.

(iii) Find the value of t.

(iv) Sketch the graph of f.

(v) Find the probability $P(-\frac{\pi}{6} \leq X \leq \frac{\pi}{6})$.

(vi) Find the mode of X.

(vii) Find the median of X.

(viii) Find the mean of X.

(ix) Find the variance of X.

Q7. Consider the function

$$f(x) = \begin{cases} -r(16x^2 - 16x + 3) & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 0 & \text{otherwise,} \end{cases}$$

where r is a positive constant. Let X be a continuous random variable defined by a probability density function f.

(i) Show that $f(x) \geq 0$ for all x where f is defined.

(ii) Evaluate $\int_{-\infty}^{+\infty} f(x)dx$ in terms of r.

(iii) Find the value of r.

(iv) Sketch the graph of f.

(v) Find the probability $P(\frac{1}{2} < X \leq \frac{3}{4})$.

(vi) Find the mode of X.

(vii) Find the median of X.

(viii) Find the mean of X.

(ix) Find the variance of X.

Q8. Consider the function

$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Let X be a continuous random variable defined by a probability density function f.

(i) Show that $f(x) \geq 0$ for all x where f is defined.

(ii) Evaluate indefinite integral $\int f(x)dx$.

(iii) Show that f represents a probability density function.

(iv) Sketch the graph of f.

(v) Find the mode of X.

(vi) Find the median of X.

(vii) Find the mean of X.

(viii) Find the variance of X.

Q9. The length of a worm of certain species is X centimetres, where X is a random variable with a probability density function defined:

$$f(x) = ax\sqrt{9 - x^2}, 0 < x \leq 3, a \in \mathbb{R}$$

(i) Find the exact value of a.

(ii) Find the mean length.

(iii) Find the exact value of the mode.

(iv) Find the median.

(v) Find the standard deviation.

Q10. Let X be a continuous random variable defined by a probability density function f:

$$f(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq 1 \\ \frac{7-3x}{t} & \text{for } 1 < x \leq \frac{7}{3} \\ 0 & \text{otherwise.} \end{cases}$$

where t is a constant.

(i) Find the vaule of t.

(ii) Find $E(X)$.

(iii) Find the value of the median.

(iv) Find $P(\frac{1}{2} < X < 2)$

(v) Write down the mode.

Q11. Let X be a continuous random variable defined by a probability density function f:

$$f(x) = \begin{cases} a & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{b}{x^3} & \text{for } \frac{1}{2} < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

where a and b are constants.
Knowing that $E(X) = 0.2$ find:

(i) the values of a and b,

(ii) the meadian,

(iii) the interquartile range.

43.4 Random variable - probability function - summing up

DISCRETE	**CONTINUOUS**
$0 \leq P(X = x_i) \leq 1$ for all values of x_i	$f(x) \geq 0$ for all $x \in \mathbb{R}$
$\sum_{i=1}^{n} P(X = x_i) = 1$	$\int_{-\infty}^{+\infty} f(x)dx = 1$ (For f defined over interval $a \leq x \leq b$, i.e. $\begin{cases} f(x) \geq 0 & \text{for } a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$, $\int_{a}^{b} f(x)dx = 1$.)
	Calculating probabilities: $P(x \leq X \leq t) = \int_{s}^{t} f(x)dx$
MODE	
most common value	maximum of $f(x)$
MEDIAN	
the value with half probabilities below and half above it	
	m such that $\int_{-\infty}^{m} f(x)dx = \int_{m}^{+\infty} f(x)dx = \frac{1}{2}$
EXPECTED VALUE - MEAN	
$E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$	$E(X) = \int_{-\infty}^{+\infty} xf(x)dx$
PROPERTIES OF EXPECTED VALUE	
$E(a) = a$, where a is a constant.	
$E(aX + b) = aE(X) + b$, where a and b are constant.	
$E(g(X)) = \sum_{i=1}^{n} g(x_i)P(X = x_i)$	non-obligatory $E(g(X)) = \int_{-\infty}^{+\infty} g(x)f(x)dx$
NOTE. $E(X^2) \neq (E(X))^2$ $E(g(X)) \neq g(E(X))$ $(g(X) = aX + b$ is an exception)	
VARIANCE	
$Var(X) = E((X - \mu)^2) = E(X^2) - (E(X))^2 = E(X^2) - \mu^2$	
$Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 P(X = x_i)$ $= \sum_{i=1}^{n} x_i^2 P(X = x_i) - \mu^2$	$Var(X) = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx$ $= \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2$
STANDARD DEVIATION	
$Sd(X) = \sqrt{Var(X)}$	
$Sd(X) = \sqrt{\sum_{i=1}^{n} x_i^2 P(X = x_i) - \mu^2}$	$Sd(X) = \sqrt{\int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2}$
PROPERTIES OF VARIANCE / STANDARD DEVIATION	
$Var(a) = 0$ $Sd(a) = 0$	a is a constant.

