Higher Level Mathematics

Krzysztof Sikora

May 29, 2018



Copyright © 2018 by Krzysztof Sikora.
This work is made available under the terms of the Creative Commons Attribution-ShareAlike 4.0 license, http://creativecommons.org/licenses/by-sa/4.0/.
If you want to use these materials and you are not a student of the author, first contact the author by email: krzyssikora@gmail.com.

Contents

I Introductory topics. 7

1 Quadratics 7
1.1 Factorisation . . . 7
1.2 Completing the square . . . 7
1.3 Quadratic function . . . 7
1.4 Quadratic inequalities . . . 9
1.5 Viette's formulae . . . 9
1.6 Applications of quadratics . . . 9

2 Lines, circles and vectors 10

3 Functions 11

II Algebra 11

4 Sequences 11
4.1 Arithmetic sequences . . . 11
4.2 Geometric sequences . . . 12

5 Binomial theorem 12

6 Logarithms and exponents 13
6.1 Algebra of logarithms . . . 13
6.2 Logarithmic equations . . . 14

7 Proof by mathematical induction 14

III Functions and equations 14

8 Graphs of rational functions 14

9 Basic transformations of graphs of functions 17

10 Polynomials 22
10.1 Algebra of polynomials . . . 22
10.2 Factor and remainder theorems . . . 22
10.3 Vieta's formulas . . . 23
10.4 Polynomial inequalities . . . 23

11 Exponential and logarithmic function 24

12 Compositions and inverses 24

13 Inequalities 26

IV Circular functions and trigonometry 26

14 Circular functions 26
14.1 Basic circular functions . . . 26
14.2 Unit circle . . . 26
14.3 Graphs of trigonometric functions . . . 28
14.4 Reciprocal circular functions . . . 28

14.5 Inverse circular functions . . . 30

15 Trigonometric identities **31**

16 Trigonometric equations and inequalities **31**

17 Arcs, sectors, sine rule, cosine rule **33**
17.1 Miscellaneous problems . . . 33

V Differential calculus **34**

18 Evaluating derivatives **34**
18.1 Definition of a derivative . . . 34
18.2 Power rule . . . 34
18.3 Derivatives of elementary functions . . . 34
18.4 Chain rule . . . 34
18.5 Derivatives of inverse functions . . . 35

19 Piecewise functions **35**

20 Lines tangent and normal to curves **36**

21 Further derivatives. Use of derivatives in sketching graphs **36**
21.1 Stationary and inflection points . . . 37
21.2 Kinds of inflection points . . . 37

22 Applications of differential calculus **38**
22.1 Implicit differentiation . . . 38
22.2 Curves defined parametrically . . . 39
22.3 Optimisation . . . 39
22.4 Applying derivatives in kinematics . . . 40
22.5 Related rates . . . 40

VI Integral calculus **40**

23 Integrals **40**
23.1 Indefinite integrals . . . 40
23.2 Definite integrals . . . 42

24 Area and volume **42**
24.1 Area under a curve . . . 42
24.2 Area between curves . . . 42
24.3 Volume of revolution . . . 43
24.4 Kinematics . . . 44

25 Further integration **44**
25.1 Integration by parts . . . 44
25.2 Integration by substitution . . . 45
25.3 Integrating rational expressions . . . 46
25.4 More trigonometric substitutions . . . 47
25.5 Differential equations . . . 47
25.6 Mixed exercise . . . 47

VII Vectors **48**

26 Introduction to vectors **48**
26.1 Basics . . . 48
26.2 Scalar product . . . 48

27 Straight lines **49**
27.1 Equations of straight lines . . . 49
27.2 Intersections of straight lines . . . 50

28 Matrices **50**
28.1 Basics . . . 50
28.2 Multiplication of matrices . . . 50
28.3 Determinant of a matrix, inverse matrix . . . 51
28.4 Solving simultaneous equations - intersections of the planes . . . 52

29 Planes in three-dimensional space **52**
29.1 Intersections of the planes . . . 52
29.2 Vector product . . . 53
29.3 Properties of scalar and vector product . . . 53
29.4 Alternate forms of equations of planes . . . 53
29.5 3D distances . . . 53
29.6 Angles between lines, planes... . . . 55

VIII Complex numbers **55**

30 Introduction to complex numbers **55**
30.1 Polar coordinates . . . 55
30.2 Introducing complex numbers . . . 55
30.3 Algebra of complex numbers . . . 56
30.4 Conjugate numbers. Polar form of a complex number . . . 56

31 De Moivre's theorem **57**
31.1 De Moivre's theorem and its generalization . . . 58

32 Conjugate roots **59**
32.1 Fundamental theorem of algebra . . . 59
32.2 Conjugate Root theorem . . . 59

IX Statistics **59**

33 Population, sample, types of data **59**

34 Presentation of data **60**

35 Measures of tendency, measures of spread **60**

36 Cumulative frequency **61**

X Probability **62**

37 Counting principles **62**

38 Introduction to probability **63**
38.1 Concept of probability . . . 63
38.2 Formal approach . . . 63
38.3 Axioms of probability . . . 63
38.4 Exemplary simple probability problems . . . 63
38.5 Counting principles . . . 63

39 Conditional probability **63**
39.1 Combined events . . . 63
39.2 Conditional probability . . . 64
39.3 Independent events . . . 64
39.4 The law of total probability . . . 64
39.5 Bayes' theorem . . . 64

40 Discrete random variables **65**
40.1 Discrete random variables . . . 65
40.2 Expected value of a discrete random variable . . . 67
40.3 The meaning of expected value . . . 67
40.4 Properties of expected value of a discrete random variable . . . 67
40.5 Variance of a discrete random variable . . . 67

40.6 Properties of variance of a discrete random variable 67

41 Binomial distribution **67**

42 Poisson distribution **68**

43 Continuous random variables **68**

44 Normal distribution **69**

PART I

INTRODUCTORY TOPICS.

CHAPTER 1

QUADRATICS

1.1 FACTORISATION

QUESTIONS 1.1.

Q1. (1) $-6, -4$
(2) $-5, -3$
(3) $-6, 7$
(4) 3.5
(5) $4, -8$
(6) $16, -2$
(7) $4.41, 1.59$
(8) $5, 0$
(9) $0, 0.6$
(10) $-\frac{8}{3}, \frac{8}{3}$
(11) $\pm\frac{\sqrt{5}}{2}$
(12) $\pm\frac{\sqrt{10}}{2}$
(13) $1.5, -4.5$
(14) $-2, 8$
(15) $-1.5, 3.5$
(16) $2.4, -2$
(17) no solutions
(18) $\pm 1, \pm 2$
(19) -1.5
(20) $-2, -3$
(21) $7, 2$
(22) $-2, 5$
(23) $1, -4$
(24) $3, -3$
(25) -2
(26) 1
(27) $-2, -5$
(28) $5, 3$
(29) $-3, 7$
(30) $3, -6$
(31) $2, -2$
(32) $3, 3$
(33) -1

Q2. (1) $-\frac{3}{2}, -1$
(2) $-\frac{3}{2}, 2$
(3) $0.5, -3$
(4) $0.5, -1.5$
(5) $-1, 2.5$
(6) $-0.5, -3$
(7) $3, \frac{1}{3}$
(8) $-0.4, -3$
(9) $2, 1.5$
(10) $0.5, -3$
(11) $-\frac{2}{3}, 1$
(12) $4.5, 0.5$
(13) $-2, \frac{1}{3}$
(14) $0.5, -1$
(15) no solutions
(16) $-0.6, -1$
(17) $3, \frac{2}{3}$
(18) $-1, \frac{3}{2}$
(19) $-\frac{1}{3}, 2$
(20) $\frac{4}{3}, \frac{1}{3}$
(21) $1, -0.5$
(22) $\frac{2}{3}, -1$
(23) no solutions
(24) $\frac{8}{3}, \frac{2}{3}$
(25) $0.4, -3$
(26) $0, \pm 3$
(27) 2
(28) $-\frac{5}{2}, 1, \frac{5}{2}$
(29) $-4, -1, 3$

1.2 COMPLETING THE SQUARE

QUESTIONS 1.2.

Q3. (1) $0.24, -4.24$
(2) $-1.59, -4.41$
(3) $6.79, 2.21$
(4) $4.68, 0.32$
(5) $1.77, 0.566$
(6) no solutions
(7) $-0.59, -1.7$
(8) no solutions

1.3 QUADRATIC FUNCTION

QUESTIONS 1.3.

Q4. (1) $0.24, -4.24$
(2) $-1.59, -4.41$
(3) $6.79, 2.21$
(4) $4.68, 0.32$
(5) $1.77, 0.566$
(6) no solutions
(7) $-0.59, -1.7$
(8) no solutions

Q5. (1) ± 2
(2) $\pm\sqrt{6}$
(3) $\pm\frac{\sqrt{10}}{2}$
(4) $\pm 2, \pm 3$
(5) $\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{3}}$
(6) $\pm\sqrt{2}$
(7) $-1, 2$
(8) $-2, 2$

Q6. $(-1,-1),(4,4)$

Q7. $(-5,4),(-1,-4)$

Q8. $(1,1),(4,-2)$

Q9. $(-4,1),(1,-1.5)$

Q10. $(-4,12),(2,0)$

Q11. $(-2,5),(2,-3)$

Q12. $(-2,1),(2,-3)$

Q13. $y=-\frac{1}{2}x^2-x+4$

Q14. $y=\frac{7}{4}x^2+7x+4$

Q15. $0,4$

Q16. $-\frac{2}{7},2$

Q17. $-2\sqrt{2}<m<2\sqrt{2}$

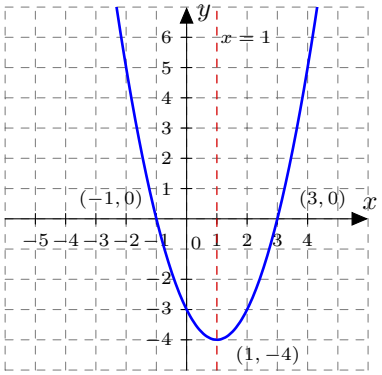
Q18. $c=3$, zeroes: $\frac{3}{2}\pm\frac{\sqrt{3}}{2}$

Q19.

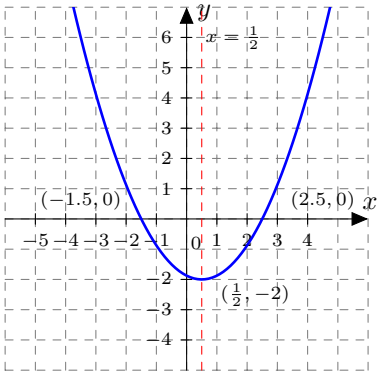
	$y=$	$y=$	$y=$			
(1)	$x^2-2x-15$	$(x-1)^2-16$	$(x-5)(x+3)$	$-3,5$	$(1,-16)$	64
(2)	$2x^2-12x+10$	$2(x-3)^2-8$	$2(x-1)(x-5)$	$1,5$	$(3,-8)$	64
(3)	$2x^2+5x-3$	$2(x+\frac{5}{4})^2-\frac{49}{8}$	$2(x-\frac{1}{2})(x+3)$	$-3,\frac{1}{2}$	$(-\frac{5}{4},-\frac{49}{8})$	49
(4)	x^2+4x+1	$(x+2)^2-3$	$(x+2-\sqrt{3})(x+2+\sqrt{3})$	$-2\pm\sqrt{3}$	$(-2,-3)$	12
(5)	$-2x^2-18x-28$	$-2(x+\frac{9}{2})^2+\frac{25}{2}$	$-2(x+2)(x+7)$	$-2,-7$	$(-\frac{9}{2},\frac{25}{2})$	100
(6)	$-4x^2+28x-49$	$-4(x-\frac{7}{2})^2$	$-4(x-\frac{7}{2})^2$	3.5	$(\frac{7}{2},0)$	0
(7)	$-\frac{1}{2}x^2-2x-3$	$-\frac{1}{2}(x+2)^2-1$	—	none	$(-2,-1)$	-2
(8)	$-4x^2+8x-2$	$-4(x-1)^2+2$	$-4(x-1+\frac{\sqrt{2}}{2})(x-1-\frac{\sqrt{2}}{2})$	$1\pm\frac{\sqrt{2}}{2}$	$(1,2)$	32
(9)	$-2x^2+2x\sqrt{2}+7$	$-2(x-\frac{\sqrt{2}}{2})^2+8$	$-2(x-\frac{\sqrt{2}}{2}+2)(x-\frac{\sqrt{2}}{2}-2)$	$\frac{\sqrt{2}}{2}\pm2$	$(\frac{\sqrt{2}}{2},8)$	64

Q20. (1) 3 (2) 16 (3) 8 (4) 11 (5) 6 (6) 2 (7) 7 (8) 14 (9) 10 (10) 12

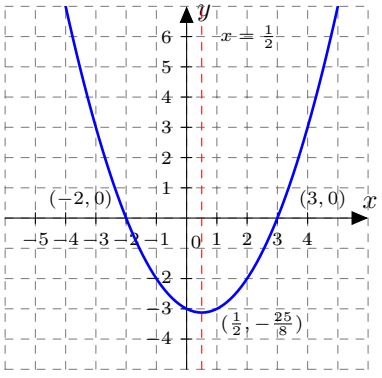
Q21. (1)



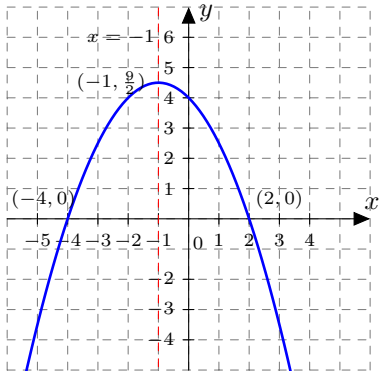
(2)



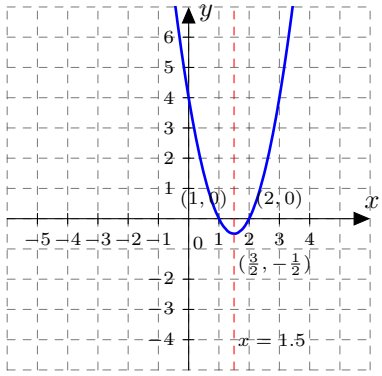
(3)



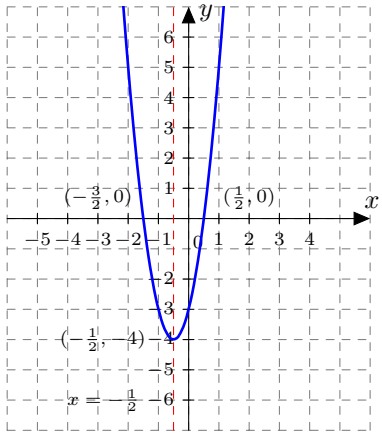
(4)



(5)



(6)



1.4 Quadratic inequalities

Questions 1.4.

Q22. (1) no solutions
(2) $x < -2$ or $x > 0$
(3) $x \in \mathbb{R}$
(4) $x < -1$ or $x > 3$
(5) $1 \leq x \leq 5$
(6) $6 < x < -4$
(7) $-4 \leq x \leq 6$
(8) $x < -3$ or $x > \frac{3}{2}$
(9) $x \leq -\frac{5}{2}$ or $x \geq \frac{2}{3}$
(10) $\frac{5}{3} \leq x \leq 3$
(11) $x \leq -4$ or $x \geq -2$
(12) $\frac{1}{3} < x < 5$

Q23. (1) $x < -2$ or $x > 1$
(2) $-4 \leq x \leq -1$
(3) $-\frac{2}{3} < x < \frac{3}{2}$
(4) $x < 1$ or $x > 2.5$
(5) $-3 \leq x \leq 0$
(6) $x < 0$ or $x > 5$
(7) $-3 \leq x \leq 2$
(8) $x \leq 2$ or $x \geq 3$
(9) $x \neq -3$
(10) $x \in \mathbb{R}$
(11) no solutions
(12) $x \neq 1$
(13) $x = \frac{1}{2}$
(14) $0 < x < \frac{2}{3}$
(15) $x < -2$ or $x > 1$
(16) $-2 \leq x \leq 0$
(17) $x < -2$ or $x > \frac{3}{2}$
(18) $x \leq \frac{1}{2}$ or $x \geq 3$
(19) no solutions
(20) $x < 0$ or $x > \frac{2}{5}$

1.5 Viette's formulae

Questions 1.5.

Q24.

	a	b	c	$\frac{c}{a}$	$-\frac{b}{a}$	x_1	x_2
1	+	−	+	+	+	+	+
2	+	?	−	−	?	+	−
3	+	?	−	−	?	−	+
4	+	+	+	+	−	−	−

	a	b	c	$\frac{c}{a}$	$-\frac{b}{a}$	x_1	x_2
5	−	+	−	+	+	+	+
6	−	?	+	−	?	+	−
7	−	?	+	−	?	−	+
8	−	−	−	+	−	−	−

Q25. (1) different signs
(2) both positive
(3) different signs
(4) different signs
(5) both negative
(6) different signs
(7) different signs
(8) both positive
(9) different signs
(10) different signs
(11) both negative
(12) both positive

Q26. $m < -6$

Q27. $m \in]0, 1[\cup]1, +\infty[$

Q28. $m < 3$

Q29. (1) different signs
(2) both positive
(3) different signs
(4) both positive
(5) both negative
(6) no solutions

Q30. (1) $\frac{5}{3}$
(2) -2
(3) $\frac{61}{9}$
(4) $\frac{395}{27}$
(5) $-\frac{5}{6}$
(6) $\frac{\sqrt{97}}{3}$
(7) $\frac{61}{36}$
(8) $-\frac{61}{18}$
(9) $\frac{5\sqrt{97}}{9}$

Q31. (1) $-\frac{8}{5}$
(2) $\frac{2}{5}$
(3) $\frac{44}{25}$
(4) $\frac{752}{125}$
(5) -4
(6) $\frac{2\sqrt{6}}{5}$
(7) 11
(8) $\frac{22}{5}$
(9) $\frac{16\sqrt{6}}{25}$

Q32. (1) $4x^2 + 7x - 8 = 0$
(2) $2x^2 - 7x - 16 = 0$
(3) $16x^2 - 113x + 64 = 0$
(4) $32x^2 - 113x + 32 = 0$
(5) $64x^2 - 1015x - 512 = 0$
(6) $8x^2 + 7x - 4$
(7) $64x^2 - 113x + 64 = 0$
(8) $32x^2 + 113x + 32 = 0$
(9) $2x^2 - 7x - 16 = 0$

Q33. (1) $-5x^2 - 3x + 9 = 0$
(2) $-5x^2 + 6x + 36 = 0$
(3) $25x^2 - 99x + 81 = 0$
(4) $100x^2 - 198x + 81 = 0$
(5) $125x^2 - 432x - 729 = 0$
(6) $9x^3 + 3x - 5 = 0$
(7) $81x^2 - 99x + 25 = 0$
(8) $45x^2 + 99x + 45 = 0$
(9) $45x^2 - 57x - 361 = 0$

1.6 Applications of quadratics

Questions 1.6.

Q34. *Hint:* when one side of a rectangle is x, the other is $20 - x$.

Q35. $10, 24, 26$

Q36. $6m \times 8m$

Q37. $15m \times 30m$

Q38. $17.6m \times 4.7m$

Q39. $12m \times 12m$

Q40. a) $16m$
b) after 6 seconds
c) after 2 seconds
d) $\frac{64}{3}m \approx 21.3m$

Q41. $5cm \times 7cm$

Q42. $5cm, 11cm$

Q43. $7cm \times 13cm$

Q44. $8cm, 15cm$

Q45. —

Q46. —

Q47. $\left(\frac{2}{3}, 0\right)$

Q48. $\left(-\frac{1}{3}, \frac{8}{9}\right)$

CHAPTER 2

LINES, CIRCLES AND VECTORS

QUESTIONS 2.1.

Q1. 54

Q2. (1) $y = -\frac{1}{2}x$
(2) $y = -\frac{2}{3}x - 1$
(3) $y = \frac{3}{2}x - 8$
(4) $5x + 2y + 1 = 0$
(5) $3x - 4y + 11 = 0$
(6) $2x + y - 3 = 0$

Q3. (1) $(4, 3), r = 6$
(2) $(-3, 1), r = \sqrt{11}$
(3) $(-8, 6), r = 10$
(4) $(-5, 0), r = 3\sqrt{2}$
(5) $(-2, \frac{1}{2}), r = \frac{5}{2}$
(6) $(\frac{3}{2}, -4), r = \frac{9}{2}$
(7) $(-\frac{7}{2}, \frac{5}{2}), r = 4$

Q4. (1) $(-4, -1), (3, 0)$
(2) $(-3, 4), (-10, -3)$

Q5. (1) $\sqrt{37} - \sqrt{13} \approx 2.48$
(2) $6\sqrt{2} - 2\sqrt{5} \approx 4.07$

Q6. 4

Q7. -7

Q8. $2q = 5p + 24$

Q9. $c = -6; (-3, 1)$ or $c = -26; (-5, 7)$

Q10. $\frac{5 \pm \sqrt{105}}{4}$

Q11. $-\frac{4}{3}$ or $\frac{3}{4}$

Q12. $\frac{1}{2}$ or 2

Q13. 12 or 22

QUESTIONS 2.2.

Q14. (1) $b - a$
(2) $b - c$
(3) $c - b$
(4) $c - a$
(5) $\frac{1}{2}b - a + \frac{1}{2}c$
(6) $-\frac{3}{4}a + \frac{1}{4}b + \frac{1}{2}c$
(7) $\frac{1}{2}a - \frac{1}{2}c$
(8) $\frac{3}{4}a + \frac{3}{4}b - \frac{1}{2}c$
(9) $\frac{1}{4}a + \frac{1}{4}b - \frac{1}{2}c$

Q15. (1) $\begin{pmatrix} anything \\ 5 \end{pmatrix}$
(2) $\begin{pmatrix} anything \\ 0 \end{pmatrix}$
(3) $\begin{pmatrix} -6 \\ anything \end{pmatrix}$

Q16. (1) $\begin{pmatrix} 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 10 \end{pmatrix}; \begin{pmatrix} t \\ 2t + 6 \end{pmatrix}, t \in \mathbb{R}$
(2) $\begin{pmatrix} 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \end{pmatrix}; \begin{pmatrix} t \\ -\frac{2}{3}t + 5 \end{pmatrix}, t \in \mathbb{R}$
(3) $\begin{pmatrix} 0 \\ 3.5 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6.5 \end{pmatrix}; \begin{pmatrix} 2t + 1 \\ 3t + 5 \end{pmatrix}, t \in \mathbb{R}$

Q17. (1) $\begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$
(2) $\begin{pmatrix} -\frac{12}{13} \\ \frac{5}{13} \end{pmatrix}$
(3) $\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} \end{pmatrix}$
(4) $\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$
(5) $\begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$
(6) $\begin{pmatrix} 0.626 \\ -0.780 \end{pmatrix}$

Q18. (1) $\begin{pmatrix} 1.8 \\ -2.4 \end{pmatrix}$
(2) $\begin{pmatrix} -24 \\ 10 \end{pmatrix}$
(3) $\begin{pmatrix} -\sqrt{3} \\ -2\sqrt{3} \end{pmatrix}$
(4) $\begin{pmatrix} 3\sqrt{6} \\ \sqrt{6} \end{pmatrix}$
(5) $\begin{pmatrix} -2\sqrt{3} \\ 4\sqrt{3} \end{pmatrix}$
(6) $\begin{pmatrix} 3.13 \\ -3.90 \end{pmatrix}$

Q19. (1) $\vec{v} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}, \vec{u} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$

(2) $\vec{v} = \begin{pmatrix} 4\sqrt{5} \\ -2\sqrt{5} \end{pmatrix}, \vec{u} = \begin{pmatrix} 2\sqrt{5} \\ 4\sqrt{5} \end{pmatrix}$

(3) $\vec{v} = \begin{pmatrix} \sqrt{10} \\ -3\sqrt{10} \end{pmatrix}, \vec{u} = \begin{pmatrix} 3\sqrt{10} \\ \sqrt{10} \end{pmatrix}$

(4) $\vec{v} = \begin{pmatrix} 2\sqrt{5} \\ -4\sqrt{5} \end{pmatrix}, \vec{u} = \begin{pmatrix} 4\sqrt{5} \\ 2\sqrt{5} \end{pmatrix}$

Q20. (1) $\begin{pmatrix} -4.5 \\ 6 \end{pmatrix}$

(2) $7.5i + 5j$

Q21. (1) $2\sqrt{13}$

(2) $5\sqrt{5}$

Q22. $\frac{5}{2}\sqrt{10}$

Questions 2.3.

Q23. (1) 93.2° (1.63 rad)
(2) 170° (2.96 rad)
(3) 14° (0.245 rad)
(4) 147° (2.57 rad)
(5) 18.4° (0.322 rad)
(6) 174° (3.03 rad)
(7) 113° (1.98 rad)
(8) 37.2° (0.65 rad)
(9) 135° (2.36 rad)
(10) 156° (2.73 rad)

Q24. 47.9° or 0.837, 20.5

Q25. 60.3° or 1.05

Q26. 59.0° or 1.03

chapter 3

Functions

Questions 3.1.

Q1. (i) none
(ii) injective, surjective, bijective
(iii) surjective
(iv) injective, surjective, bijective
(v) injective

Q2. (a) $a \leq -3$ or $a \geq -2$ (b) $-3 \leq a \leq -2$ (c) $a = -3$ or $a = -2$

Q3. (a) $a \leq -3$ (b) $a \geq -3$ (c) $a = -3$

part II

Algebra

chapter 4

Sequences

4.1 Arithmetic sequences

Questions 4.1.

Q1. 104 **Q2.** −20 **Q3.** 90.5 **Q4.** −10 **Q5.** −3

Q6.

	n	k	a_n	a_k	$\frac{n+k}{2}$	$a_{\frac{n+k}{2}}$
(1)	1	5	7	12	3	9.5
(2)	2	10	3	15	6	9
(3)	6	14	28	3	10	15.5
(4)	13	25	7	11	19	9
(5)	7	15	8	-2	11	3
(6)	12	18	-3	28	15	12.5

$a_{\frac{n+k}{2}} = \frac{a_n + a_k}{2}$

Q7. 620 **Q8.** 285 **Q9.** $u_1 = 11, S_{13} = -91$

Q10. (1) 2550 (3) 2300 (5) 5040 (7) 3075
(2) 1225 (4) 3925 (6) 3131 (8) -1767

Q11. 20 **Q16.** 40 or 125 **Q21.** 80

Q12. $-3, 2$ **Q17.** 360 **Q22.** 17

Q13. 253 **Q18.** $12.5, 0.25$ **Q23.** 100

Q14. 385 **Q19.** -0.2 **Q24.** $3; 168$

Q15. 13 **Q20.** $10, -\frac{1}{3}$ **Q25.** $1; 4, 7$(or $7; 4; 1$)

4.2 Geometric sequences

Questions 4.2.

Q26. $\frac{6561}{64} = 102.515625 \approx 103$ **Q28.** 41984

Q27. -1024 **Q29.** 32768

Q30.

	n	k	m	a_n	a_k	a_m	R	S_m
(1)	1	5	8	4	0.25	± 0.03125	± 0.5	$\frac{255}{32} \approx 7.97$ or $\frac{85}{32} \approx 2.66$
(2)	2	6	7	-32	-2	1	-0.5	43
(3)	3	5	8	$\frac{2}{3}$	6	162	3	$\frac{6560}{27}$
(4)	2	3	6	2	3	10.125	1.5	$\frac{665}{24}$
(5)	4	7	9	$-\frac{2}{3}$	18	162	-3	$\frac{9842}{81}$
(6)	1	6	8	$\pm\frac{32}{81}$	96	864	± 3	$\frac{104960}{81}$ or $\frac{52480}{81}$
(7)	2	5	10	-6	48	-1536	-2	-1023

Q31. $0.5; 3$ or $-1; -3$

Q32. $\frac{2}{7}; 2$

Q33. $u_1 = 2, r = 3$ or $u_1 = -4, r = -3$

Q34. $r = \frac{2}{3}, u_1 = 6$

Q35. $u_1 = 2, r = \pm\frac{3}{2}$

Q36. (1) $a = \frac{x+y}{2}$
(2) $a = \pm\sqrt{xy}$

Q37. (i) 927 PLN
(ii) after 62 years

Q38. (i) 929 PLN
(ii) after 61 years and 2 months

Q39. (i) A: 290 PLN, B:118 PLN
(ii) in the 24^{th} month
(iii) in the 35^{th} month

Q40. $x = 6, y = 4$ or $x = \frac{8}{3}, y = -\frac{8}{3}$

Q41. $r = -\frac{2}{3}, u_1 = 3$

Chapter 5

Binomial theorem

Questions 5.1.

Q1. (1) 6 (2) 24 (3) 120 (4) 720 (5) 5040

Q2. (1) 7 (2) 8 (3) 12 (4) 110 (5) 380

Q3. (1) 7 (2) 5 (3) 8

Q4. (1) 3 (4) 10 (7) 6 (10) 15 (13) 35
(2) 4 (5) 10 (8) 15 (11) 21 (14) 56
(3) 6 (6) 5 (9) 20 (12) 35 (15) 28

Q5. (1) $\frac{n(n+1)}{2}$ (3) $\frac{n(n-1)}{2}$ (5) $\frac{2n(2n-1)}{2}$
(2) $\frac{3n(3n-1)}{2}$ (4) $\frac{n(n+1)}{2}$ (6) $\frac{(n+2)(n+1)n}{6}$

Q6. (1) $x^4+4x^3+6x^2+4x+1$
(2) $x^4-4x^3+6x^2-4x+1$
(3) $x^5+10x^4+40x^3+80x^2+80x+32$
(4) $32x^5-80x^4+80x^3-40x^2+10x-1$
(5) $32a^5-80a^4b+80a^3b^2-40a^2b^3+10ab^4-b^5$
(6) $a^8-12a^6b+54a^4b^2-108a^2b^3+81b^4$
(7) $x^6-6x^5y+12x^4y^2-8x^3y^3$
(8) $49-20\sqrt{6}$
(9) $2\sqrt{2}a^3-6a^2b+3\sqrt{2}ab^2-b^3$

Q7. 80.

Q8. -945.

Q9. 160.

Q10. 5376.

Q11. $8-52x+102x^2+...$

Q12. $16+208x+1126x^2+...$

Q13. -2

Q14. *Hint:* $2^n=(1+1)^n$.

CHAPTER 6

LOGARITHMS AND EXPONENTS

6.1 ALGEBRA OF LOGARITHMS

QUESTIONS 6.1.