DISCRETE		CONTINUOUS
	$Var(aX+b) = a^2 Var(X)$ $Sd(aX+b) = \lvert a \rvert Sd(X)$	a and b are constant.
Notation.	$Var(X) = \sigma^2$	
	$Sd(X) = \sqrt{\sigma^2} = \sigma$	

Chapter 44

Normal distribution

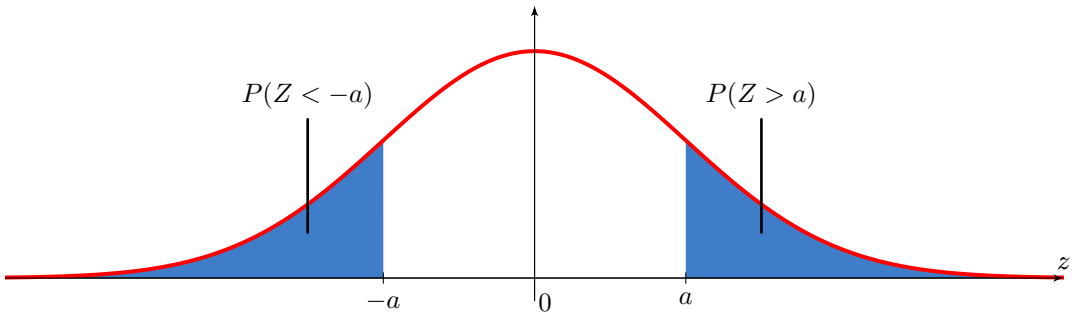
Notation: $X \sim N(\mu, \sigma^2)$ means that a random variable X follows the normal distribution with

- mean μ and
- standard deviation σ (variance σ^2).

$Z \sim N(0, 1)$ means that the random variable Z follows the normal distribution with

- mean $\mu = 0$ and
- standard deviation $\sigma = 1$ (variance $\sigma^2 = 1$).

$Z \sim N(0, 1)$ is called **standard** normal variable.



Questions 44.1.

Q1. For $Z \sim N(0, 1)$:

(i) evaluate the following probabilities:

(1) $P(Z < 0.5)$
(2) $P(Z < 2.87)$
(3) $P(Z < 3.82)$
(4) $P(Z < 0)$
(5) $P(Z > 0.62)$
(6) $P(Z > 1.85)$
(7) $P(Z > -1)$
(8) $P(Z > -2.04)$
(9) $P(Z > -0.89)$
(10) $P(Z < -2.3)$
(11) $P(Z < -1.41)$
(12) $P(Z < -3)$
(13) $P(1 < Z < 2.5)$
(14) $P(1.03 < Z < 2.21)$
(15) $P(-2.1 < Z < -0.76)$
(16) $P(-1 < Z < 1)$
(17) $P(-1.78 < Z < 0.62)$
(18) $P(-2.67 < Z < 1.08)$

(ii) find the value of a such that:

(1) $P(Z < a) = 0.635$
(2) $P(Z < a) = 0.824$
(3) $P(Z < a) = 0.999$
(4) $P(Z < a) = 0.424$
(5) $P(Z < a) = 0.307$
(6) $P(Z < a) = 0.08$
(7) $P(Z > a) = 0.74$
(8) $P(Z > a) = 0.685$
(9) $P(Z > a) = 0.9$
(10) $P(Z > a) = 0.25$
(11) $P(Z > a) = 0.341$
(12) $P(Z > a) = 0.025$

Q2. Let $0 < p_1 < p_2 < 0.5 < p_3 < p_4 < 1$. Let $Z \sim N(0, 1)$.
Let then $P(Z < a_k) = p_k$ for $k \in \{1, 2, 3, 4\}$.