Q1. (1) 3 (2) $\frac{1}{4}$ (3) $\frac{\sqrt{2}}{2}$ (4) $\frac{25}{2}$ (5) $4^6=2^{12}$ (6) 11×3^n

Q2. (1) 3 (6) $-\frac{1}{2}$ (11) 2.5 (16) $-\frac{3}{2}$ (21) $\frac{3}{4}$
(2) $\frac{3}{2}$ (7) $-\frac{3}{2}$ (12) $\frac{5}{4}$ (17) $\frac{3}{5}$ (22) $-\frac{7}{4}$
(3) $\frac{2}{3}$ (8) -3 (13) -5 (18) $\frac{1}{2}$ (23) $\frac{1}{12500}$
(4) $\frac{3}{2}$ (9) $-\frac{7}{4}$ (14) 5 (19) $-\frac{8}{3}$ (24) $\frac{2}{\sqrt{3}}$
(5) $\frac{1}{4}$ (10) 3.5 (15) -2 (20) $\frac{5}{6}$ (25) $2\times 3^{\frac{1}{4}}$

Q3. (1) $2b$ (3) $2a+b$ (5) $2+a+b$ (7) $3a$
(2) $a+b$ (4) $3+b$ (6) $3+a+3b$ (8) $3+2a$

Q4. (1) $a+2b$ (3) $\frac{1}{2}a+b$ (5) $a+6b$ (7) $\frac{1}{2}a+4b$
(2) $2a+4b$ (4) $a+2b$ (6) $6b$ (8) a

Q5. (1) $-a$ (3) $\frac{1}{a}$ (5) $\frac{1}{2}a$ (7) $\frac{2+5a}{3}$
(2) $2a$ (4) $1+\frac{3}{a}$ (6) $\frac{1}{3}a$ (8) $\frac{5}{2a}$

Q6. (1) $-a-3b$ (2) $\frac{1}{2}a-b$ (3) $a-\frac{1}{2}b$ (4) $1-4a-\frac{3}{2}b$

Q7. (1) $a-2b-\frac{1}{2}c$ (2) $3a+\frac{1}{2}b-2c)$ (3) $2.5a-1.5b-c$ (4) $-4a-2.5b-\frac{4}{3}c$

Q8. $a=2, b=2$

Q9. $a=2, b=1$

Q10. $a=4, b=1$

Q11. $a=3, b=3$

Q12. $a=3, b=1$

Q13. $a=4, b=2$

Q14. (1) $\frac{3}{4}(2a-1)$ (2) $\frac{2a+3}{4a-2}$ (3) $\frac{3}{2a-1}$ (4) $\frac{14-8a}{3a-3}$

Q15.

(1) $\sqrt{3}$ (3) 8 (5) $2\sqrt{2}$ (7) $\frac{1}{4}$
(2) 25 (4) $3^{(\frac{4}{3})}$ (6) $\frac{1}{9}$ (8) 8

Q16. (1) $2a$ (7) a^3 (13) $\frac{1}{16}a^2$ (19) $\frac{256}{a^{12}}$
(2) $\frac{a}{2}$ (8) $\frac{1}{b^2}$ (14) $\frac{1}{c}$ (20) $\frac{b^6}{243}$
(3) $32a$ (9) c^2 (15) $\frac{16}{a}$ (21) $\frac{1}{3}$
(4) $8a^2$ (10) a^4 (16) $\frac{729}{b^2}$ (22) $\frac{25}{c^5}$
(5) $\frac{1}{3}b^3$ (11) $\frac{1}{a^4}$ (17) c^4 (23) ac
(6) a^2 (12) b^3 (18) $\frac{1}{8}a^6$ (24) $\frac{a^2b^4}{18}$

6.2 Logarithmic equations

Questions 6.2.

Q17. (1) $\frac{27}{64}$ (3) 8 (5) $5^5 = 3125$ (7) $\sqrt[3]{2}$
(2) 27 (4) 9 (6) $\sqrt[3]{2}$ (8) $\frac{1}{2}$

Q18. (1) 4 or $\frac{1}{2}$ (4) 3 (8) -1 or 2 (12) 0
(2) 125 or $\sqrt{5}$ (5) $\frac{1}{9}$ or $\sqrt[3]{3}$ (9) 1 (13) $\frac{1}{\sqrt{3}}$ or $\frac{1}{3}$ or 3
(3) $\sqrt[3]{2}$; $\frac{1}{\sqrt[3]{4}}$ (6) no solutions (10) 0 or 1
(7) 0 (11) 0 or $\frac{1}{2}\log_3 2 - \frac{1}{2}$

Q19. (1) 1 or 3 (2) 3 (3) e (4) $e^{\sqrt{1.5}}$ or $e^{-\sqrt{1.5}}$ or e^3

Q20. (1) 128 (2) 100

Q21. (1) $\frac{5}{8}$ or $\frac{5}{4}$ (4) 2 or -0.4 (7) -3 or 1 (9) $\sqrt{10}$ or 0.001
(2) 9 (5) -2 or 1
(3) 0.1 (6) no solutions (8) 4 or $\frac{1}{2\sqrt{2}}$ (10) $\sqrt{2}$ or 8

Q22. a. $t_0 = 6.6$
b. $k = 0.1 \ln 0.75 \approx -0.0288$
c. $6.6(0.75^{10}) \approx 0.372$
d. $\frac{10 \ln 2}{\ln 0.75} = -\frac{10}{\log_2 0.75} \approx 24.1$

Q23. (1) $x < \frac{64}{\pi^3}$ (4) $x \leq -1.25$ (7) $\frac{1}{5} \leq x \leq 5$
(2) $x \geq \frac{1}{8}$ (5) $x \in \mathbb{R}$
(3) no solutions (6) $x \in \mathbb{R}$ (8) $x \geq 2^{\frac{45}{7}}$

Q24. (i) $0 < x < 1$ or $x > 1$
(ii) 1

chapter 7

Proof by mathematical induction

part III

Functions and equations

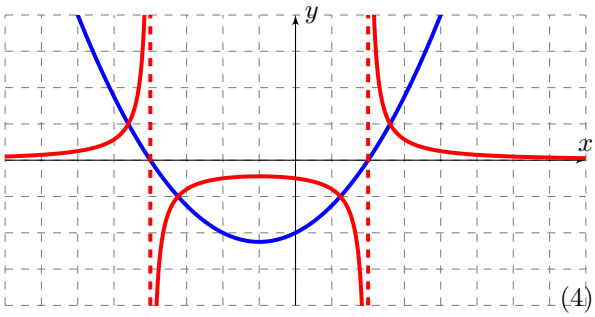
chapter 8

Graphs of rational functions

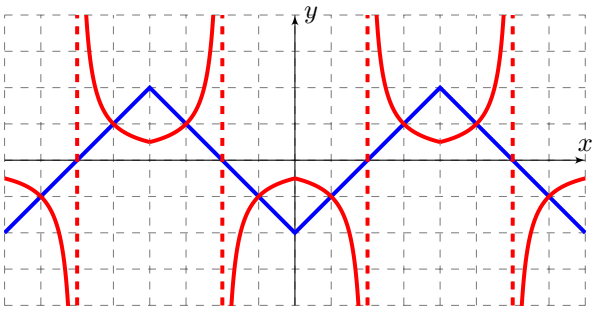
Questions 8.1.

Q1. Graphs of: $y = f(x)$ and $y = \frac{1}{f(x)}$.

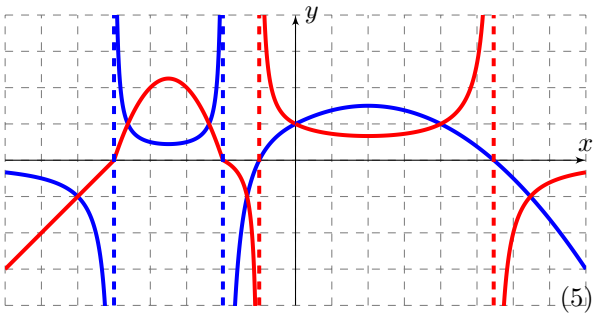
(1)



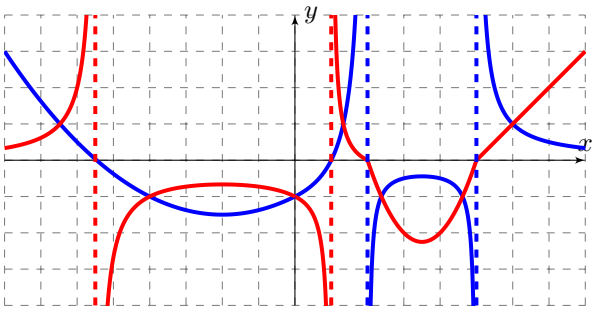
(4)



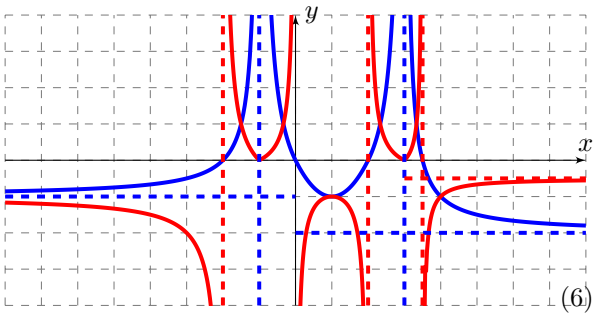
(2)



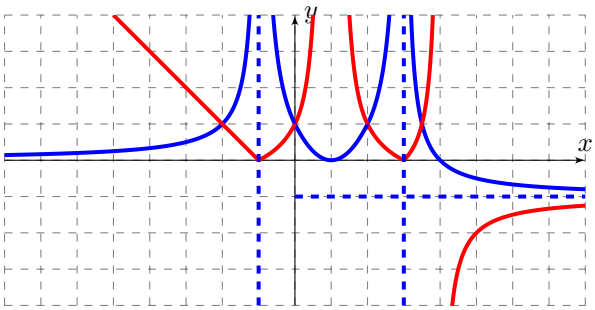
(5)



(3)

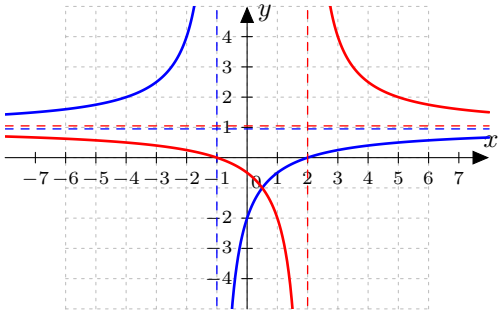


(6)

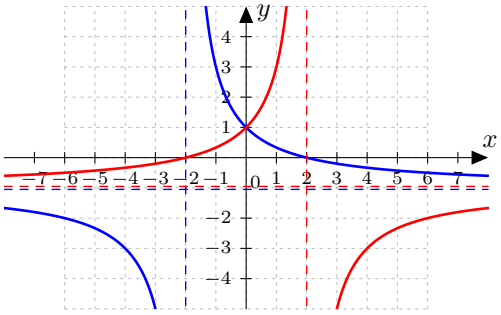


Q2. Graphs of: $x \longmapsto y$ and $x \longmapsto \frac{1}{y}$.

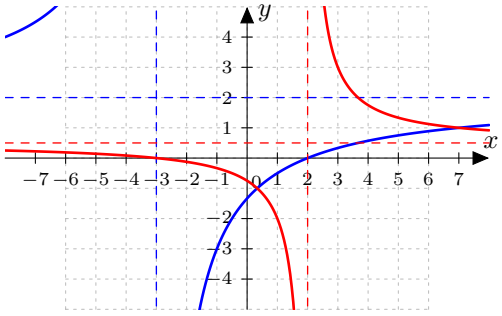
(1)



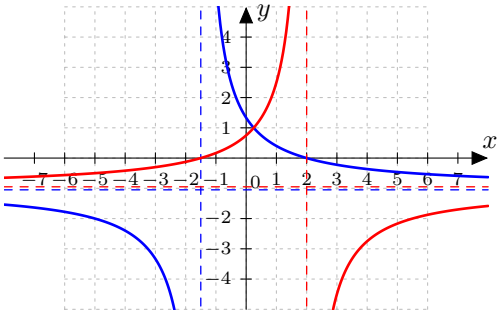
(3)



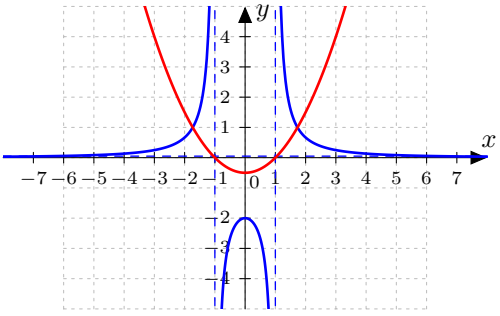
(2)



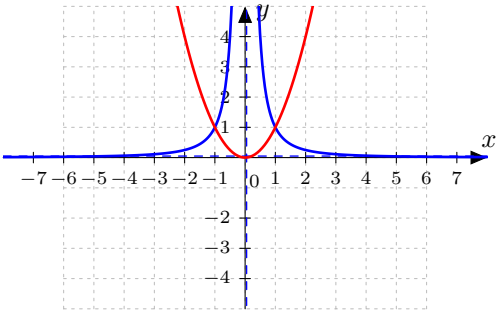
(4)



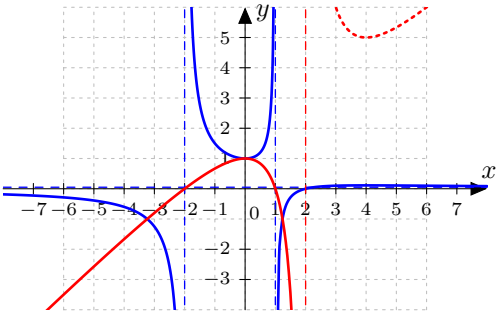
(5)



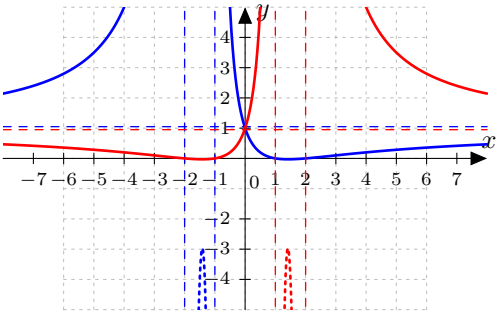
(6)



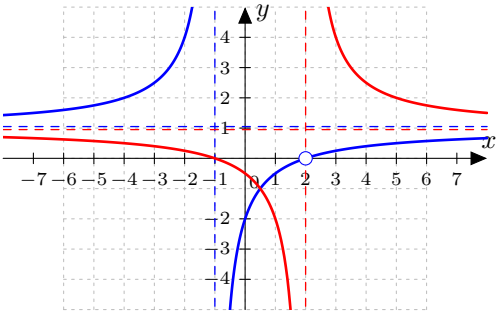
(7)



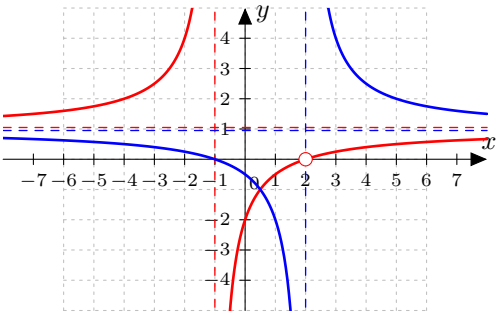
(8)



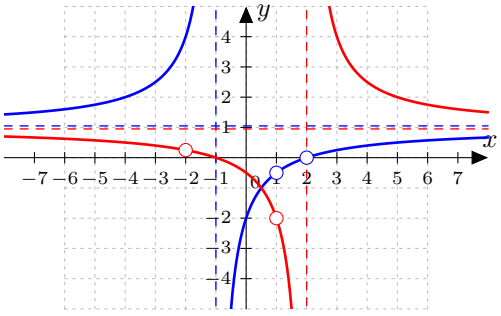
(9)



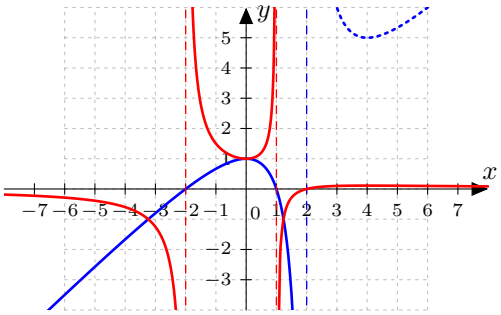
(10)



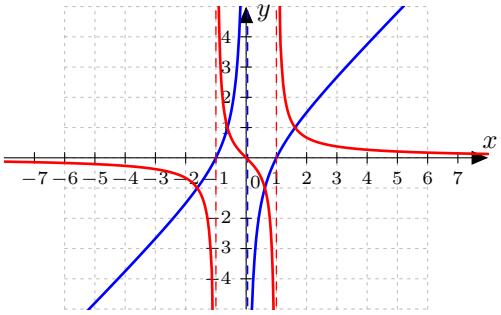
(11)



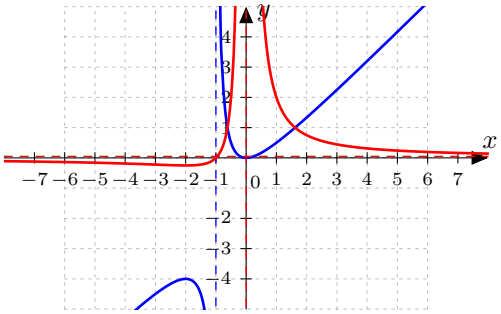
(12)



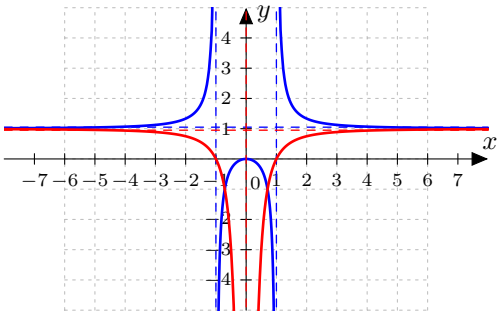
(13)



(14)



(15)



CHAPTER 9

BASIC TRANSFORMATIONS OF GRAPHS OF FUNCTIONS

QUESTIONS 9.1.

Q1. dotted graph→(transformation$_1$, transformation$_2$ ...)→solid graph

(1) $y = x^2$ →(translation by $\binom{-3}{-4}$)→ $y = (x+3)^2 - 4$

(2) $y = x^2$ →(vertical dilation by $k = 2$, translation by $\binom{1}{-6}$)→ $y = 2(x-1)^2 - 6$

(3) $y = x^2$ →(vertical dilation by $k = -\frac{1}{2}$, translation by $\binom{-2}{3}$)→ $y = -\frac{1}{2}(x+2)^2 + 3$

(4) $y = 2^x$ →(translation by $\binom{-3}{-2}$)→ $y = 2^{x+3} - 2$

(5) $y = 2^x$ →(reflection in x-axis, translation by $\binom{1}{-2}$)→ $y = -2^{x-1} - 2$

(6) $y = \log_{\frac{1}{2}} x$ →(reflection in y-axis, translation by $\binom{-2}{0}$)→ $y = \log_{\frac{1}{2}}(-x-2)$

(7) $y = x^2$ →(vertical dilation by $k = 4$, translation by $\binom{2}{-1}$)→ $y = 4(x-2)^2 - 1$

(8) $y = \sin x$ →(translation by $\binom{\frac{\pi}{2}}{}$)→ $y = \sin(x - \frac{\pi}{2}) + 2$

(9) $y = \sin x$ →(vertical dilation by $k = 2$, translation by $\binom{-\frac{\pi}{3}}{-1}$)→ $y = 2\sin(x + \frac{\pi}{3}) - 1$

(10) $y = \sin x$ →(vertical dilation by $k_y = 2$, horizontal by $k_x = -\frac{1}{2}$)→ $y = 2\sin(-2x)$

(11) $y = \sin x$ →(vertical dilation by $k = 4$, translation by $\binom{-\frac{\pi}{2}}{0}$)→ $y = 4\sin(x + \frac{\pi}{2})$

(12) $y = \cos x$→(vertical dilation by $k = 3$, translation by $\binom{\frac{\pi}{6}}{0}$)→ $y = 3\cos(x - \frac{\pi}{6})$

(13) $y = \cos x$→(vertical dilation by $k = 4$, translation by $\binom{\frac{\pi}{6}}{2}$)→ $y = 4\cos(x - \frac{\pi}{6}) + 2$

(14) $y = \cos x$→(vertical dilation by $k = 4$, translation by $\binom{-\frac{\pi}{3}}{-2}$)→ $y = 4\cos(x + \frac{\pi}{3}) - 2$

(15) $y = \cos x$→(vertical dilation by $k_y = 4$, horizontal by $k_x = 2$, translation by $\binom{-\frac{\pi}{3}}{0}$)→ $y = 4\cos(\frac{1}{2}(x + \frac{\pi}{3}))$

(16) $y = \cos x$→(vertical dilation by $k_y = 5$, horizontal by $k_x = \frac{1}{2}$, translation by $\binom{-\frac{\pi}{4}}{-2}$)→ $y = 5\cos(2(x + \frac{\pi}{4})) - 2$

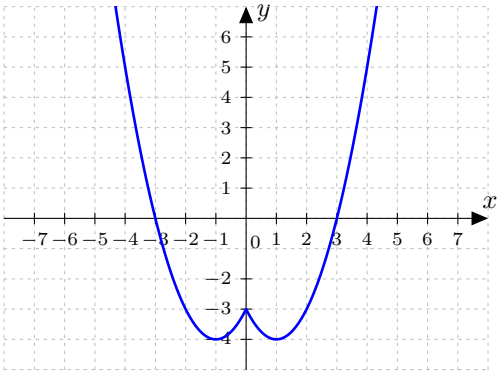
Q2. (1) $y = f(x) \to y = |f(x)|$
(2) reflection in y-axis
(3) horizontal dilation by 2
(4) translation by $\binom{-2}{2}$
(5) vertical dilation by $\frac{1}{2}$
(6) reflection in x-axis
(7) $y = f(x) \to y = f(|x|)$

Q3. (1) $\binom{4}{0}$ (2) $\binom{2}{0}$ (3) $\binom{2}{-2}$ (4) $\binom{-2}{0}$ (5) $\binom{-\frac{\pi}{2}}{1}$ (6) $\binom{\frac{\pi}{4}}{0}$ (7) $\binom{-2}{0}$ (8) $\binom{1}{1}$

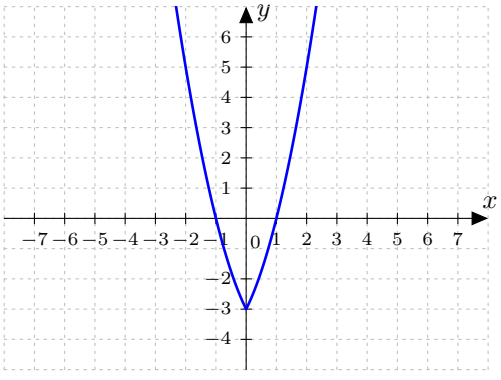
Q4. (1) 2 (2) -2 (3) 3 (4) $-\frac{1}{4}$ (5) -2

Q5. (1) $\frac{1}{2}$ (2) 4 (3) 3 (4) -2 (5) $-\frac{3}{2}$ (6) $-\frac{2}{3}$ (7) 4 (8) $\frac{1}{2}$

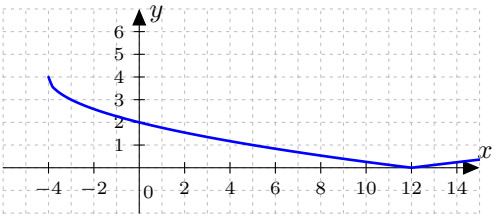
Q6. (1)



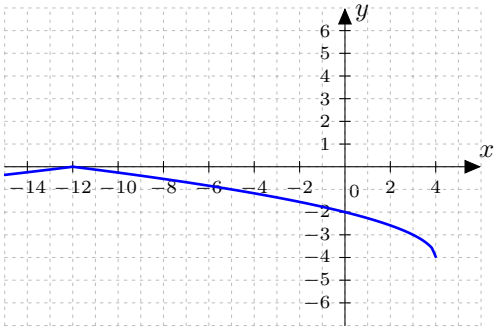
(2)



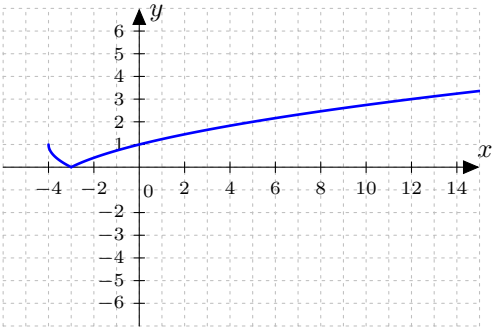
(3)



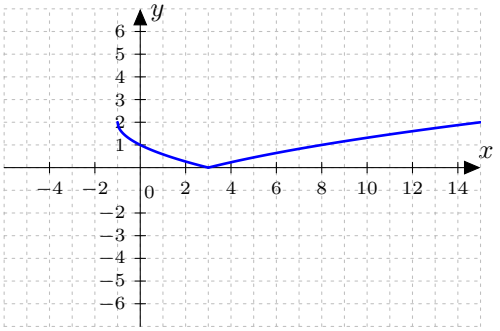
(4)



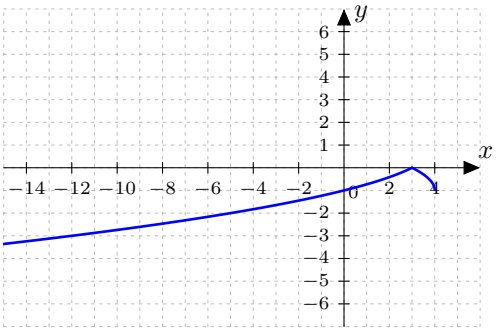
(5)



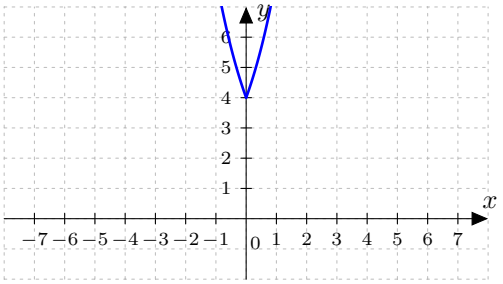
(6)



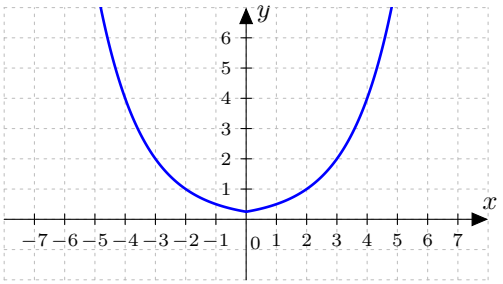
(7)



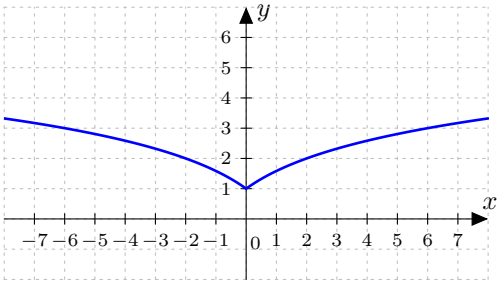
(8)



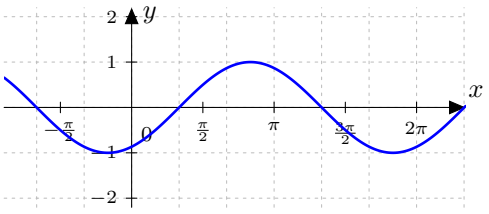
(9)



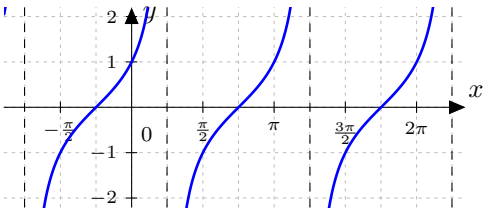
(10)



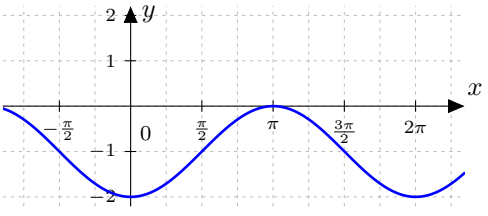
Q7. (1)



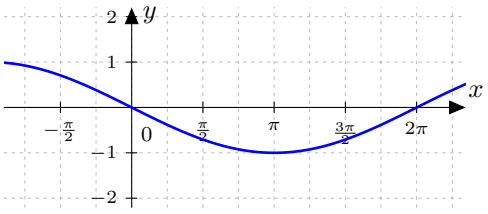
(2)



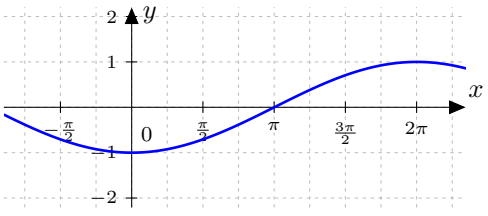
(3)



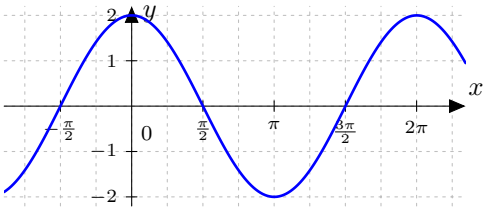
(4)

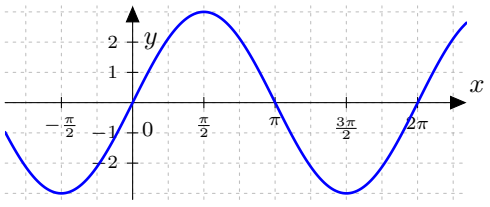


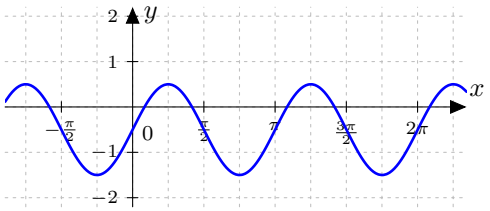
(5)

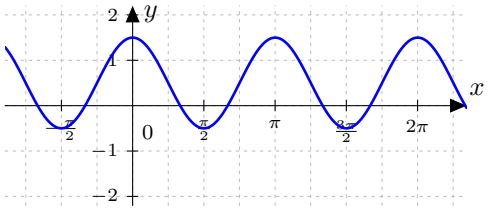


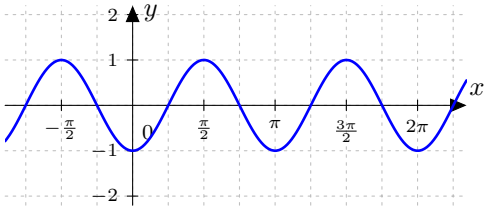
(6)