(i) Sketch the standard normal curve showing the position of a_1, a_2, a_3 and a_4.

(ii) Express in terms of p_1, p_2, p_3 and p_4:

(1) $P(Z > a_1)$
(2) $P(Z > a_4)$
(3) $P(a_2 < Z < a_4)$
(4) $P(a_2 < Z < a_3)$
(5) $P(Z > -a_1)$
(6) $P(Z < -a_4)$
(7) $P(-a_3 < Z < a_3)$
(8) $P(a_1 < Z < -a_1)$
(9) $P(-a_4 < Z < -a_3)$
(10) $P(-a_2 < Z < -a_1)$
(11) $P(-a_4 < Z < a_3)$
(12) $P(-a_3 < Z < a_4)$

(iii) Express b_1 and b_2 in terms of $-\infty, a_1, a_2, a_3, a_4$ and $+\infty$:

(1) $P(b_1 < Z < b_2) = p_1$
(2) $P(b_1 < Z < b_2) = p_4$
(3) $P(b_1 < Z < b_2) = p_2 - p_1$
(4) $P(b_1 < Z < b_2) = 1 - p_1$
(5) $P(b_1 < Z < b_2) = 1 - p_3$
(6) $P(b_1 < Z < b_2) = p_4 - p_1$
(7) $P(b_1 < Z < b_2) = p_3 - p_2$
(8) $P(b_1 < Z < b_2) = p_4 - p_3$

Q3. Let $X \sim N(5, 2^2)$. (I.e. $E(X) = 5$, $Var(X) = 4$)
Let $Z = \frac{X-5}{2}$.

(1) Find $E(Z)$.
(2) Find $Var(Z)$.
(3) Express inequality "$X < 7$" in terms of Z and hence find $P(X < 7)$.
(4) Express inequality "$X < a$" in terms of Z and hence find a for which $P(X < a) = 0.43$.

Q4. Let $X \sim N(12, 3^2)$.
Find:

(1) $P(X < 15)$
(2) $P(X < 16.5)$
(3) $P(X < 10.5)$
(4) $P(X > 11)$
(5) $P(13 < X < 15)$
(6) $P(9 < X < 12)$
(7) $P(7 < X < 11)$
(8) $P(8 < X < 14)$
(9) $P(10 < X < 14)$

Q5. Let $X \sim N(168, 12^2)$.
Find:

(1) $P(X < 195)$
(2) $P(X < 176)$
(3) $P(X < 145)$
(4) $P(X > 151)$
(5) $P(173 < X < 185)$
(6) $P(168 < X < 182)$
(7) $P(130 < X < 160)$
(8) $P(135 < X < 175)$
(9) $P(150 < X < 175)$

Q6. Let $X \sim N(7.42, 3)$.
Find:

(1) $P(X < 8.55)$
(2) $P(X < 6.5)$
(3) $P(X > 6.3)$
(4) $P(X > 9.5)$
(5) $P(5 < X < 7)$
(6) $P(5.43 < X < 8)$

Q7. Let $X \sim N(7.5, 2.5^2)$.
Find the value of a such that:

(1) $P(X < a) = 0.635$ (3) $P(X > a) = 0.235$ (5) $P(X > a) = 0.847$
(2) $P(X < a) = 0.75$ (4) $P(X < a) = 0.25$ (6) $P(X > a) = 0.975$

Q8. Let $X \sim N(63.5, 10)$.
Find the value of a such that:

(1) $P(X < a) = 0.765$ (3) $P(X > a) = 0.365$ (5) $P(X > a) = 0.678$
(2) $P(X < a) = 0.75$ (4) $P(X < a) = 0.25$ (6) $P(X > a) = 0.84$

Q9. Weight of bags of sugar is normally distributed with mean 1002 grams and standard deviation 8 grams.

(1) Find the percentage of bags that weigh less than 1000 grams.
(2) Find the percentage of bags that weigh more than 1020 grams.
(3) Find the percentage of bags with weight between 985 and 1015 grams.
(4) $\frac{1}{3}$ of bags weigh more than m grams. Find the value of m correct to 1 gram.
(5) $\frac{1}{3}$ of bags weigh less than d grams. Find the value of d correct to 1 gram.