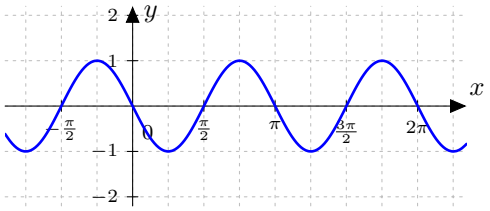


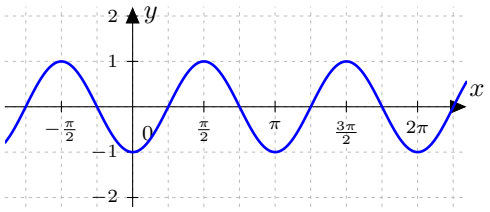
(7)


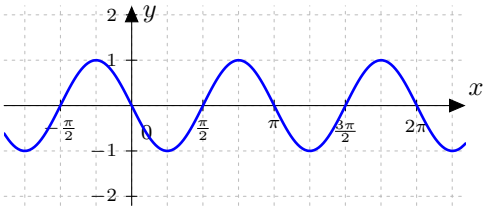
(8)


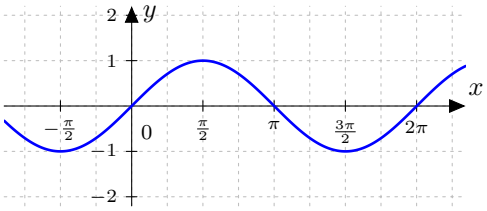
(9)


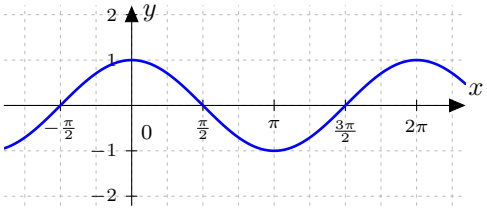
(10)


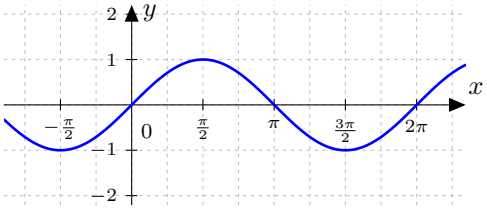
(11)


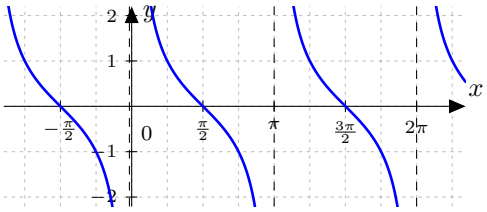
(12)


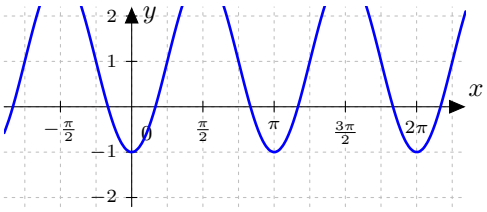
(13)


(14)


(15)


(16)


(17)


(18)


Q8. (exemplary answers)

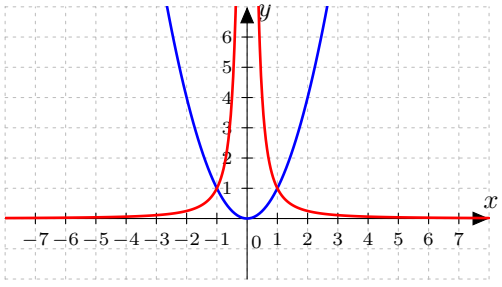
(1) reflection in x-axis (or y-axis) followed by translation by $\binom{\pm\frac{\pi}{2}}{0}$

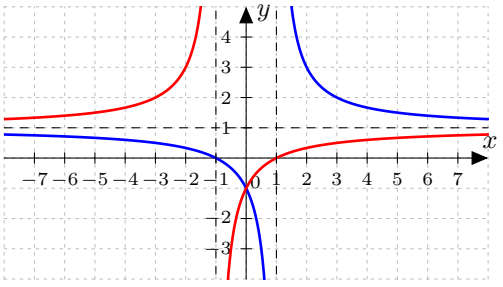
(2) translation by $\binom{\pm\frac{\pi}{2}}{0}$ followed by reflection in x-axis (or y-axis)

(3) translation $\binom{-2}{0}$ followed by reflection in $y = x$ followed by translation $\binom{2}{0}$ (or reflection in $y = x - 2$)

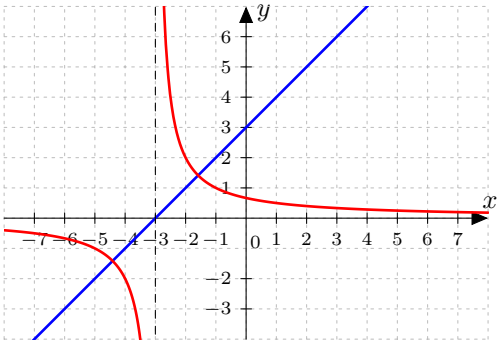
(4) translation $\binom{0}{2}$ followed by reflection in $y = x$ followed by translation $\binom{0}{-2}$ (or reflection in $y = x - 2$)

Q9. Graphs of $x \longmapsto f(x)$ and $x \longmapsto \frac{1}{f(x)}$.

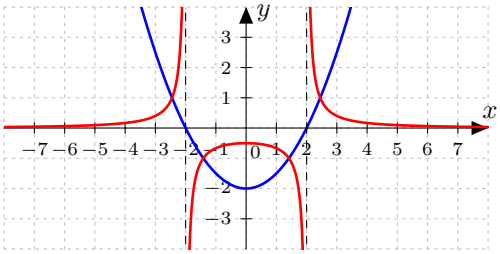
(1)


(2)


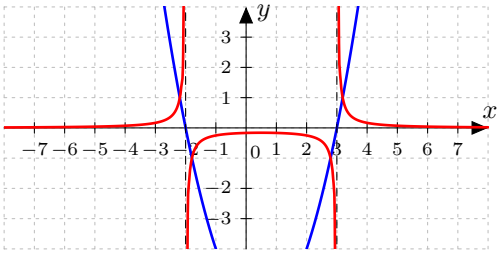
(3)



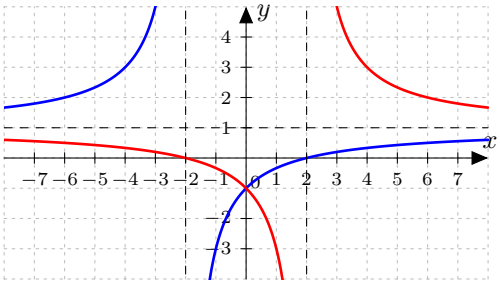
(4)



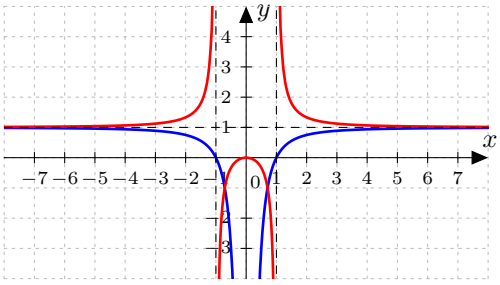
(5)



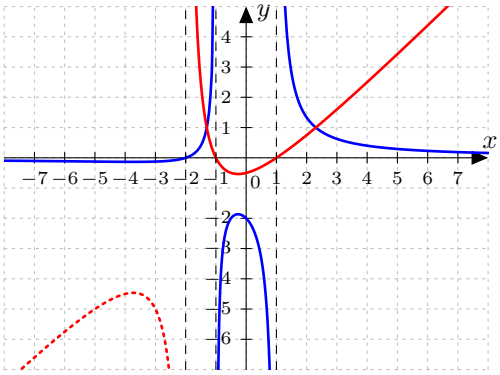
(6)



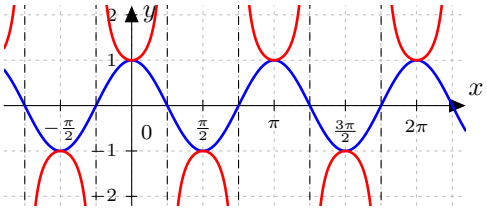
(7)



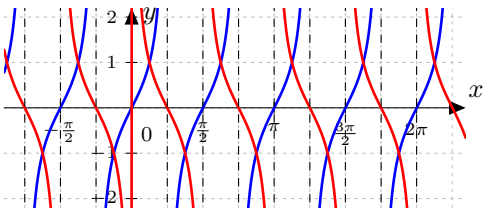
(8)



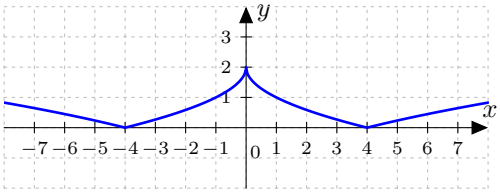
(9)



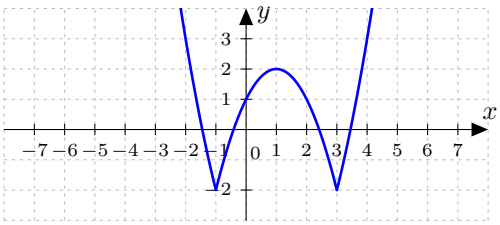
(10)



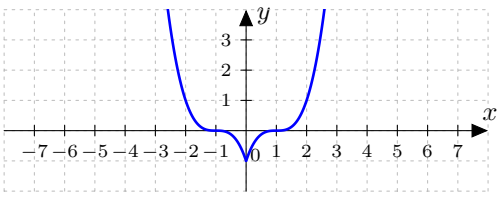
Q10. (1)



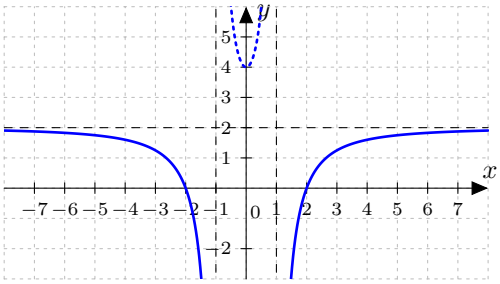
(2)



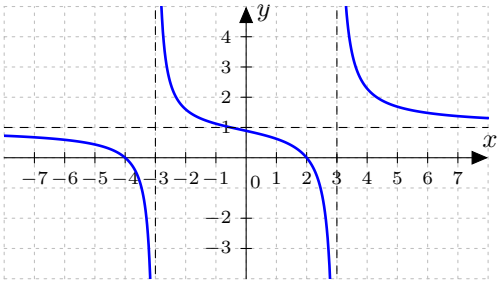
(3)



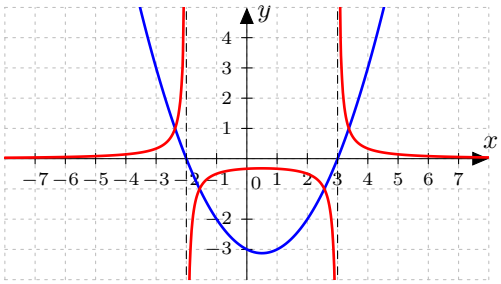
(4)



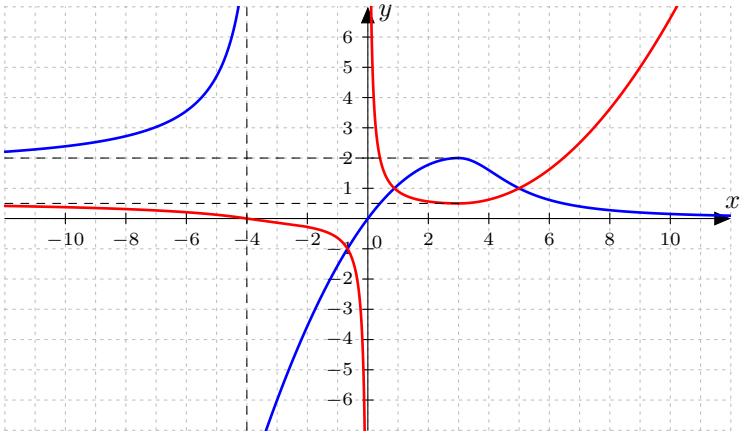
(5)



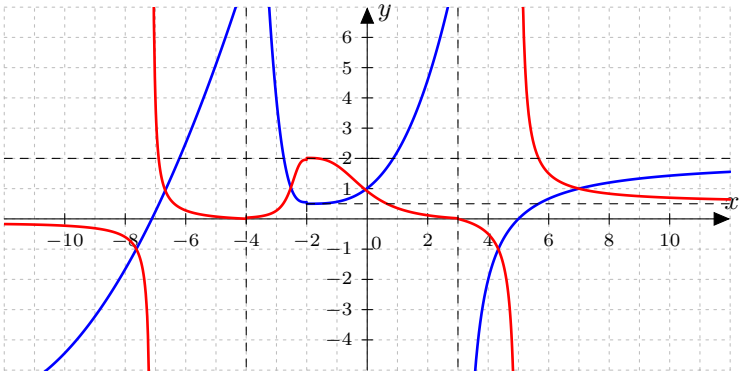
Q11. (1) Graphs of $x \longmapsto f(x)$ and $x \longmapsto \frac{1}{f(x)}$, where $f(x) = \frac{(x-3)(x+2)}{2}$.



(2) Graphs of $x \longmapsto f(x)$ and $x \longmapsto \frac{1}{f(x)}$.

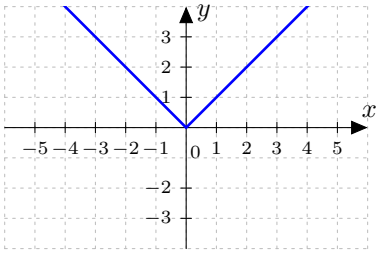


Q12. Graphs of $x \longmapsto f(x)$ and $x \longmapsto \frac{1}{f(x)}$.

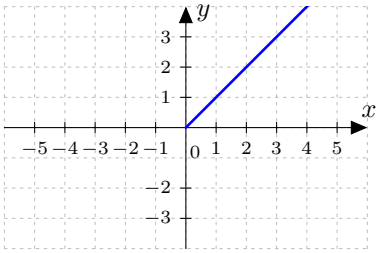


Q13.

(1)

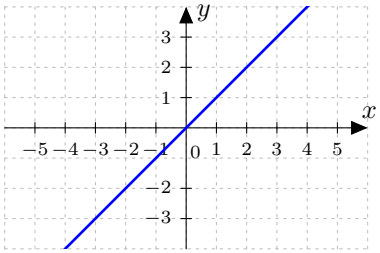


$y = \sqrt{x^2}$

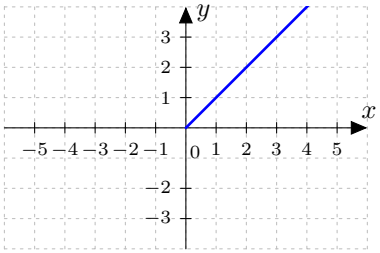


$y = (\sqrt{x})^2$

(2)

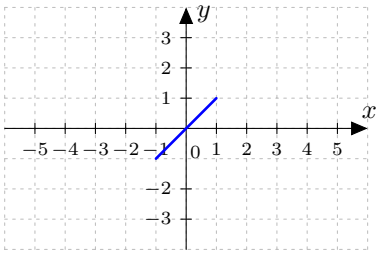


$y = \ln e^x$

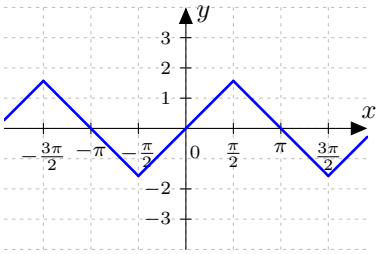


$y = e^{\ln x}$

(3)

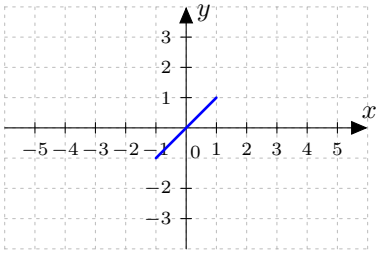


$y = \sin \arcsin x$

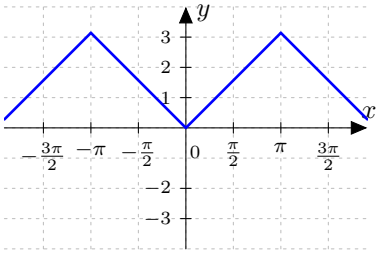


$y = \arcsin \sin x$

(4)

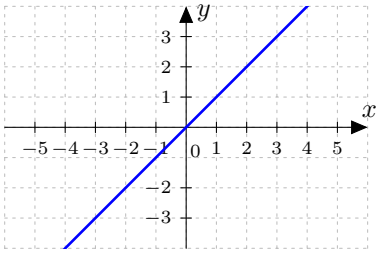


$y = \cos \arccos x$

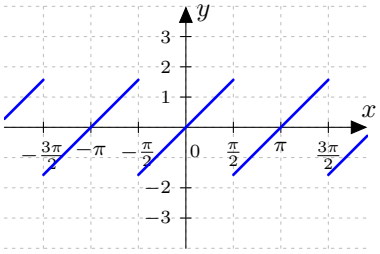


$y = \arccos \cos x$

(5)



$y = \tan \arctan x$



$y = \arctan \tan x$

Q14. $A =]-\infty, 0], B =]0, 1], C =]0, \frac{\pi}{2}]$

Q15. $A =]\frac{\pi}{3}, \frac{2\pi}{3}[, B =]0, 1], C =]-\infty, 0]$

CHAPTER 10

POLYNOMIALS

10.1 ALGEBRA OF POLYNOMIALS

QUESTIONS 10.1.

Q1. (1) $Q(x) = 3x^3 + 2x^2 - x + 3$;
(2) $Q(x) = 4x^3 - 3x^2 + 2$;
(3) $Q(x) = 2x^2 + 5; R = 9$;
(4) $Q(x) = 4x^3 - 2x^2 - x + 2; R = 2$.

Q2. (1) $Q(x) = -2x^4 + x^2 + 2x - 3$;
(2) $Q(x) = x^4 + 3x^3 - 2x^2 + x$;
(3) $Q(x) = -2x^3 + 2x^2 - x + 1$;
(4) $Q(x) = x^3 - 3x^2 + 2x + 0.5$;
(5) $Q(x) = 2x^2 - x + 1; R(x) = x$;
(6) $Q(x) = 3x^2 - \frac{3}{2}; R(x) = \frac{3}{2}x + \frac{1}{2}$.

Q3. (1) $R = 2$;
(2) $R = -58$;.
(3) $R = -60$;
(4) $R = 2$;
(5) $R = 7$;
(6) $R = 4.25$

10.2 REMAINDER THEOREM AND FACTOR THEOREM

QUESTIONS 10.2.

Q4. $R(x) = -2x + 2$

Q5. $R(x) = x$

Q6. $R(x) = -0.2x + 2.4$

Q7. $R(x) = x + 4$

Q8. $R(x) = 3x - 1$

Q9. $R(x) = -2x + 3$

Q10. $a = 2$

Q11. $a = -1; b = 3$

Q12. $a = 3; b = 2$

Q13. $a = -1; b = 5$

Q14. $a = 2; b = 1$

Q15. $a = 4; b = 3$

Q16. $a = 5; b = -5$

Q17. $a = 1; b = 6$

Q18. $a = -1; b = 3$

Q19. $a = 1, b = 2$

Q20. $a = 3, b = -1$

10.3 Vieta's formulas

Questions 10.3.

Q21.
(1) sum$= -2$, product$= 5$
(2) sum$= \frac{1}{2}$, product$= \frac{3}{2}$
(3) sum$= 0$, product$= -2$
(4) sum$= -\frac{3}{4}$, product$= \frac{1}{4}$
(5) sum$= \frac{3}{2}$, product$= 0$
(6) sum$= 0$, product$= -4$
(7) sum$= 0$, product$= 0$
(8) sum$= -2$, product$= -1$

Q22. $a = -6$, product$= \frac{1}{3}$

Q23. $a = 2$, product$= \frac{1}{2}$

Q24. $b = 12$, sum$= \frac{3}{2}$

Q25. $b = -4$, sum$= -3$

Q26. $(0, -6)$

Q27. even

Q28. 4

Q29. $-\frac{1}{2}, 3$

Q30. $\frac{5}{2}, -\frac{3}{2}$

Q31. $a = 4, b = -6$

Q32. $P(x) = x^3 + 5x^2 + 2x - 8$

Q33. solutions: $-1, -2, -4$; $a = 7, b = 14$

Q34. solutions: $1, 5, 9$; $a = -15, b = 59$ or: $-1, -5, -9$; $a = 15, b = 59$

Q35. (1) $4\frac{8}{45}$ (2) $-1\frac{1}{9}$ (3) $-3\frac{13}{27}$ (4) $2\frac{73}{81}$ (5) $7\frac{23}{135}$ (6) $\frac{2}{15}\sqrt{370}$

Q36. (1) $18x^2 + 20x - 15 = 0$ (2) $225x^2 - 940x + 324 = 0$ (3) $27x^2 + 94x + 27 = 0$

10.4 Polynomial inequalities

Questions 10.4.

Q37.
(1) —
(2) $x \leq -4$ or $-\frac{1}{2} \leq x \leq \frac{3}{2}$
(3) $x^2 + x + 1$

Q38.
(1) $1 < x < 2$ or $x > 3$
(2) $x < -4$ or $-1 < x < 2$
(3) $x \leq -5$ or $x = 1$
(4) $x > \frac{3}{2}$
(5) $x < -\frac{1}{2}$
(6) $x \leq -\frac{2}{3}$ or $x \geq 1$
(7) $-2 < x < -\frac{1}{2}$ or $1 < x < 3$
(8) $-2 \leq x \leq 3$
(9) $x \leq -2$ or $x = -\frac{1}{2}$ or $x \geq 3$
(10) $-2 < x < \frac{2}{3}$
(11) $-\frac{1}{2} \leq x \leq \frac{2}{3}$ or $x \geq 2$
(12) $x \leq \frac{1}{3}$ or $x \geq 1$

Q39.
(1) $x \in]-1, 2[\cup]3, +\infty[$
(2) $x \in]-\infty, -4] \cup \{2\}$
(3) $x \in]-\infty, -5[\cup]1, 3[$
(4) $x \in [-2, -1] \cup [4, +\infty[$
(5) $x \in]-\infty, -3[\cup]-2, 3[$
(6) $x \in]-\infty, -2] \cup [3, 5]$
(7) $x \in]-1, 3[\cup]5, +\infty[$
(8) $x \in \{-2\} \cup [2, +\infty[$
(9) $x \in]-1, 1[\cup]2, +\infty[$
(10) $x \in [-4, 2] \cup [3, +\infty[$
(11) $x \in]-\infty, -5[\cup]-1, 3[$
(12) $x \in \{-\frac{7}{3}\} \cup [\frac{3}{2}, +\infty[$
(13) $x \in]-\frac{5}{2}, \frac{5}{3}[\cup]\frac{5}{3}, +\infty[$
(14) $x \in]-\infty, -2] \cup [-1, 2] \cup [3, +\infty[$
(15) $x \in]-\infty, -4] \cup [-1, 1] \cup [2, +\infty[$
(16) $x \in [-5, 1] \cup \{3\}$
(17) $x \in]-1, 4[$
(18) $x \in]-\infty, -3] \cup [-2, 2] \cup [3, +\infty[$
(19) $x \in]2, 3[$
(20) $x \in]-\infty, -2] \cup [0.25, +\infty[$
(21) $x \in]-\infty, -2] \cup [5, +\infty[$
(22) $x \in [-2, -1] \cup [3, 5]$
(23) $x \in]-3, -\frac{1}{2}[\cup]\frac{1}{3}, 2[$
(24) $x \in]-\infty, -3] \cup [-0.5, 1.5] \cup [2, +\infty[$

(25) $x \in]-\infty, -2[\cup]-2, 2[\cup]3, +\infty[$
(26) $x \in]-\infty, 0] \cup \{1.5\} \cup [4, +\infty[$
(27) $x \in]-\frac{1}{2}, \frac{1}{2}[$
(28) $x \in]-\infty, -3] \cup [\frac{1}{2}, +\infty[$
(29) $x \in \emptyset$
(30) $x \in]-\infty, -\frac{5}{2}] \cup [\frac{2}{3}, +\infty[$
(31) $x \in]-\frac{5}{2}, \frac{4}{3}[\cup]\frac{4}{3}, \frac{5}{2}[$

CHAPTER 11

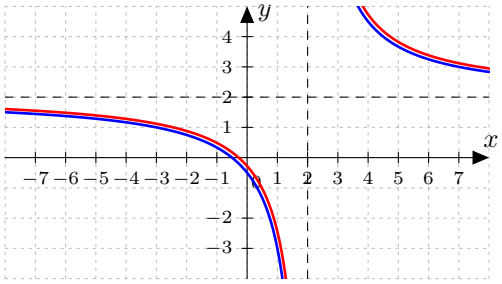
EXPONENTIAL AND LOGARITHMIC FUNCTION

CHAPTER 12

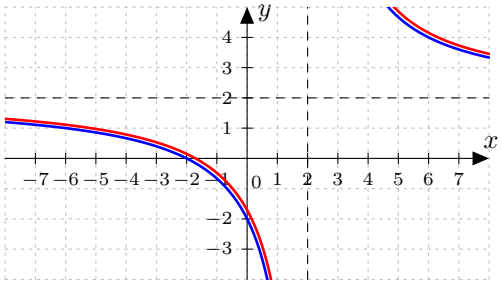
COMPOSITIONS AND INVERSES

QUESTIONS 12.1.

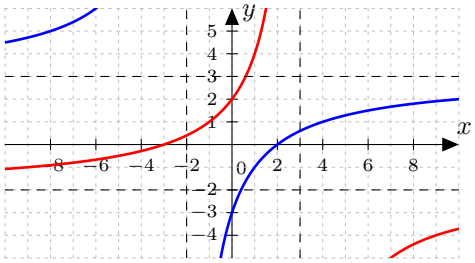
Q1. Graphs of $x \mapsto f(x)$ and $x \mapsto f^{-1}(x)$. D - domain of f / range of f^{-1}, D^{-1} - domain of f^{-1} / range of f.

(1)


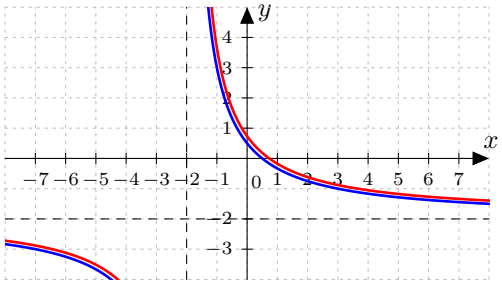
$f^{-1}(x) = f(x)$, $D = D^{-1} = \mathbb{R} - \{2\}$

(2)


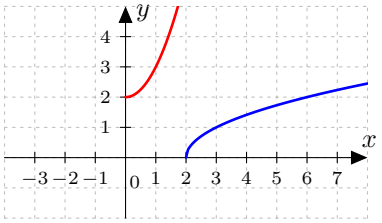
$f^{-1}(x) = f(x)$, $D = D^{-1} = \mathbb{R} - \{2\}$

(3)


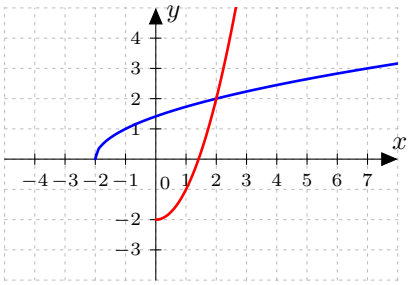
$f^{-1}(x) = \frac{-2x-6}{x-3}$, $D = \mathbb{R} - \{-2\}, D^{-1} = \mathbb{R} - \{3\}$

(4)


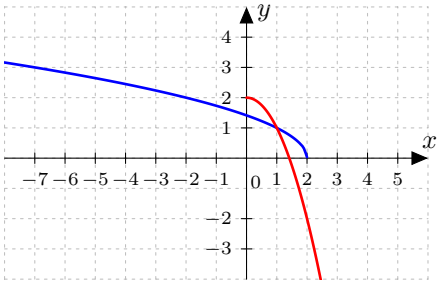
$f^{-1}(x) = f(x)$, $D = D^{-1} = \mathbb{R} - \{-2\}$

(5)


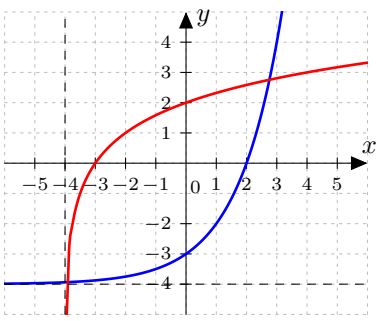
$f^{-1}(x) = x^2 + 2$, $D = [2, +\infty[, D^{-1} = [0, +\infty[$

(6)


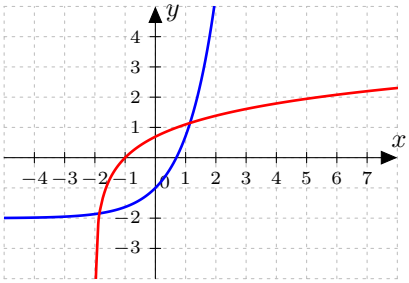
$f^{-1}(x) = x^2 - 2$, $D = [-2, +\infty[, D^{-1} = [0, +\infty[$

(7)


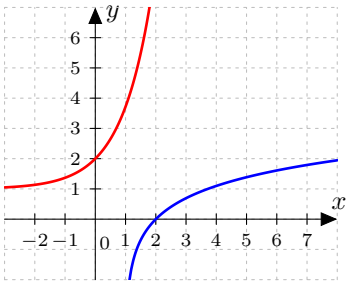
$f^{-1}(x) = 2 - x^2$, $D =]-\infty, 2[, D^{-1} = [0, +\infty[$

(8)


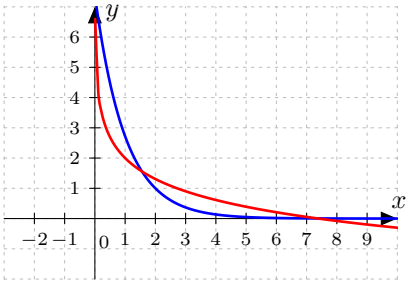
$f^{-1}(x) = \log_2(x+4)$, $D = \mathbb{R}, D^{-1} =]-4, +\infty[$