Q10. Times of 1 km run of middle school kids are normally distributed with mean 3 minutes 52 seconds and standard deviation 34 seconds.

(1) Find the percentage of kids whose time is less than 4 minutes.
(2) What is the probability that the time of a randomly chosen kid is longer than 5 minutes?
(3) What is the probability that the time of a randomly chosen kid is between 3 and 4 minutes?
(4) Find the time t such that 25% of kids run longer than t.
(5) Find the time s such that 25% of kids run shorter than s.

Q11. Heights of certain type of plant is normally distributed with mean $28.3cm$ and variance $5.2cm^2$.

(1) 25% of plants are lower than h_1. Find h_1.
(2) 25% of plants are higher than h_2. Find h_2.
(3) Hence, find the interquartile range of the heights of plants.

Q12. Amount of drink poured into 500ml mugs in certain bar is normally distributed with mean 506 ml and standard deviation 10.1ml.

(1) What is the probability that a mug contains less than 500 ml of a drink?
(2) What is the probability that a mug contains more than 550 ml of a drink?

The owner of the bar wants bartenders to pour not less than 495 and not more than 510 ml of a drink.

(3) What is the probability that a bartender fulfils the requirements?
(4) 40% of mugs are filled with less than V ml of a drink. Find the value of V.

Q13. The examination results are normally distributed with mean 52 and standard deviation 8. 85% of students passed the test.

(1) What is the lowest pass grade?
(2) What is the percentage of students who scored more than 70 marks?
(3) What are the marks of the top 10% of students?
(4) What is the percentage of students who scored at least twice more than lowest pass grade?

Q14. Heights of men are normally distributed with standard deviation 14 cm.
It is known that 14.2% of men are higher than 187 cm. Find the mean height.

Q15. Weights of women are normally distributed with standard deviation 4.2 kg.
21.6% of women weigh less than 54 kg. Find the mean weight.

Q16. Lengths of candles are normally distributed with mean 14.5 cm.
90% of the candles are longer than 13 cm. Find the standard deviation.

Q17. A manager of a taxi company makes a record of the average usage of fuel. The data are obtained each time when a driver refuels his car. The manager believes that the usage follows the normal distribution with mean 6.8 litres per 100 km. He discovers that 8.3% of drivers use on average less than 5 litres of fuel per 100 km. Find the variance of the distribution.

Q18. The examination board decided that students who scored less than 40 marks fail the exam. The board decided that students with scores higher than 85 marks get the distinctions. It turned out that the two groups make 12% and 8% of the students respectively.

Assuming the results follow normal distribution find the mean and standard deviation.

Q19. A continuous variable X is normally distributed with mean μ and variance σ^2. It is known that $P(X < 85) = 0.7$ and $P(X < 72) = 0.52$. Find the values of μ and σ.

Q20. A continuous variable X is normally distributed with mean μ and variance σ^2. It is known that $P(X > 2.07) = 0.271$ and $P(X < 1.96) = 0.671$. Find the values of μ and σ.

Q21. A continuous variable X is normally distributed with mean μ and variance σ^2. It is known that $P(X < 800) = 0.3$ and $P(X > 1000) = 0.1$. Find the values of μ and σ.

Q22. A continuous variable X is normally distributed with mean μ and variance σ^2. It is known that $P(X > 15.2) = 0.77$ and $P(X < 16.3) = 0.34$. Find the values of μ and σ.

Q23. Let $X \sim N(\mu, \sigma^2)$. Knowing that $P(X < 15) = P(X > 25) = 0.315$ find the values of μ and σ.

Q24. Let $X \sim N(\mu, \sigma^2)$. Knowing that $P(14 < X < \mu) = P(X > 22) = 0.16$ find the values of μ and σ.

Q25. Let $X \sim N(\mu, \sigma^2)$. Knowing that $P(X < 124) + P(\mu < X < 138) = 0.5$ and $P(127 < X < 129) + P(X > 135) = 0.43$ find the values of μ and σ.

Q26. Let $X \sim N(\mu, \sigma^2)$. Knowing that $P(X < 29) = P(X > 21)$ and $P(X < 17) + P(X > 33) = 0.3$ find the values of μ and σ.

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