(9) 

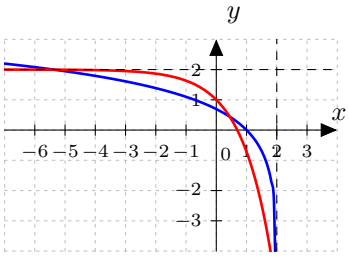
$f^{-1}(x) = \ln(x+2),\ D = \mathbb{R}, D^{-1} =]-2, +\infty[$

(11) 

$f^{-1}(x) = e^x + 1,\ D =]0, +\infty[, D^{-1} = \mathbb{R}$

(10) 

$f^{-1}(x) = 2 - \ln x,\ D = \mathbb{R}, D^{-1} =]0, +\infty[$

(12) 

$f^{-1}(x) = 2 - e^x,\ D =]-\infty, 2[, D^{-1} = \mathbb{R}$

Q2. *(check the graphs with an approprite software)*

(1) (i) $\frac{x+1}{x+2}$ (ii) $x \neq -2,\ y \neq 1$

(2) (i) $\frac{2x+6}{x-3}$ (ii) $x \neq 3,\ y \neq 2$

(3) (i) $\frac{x+2}{-2x+1}$ (ii) $x \neq \frac{1}{2},\ y \neq -\frac{1}{2}$

(4) (i) $\frac{2x+1}{x+1}$ (ii) $x \neq -1,\ y \neq 2$

(5) (i) $\frac{1}{2}\ln(x-1)$ (ii) $x > 1,\ y \in \mathbb{R}$

(6) (i) $1 - \ln(2-x)$ (ii) $x < 2,\ y \in \mathbb{R}$

(7) (i) $3 - e^{x-1}$ (ii) $x \in \mathbb{R},\ y < 3$

(8) (i) $2 - 2^{1-x}$ (ii) $x \in \mathbb{R},\ y < 2$

(9) (i) $3 - (x-2)^2$ (ii) $x \geq 2,\ y \leq 3$

(10) (i) $(x-2)^2 - 1$ (ii) $x \leq 2,\ y \geq -1$

(11) (i) $\sqrt{-x+4} - 1$ (ii) $x \geq -4,\ y \geq -1$

(12) (i) $-\sqrt{2x+4} + 2$ (ii) $x \geq -2,\ y \leq 2$

Q3. (1) $(f \circ g)(x) = 2x^2 + 5$; $(g \circ f)(x) = 4x^2 + 12x + 10$

(2) $(f \circ g)(x) = \sqrt{x^2 + 2x}$; $(g \circ f)(x) = x + 2\sqrt{x-1}$

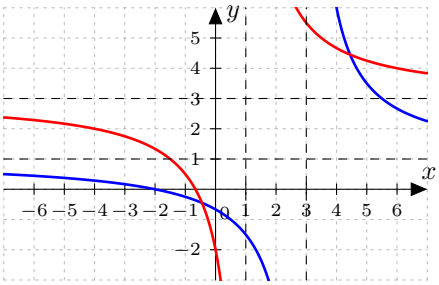
(3) $(f \circ g)(x) = \sqrt[3]{x}$; $(g \circ f)(x) = \sqrt[3]{x-2} + 2$

Q4.

(a) $(f \circ g)(x) = \frac{4x-1}{-x-6}, x \neq -6, x \neq -1$

(b) $f^{-1}(x) = \frac{3x+2}{x-1}$

(c) $x \longmapsto f(x)$, $x \longmapsto f^{-1}(x)$



(d) $x \in]-\infty, -2[\cup]3, +\infty[, y \in \mathbb{R}^*$

(e) $h^{-1}(x) = \frac{3e^x+2}{e^x-1}$.

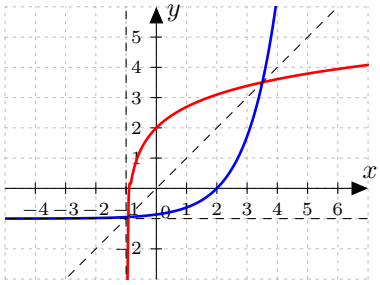
(f) —

(g) $x \neq 1,\ y \in]0, e^2[\cup]e^2, +\infty[$

(h) $k^{-1}(x) = \frac{\ln x + 3}{2 - \ln x}$.

(i) —

Q5. (a) $f^{-1} = \ln(x+1) + 2, x > -1, y \in \mathbb{R}$



(b) $x \longmapsto f(x)$, $x \longmapsto f^{-1}(x)$

(c) $g(x) = \ln(x+4) + 2$ (d) $h(x) = 1 - \ln(x+1)$

Q6. $A = [0, 4[,\ B =]0, 4],\ C =]-\infty, 2]$

Q7. $A =]0, e^3],\ B =]-\infty, 3],\ C = [0, +\infty[$

Q8. $A = [-1, 1], B = [-\frac{\pi}{2}, \frac{\pi}{2}], C = [e^{\frac{\pi}{2}}, e^{-\frac{\pi}{2}}]$

Q9. $A =]-\infty, 0], B =]0, 1], C =]0, \frac{\pi}{2}]$

Q10. $A = \bigcup_{k\in\mathbb{Z}}]\frac{(2k-1)\pi}{3}, \frac{2k\pi}{3}[,\ B =]0, 2], C =]-\infty, \ln 2]$

Q11. $x > 0,\ x \neq 1$, product $= 1$, sum $= \frac{11\sqrt{10}}{10}$

Q12. domain:

(i) $x \leq 1$ (ii) $\mathbb{R}$ (iii) $x \leq 0$ (iv) $x \leq 1$

range:

(i) $y \geq 0$ (ii) $y > 0$ (iii) $0 \leq y < 1$ (iv) $y \geq 1$

Q13. (i) $a = 2$ (ii) $y \geq -1$ (iii) $f^{-1}(x) = \sqrt{x+1} + 2$

Q14. (1) $f^{-1}(x) = \sqrt{\ln(x+2)},\ x \geq -1\ y \geq 0$
(2) $f^{-1}(x) = -\sqrt{\ln(x-1)},\ x \geq 2,\ y \leq 0$
(3) $f^{-1}(x) = -\sqrt{e^{x+1} - 1},\ x \geq -1,\ y \leq 0$
(4) $f^{-1}(x) = \sqrt{1 - e^x},\ x \leq 0,\ 0 \leq y < 1$
(5) $f^{-1}(x) = 3\sqrt{4 - x^2},\ 0 \leq x \leq 2,\ 0 \leq y \leq 6$
(6) $f^{-1}(x) = -\sqrt{x^2 + 1},\ x \geq 0,\ y \leq -1$

Q15. $f^{-1}(x, y) = (\frac{2x+y}{3}, \frac{x-y}{3})$

CHAPTER 13

INEQUALITIES

QUESTIONS 13.1.

Q1. (1) $0 < x < 1$ or $x > 3$
(2) $-\frac{3}{2} \leq x < 2$
(3) $x < 0$ or $1 < x < 2$
(4) $\frac{5}{3} < x < 2$
(5) $1 < x < 2$
(6) $-1 < x \leq 4$
(7) $x > 1$
(8) $x < -1$ or $\frac{1}{2} < x < 2$
(9) $-2 < x < \frac{3}{2}$
(10) $-1 < x \leq 0$ or $\frac{1}{2} < x \leq 2$
(11) $x < -4$ or $0 \leq x < 4$ or $x \geq 16$
(12) $-2 < x < 0$
(13) $-\frac{1}{3} \leq x < \frac{1}{2}$ or $\frac{1}{2} < x \leq 3$
(14) $-\frac{1}{5} \leq x \leq 3$
(15) $-1 < x < 2$
(16) $x < -10$ or $x > 2$
(17) $x \leq \frac{5}{3}$ or $x \geq 3$

Q2. (1) $x < 0.589$ or $3.10 < x < 4$
(2) $0 < x < 0.832$ or $2.61 < x < 3.58$ or $6.06 < x < 6.47$
(3) $2.57 \leq x \leq 7.40$ or $x \geq 8.22$
(4) $x < -1.71$ or $0 < x < 1.71$

PART IV

CIRCULAR FUNCTIONS AND TRIGONOMETRY

CHAPTER 14

CIRCULAR FUNCTIONS

14.1 BASIC CIRCULAR FUNCTIONS

14.2 UNIT CIRCLE

Questions 14.1.

Q1.
(1) $\frac{\sqrt{3}}{2}$
(2) $-\frac{\sqrt{3}}{2}$
(3) 1
(4) none
(5) $-\sqrt{3}$
(6) $-\frac{\sqrt{3}}{2}$
(7) 0
(8) $-\sqrt{3}$
(9) $-\frac{\sqrt{3}}{2}$
(10) $-\frac{1}{2}$
(11) none
(12) $\frac{1}{2}$
(13) -1
(14) 0
(15) $\sqrt{3}$
(16) $-\frac{1}{2}$
(17) $-\frac{1}{2}$
(18) $-\frac{\sqrt{3}}{2}$
(19) $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$
(20) $\frac{1}{2}$
(21) $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Q2.

	$\sin A$	$\cos A$	$\tan A$
(1)	given	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
(2)	given	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
(3)	given	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
(4)	given	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
(5)	$\frac{3}{5}$	given	$\frac{3}{4}$
(6)	$\frac{2\sqrt{2}}{3}$	given	$-2\sqrt{2}$
(7)	Not possible	Not possible	Not possible
(8)	$-\frac{2\sqrt{2}}{3}$	given	$-2\sqrt{2}$
(9)	$-\frac{4}{5}$	given	$\frac{4}{3}$
(10)	$\frac{3}{5}$	$\frac{4}{5}$	given
(11)	$-\frac{4}{5}$	$-\frac{3}{5}$	given
(12)	$\frac{3}{\sqrt{10}}$	$-\frac{1}{\sqrt{10}}$	given
(13)	$-\frac{3}{\sqrt{10}}$	$-\frac{1}{\sqrt{10}}$	given
(14)	Not possible	Not possible	Not possible

Q3.

	$\sin A$	$\cos A$	$\tan A$
(1)	given $(t \geq 0)$	$\sqrt{1-t^2}$	$\frac{t}{\sqrt{1-t^2}}$
(2)	given $(t \geq 0)$	$-\sqrt{1-t^2}$	$-\frac{t}{\sqrt{1-t^2}}$
(3)	given $(t < 0)$	$\sqrt{1-t^2}$	$\frac{t}{\sqrt{1-t^2}}$
(4)	given $(t < 0)$	$-\sqrt{1-t^2}$	$-\frac{t}{\sqrt{1-t^2}}$
(5)	$\sqrt{1-t^2}$	given $(t \geq 0)$	$\frac{\sqrt{1-t^2}}{t}$
(6)	$\sqrt{1-t^2}$	given $(t < 0)$	$\frac{\sqrt{1-t^2}}{t}$
(7)	$-\sqrt{1-t^2}$	given $(t \geq 0)$	$-\frac{\sqrt{1-t^2}}{t}$
(8)	$-\sqrt{1-t^2}$	given $(t < 0)$	$-\frac{\sqrt{1-t^2}}{t}$
(9)	$\frac{t}{\sqrt{t^2+1}}$	$\frac{1}{\sqrt{t^2+1}}$	given $(t \geq 0)$
(10)	$-\frac{t}{\sqrt{t^2+1}}$	$-\frac{1}{\sqrt{t^2+1}}$	given $(t < 0)$
(11)	$\frac{t}{\sqrt{t^2+1}}$	$\frac{1}{\sqrt{t^2+1}}$	given $(t < 0)$
(12)	$-\frac{t}{\sqrt{t^2+1}}$	$-\frac{1}{\sqrt{t^2+1}}$	given $(t \geq 0)$
(13)	given $(t < 0)$	$-\sqrt{1-4t^2}$	$-\frac{2t}{\sqrt{1-4t^2}}$
(14)	$\sqrt{1-9t^2}$	given $(t \geq 0)$	$\frac{\sqrt{1-9t^2}}{3t}$
(15)	$\frac{1}{2}\sqrt{4-t^2}$	given $(t < 0)$	$\frac{\sqrt{4-t^2}}{t}$
(16)	$\frac{5t}{\sqrt{25t^2+4}}$	$\frac{2}{\sqrt{25t^2+4}}$	given $(t < 0)$

Q4. (All answers given to 3 s.f.)
(1) 0.201
(2) 2.37
(3) -2.94
(4) 2.69
(5) -2.69
(6) -1.47
(7) 2.03
(8) -2.85

Q5. (1) $-\frac{\pi}{3}$ (2) $\frac{5\pi}{6}$ (3) $\frac{\pi}{2}$ (4) $\frac{5\pi}{6}$ (5) $-\frac{3\pi}{4}$ (6) $-\frac{2\pi}{3}$ (7) $\frac{5\pi}{6}$

Q6. (1) $\sqrt{1-q^2}$ (2) $\frac{\sqrt{1-q^2}}{q}$ (3) $-\sqrt{1-p^2}$ (4) $-q$ (5) $\sqrt{1-q^2}$ (6) $\sqrt{1-p^2}$ (7) $\frac{p}{\sqrt{1-p^2}}$

14.3 Graphs of trigonometric functions

Questions 14.2.

Q7. Check your answers with a GDC or at *wolframalpha.com*.

Q8. $A = -5, B = 2, C = 1$

Q9. $p = -4, q = \frac{\pi}{2}, s = 2$

Q10. (i) $a = 3, b = \pi, c = \pi, d = 2$
(ii) $m = -1$.
(iii) $-1 < k < 2$.

Q11. (i) $y = 12\sin(\frac{\pi}{4}x - \frac{5\pi}{4}) + 5$
(ii) $y = 12\cos(\frac{\pi}{4}x + \frac{\pi}{4}) + 5$

Q12. (i) $y = 0.6\sin(\frac{\pi}{8}x + \frac{\pi}{4}) - 0.2$
(ii) $y = 0.6\cos(\frac{\pi}{8}x - \frac{\pi}{4}) - 0.2$
(iii) $y = 0.6\sin(\frac{\pi}{4}x + \frac{9\pi}{40}) + 0.8$

Q13. (i) $y = 70\sin(\frac{3}{2}x - \frac{\pi}{4}) - 30$
(ii) $y = 70\cos(\frac{3}{2}x - \frac{3\pi}{4}) - 30$
(iii) $y = 35\sin(\frac{3}{2}x - \frac{\pi}{2}) + 5$
or $y = 35\cos(\frac{3}{2}x - \pi) + 5$

Q14.

	expression	period	range
$f \circ g(x)$	$\frac{1}{6}\sin(6x) + \frac{2}{3}$	$\frac{\pi}{3}$	$[\frac{1}{2}, \frac{5}{6}]$
$g \circ f(x)$	$\frac{1}{4}\sin(4x) + 1$	$\frac{\pi}{2}$	$[\frac{3}{4}, \frac{5}{4}]$

Q15. (i) $d = 14, t = 8$
(ii) $d = 2, t = 20$
(iii) $d = 6\sin(\frac{\pi}{12}t - \frac{\pi}{6}) + 8$
(iv) $[1.36, 14.6]$ or from 1 : 22AM to 2 : 38PM

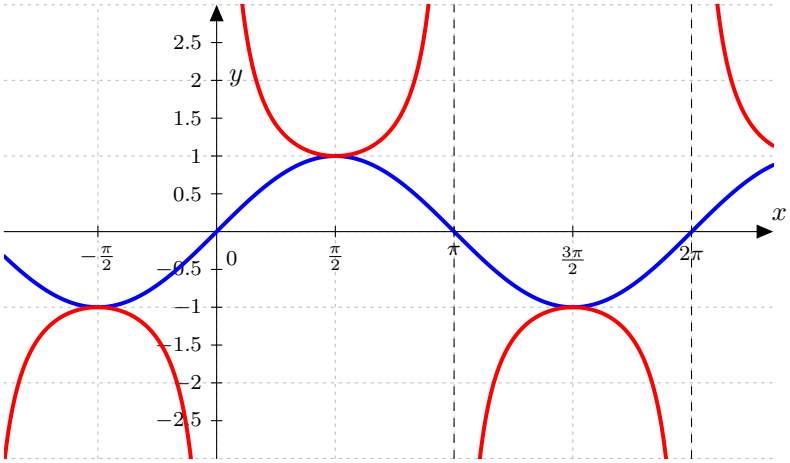
Q16. (i) $A = 5.5, B = \frac{\pi}{6}, C = -\frac{\pi}{6}, D = 12.5$
(ii) 41.2%

Q17. (i) $210cm$
(ii) 2 seconds
(iii) $206cm$
(iv) $t = \frac{7}{6}s \approx 1.17s$

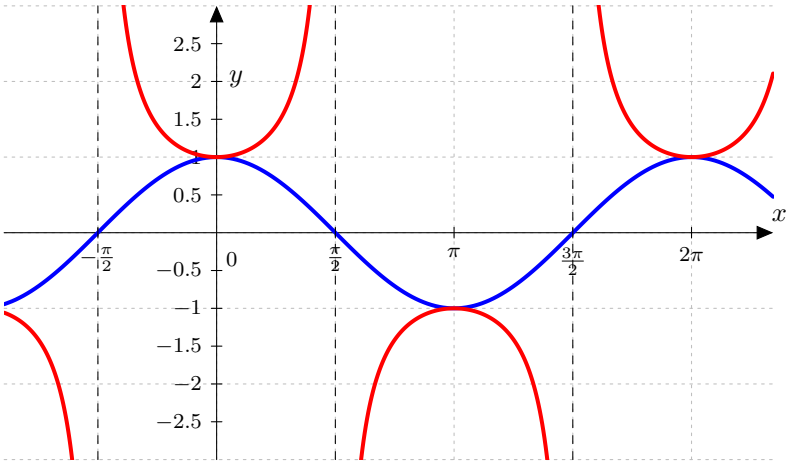
14.4 Cosecant, secant and cotangent - reciprocal circular functions

Questions 14.3.

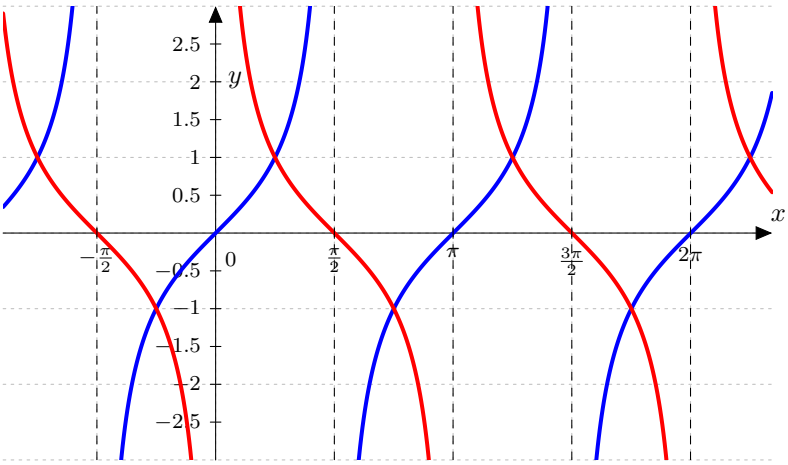
Q18. (1) $x \longmapsto \sin x$ and $x \longmapsto \csc x$



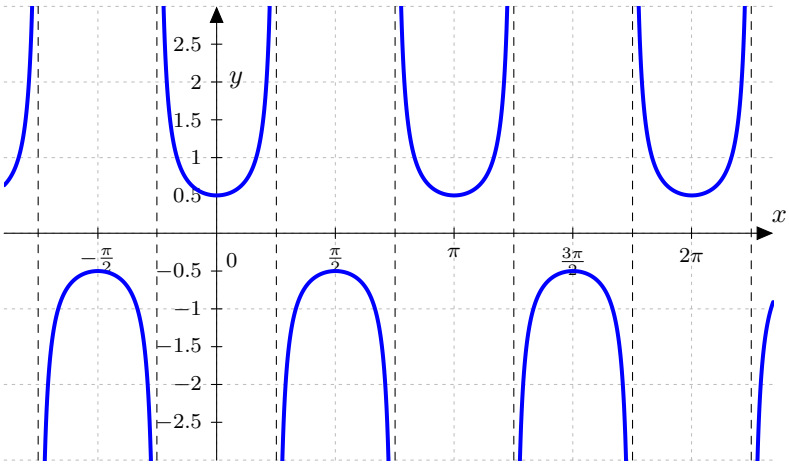
(2) $x \longmapsto \cos x$ and $x \longmapsto \sec x$



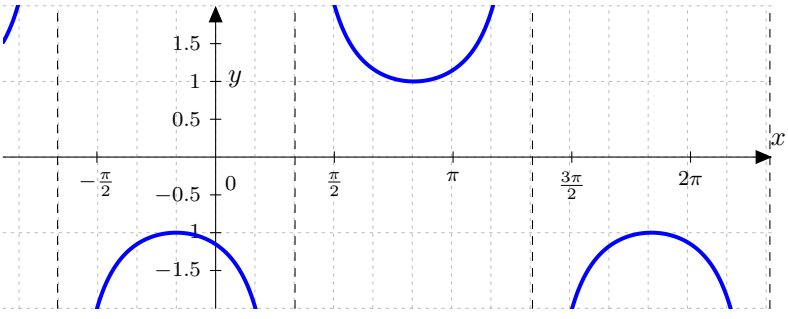
(3) $x \longmapsto \tan x$ and $x \longmapsto \cot x$



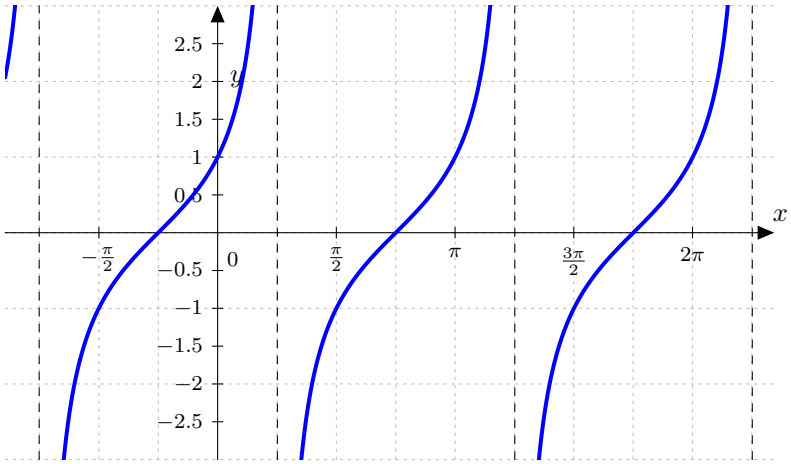
Q19. (1)



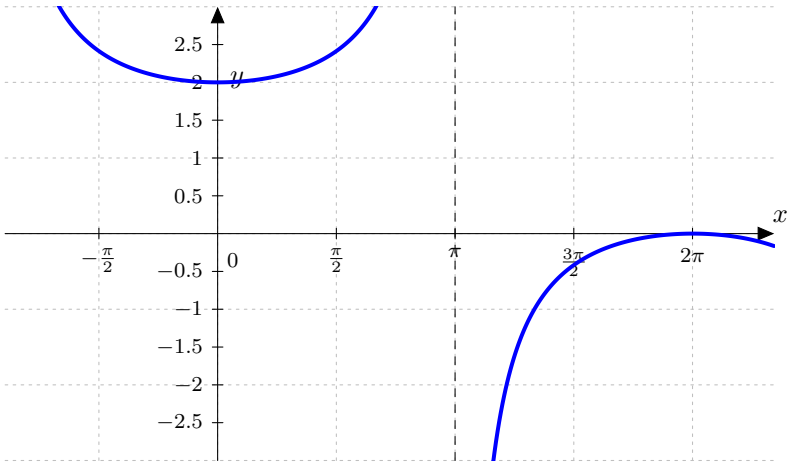
(2)



(3)



(4)



Q20. (1) $-\frac{\pi}{4}$ (2) $-\frac{2\pi}{3}$ (3) $-\frac{\pi}{6}$

Q21. (1) $x \in \{\frac{\pi}{3}, \frac{5\pi}{3}\}$
(2) $x \in \{\frac{\pi}{3}, \frac{2\pi}{3}\}$
(3) $x \in \{\frac{\pi}{6}, \frac{7\pi}{6}\}$
(4) $x = \frac{4\pi}{3}$
(5) $x = \frac{3\pi}{2}$
(6) $x \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$

Q22. **hint**: use Pythagoras' theorem to show the first identity and then divide its both sides by a square of sine or cosine.

Q23.

	$\csc x$	$\sec x$	$\cot x$
1	given	$-3/(2\sqrt{2})$	$-2\sqrt{2}$
2	–	–	–
3	$\sqrt{10}$	$-\frac{\sqrt{10}}{3}$	given
4	–	–	–
5	$-\frac{13}{12}$	given	$\frac{5}{12}$
6	given	$\frac{5}{\sqrt{21}}$	$-\frac{\sqrt{21}}{2}$

14.5 Inverse circular functions

Questions 14.4.

Q24.

function	domain	range
$x \mapsto \arcsin x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$x \mapsto \arccos x$	$[-1, 1]$	$[0, \pi]$
$x \mapsto \arctan x$	$\mathbb{R}$	$]-\frac{\pi}{2}, \frac{\pi}{2}[$

Q25.

	$\sin A$	$\cos A$	$\tan A$
(1)	$\frac{\sqrt{7}}{3}$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{14}}{2}$
(2)	$\frac{3}{4}$	$-\frac{\sqrt{7}}{4}$	$-\frac{3}{\sqrt{7}}$
(3)	$\frac{2}{3}$	$\frac{\sqrt{5}}{3}$	$\frac{2}{\sqrt{5}}$
(4)	$-\frac{3}{5}$	$\frac{4}{5}$	$-\frac{3}{4}$
(5)	$\frac{3}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	3
(6)	$-\frac{2}{\sqrt{13}}$	$\frac{3}{\sqrt{13}}$	$-\frac{2}{3}$

Q26. (All answers given to 3 s.f.)

	$A \in$	$s = \sin A$	$c = \cos A$	$t = \tan A$	$\arcsin s$	$\arccos c$	$\arctan t$	A
(1)	$[\frac{\pi}{2}, \pi]$	0.768	−0.64	−1.2	0.876	2.27	−0.876	2.27
(2)	$[\frac{\pi}{2}, \pi]$	0.908	−0.42	−2.16	1.14	2	−1.14	2
(3)	$[-\pi, -\frac{\pi}{2}]$	−0.253	−0.967	0.262	−0.256	2.89	0.256	−2.89
(4)	$[-\frac{\pi}{2}, 0]$	−0.419	0.908	−0.461	−0.432	0.432	−0.432	−0.432
(5)	$[-\pi, -\frac{\pi}{2}]$	−0.954	−0.298	3.2	−1.27	1.87	1.268	−1.87
(6)	$[\frac{\pi}{2}, \pi]$	0.196	−0.981	−0.2	0.197	2.94	−0.197	2.94
(7)	$[-\pi, -\frac{\pi}{2}]$	−0.029	−1	0.029	−0.029	3.11	0.029	−3.11

Q27. (1) $\sqrt{1-a^2}$ (2) $\frac{a}{\sqrt{1+a^2}}$ (3) $\sqrt{1-a^2}$ (4) $\frac{1}{\sqrt{1+a^2}}$ (5) $\frac{a}{\sqrt{1-a^2}}$ (6) $\frac{\sqrt{1-a^2}}{a}$ (7) $\frac{a}{\sqrt{1-a^2}}$ (8) $\frac{1}{a}$ (9) $\frac{\sqrt{1-a^2}}{a}$ (10) $\frac{1}{a}$ (11) $\sqrt{1+a^2}$ (12) $\frac{1}{\sqrt{1-a^2}}$ (13) $\frac{1}{\sqrt{1-a^2}}$ (14) $\frac{\sqrt{1+a^2}}{a}$ (15) $\frac{1}{a}$

Q28. (1) $-\frac{1}{2} \le x <\le frac12; -\frac{\pi}{2} \le y \le \frac{\pi}{2}$ (2) $-2 \le x \le 2; -3\pi \le y \le 0$ (3) $x \ge 0; 0 \le y < \sqrt{\frac{\pi}{2}}$ (4) $x \le 1, 0 \le y < \frac{\sqrt{3\pi}}{2}$ (5) $-1 \le x \le 1, x \ne 0; y \le -\frac{2}{\pi}$ or $y \ge \frac{2}{\pi}$ (6) $-1 \le x < 1; y \ge \frac{2}{\pi}$ (7) $-1 \le x \le \frac{1}{2}; 0 \le y \le \sqrt{\frac{2\pi}{3}}$ (8) $\frac{\sqrt{2}}{2} \le x \le 1; 0 \le y \le \sqrt{\frac{\pi}{4}}$ (9) $-1 \le x \le \frac{\sqrt{2}}{2}; 0 \le y \le \sqrt{\frac{3\pi}{4}}$

CHAPTER 15

TRIGONOMETRIC IDENTITIES

QUESTIONS 15.1.

Q1. $\sin(\frac{\pi}{8}) = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\cos(\frac{5\pi}{12}) = \frac{\sqrt{6}-\sqrt{2}}{4}$, $\tan(\frac{\pi}{12}) = 2 - \sqrt{3}$.

Q2. $\frac{\sqrt{3}-2\sqrt{2}}{6}$ **Q3.** $-\frac{\sqrt{15}+2}{6}$ **Q4.** -7 **Q5.** — **Q6.** — **Q7.** — **Q8.** —

Q9. (1) 1 (2) 1 (3) $\frac{1}{5}$ (4) $\frac{1}{2}$ (5) $\frac{1}{50}$ (6) $\frac{19}{9}$ (7) $\frac{1}{3}$ (8) 1 (9) $\frac{7}{40}$ (10) $\frac{1}{2}$

Q10. (i) $R = \sqrt{5}, \alpha \approx 0.464$ (ii) min $= -2\sqrt{5}$, max $= 4\sqrt{5}$ (iii) min $= \frac{2}{3+\sqrt{5}}$, max $= \frac{2}{3-\sqrt{5}}$

CHAPTER 16

TRIGONOMETRIC EQUATIONS AND INEQUALITIES

QUESTIONS 16.1.

Q1. (1) $x \in \{\frac{\pi}{6}, \frac{5\pi}{6}\}$
(2) $x \in \{-\frac{11\pi}{6}, -\frac{7\pi}{6}\}$
(3) $x \in \{-\frac{5\pi}{6}, -\frac{\pi}{6}\}$
(4) $x \in \{\frac{7\pi}{6}, \frac{11\pi}{6}\}$
(5) $x \in \{-\frac{\pi}{2}, \frac{3\pi}{2}\}$
(6) $x \in \{0.412, 2.73\}$
(7) $x \in \{3.55, 5.87\}$
(8) $x \in \{-5.87, -3.55\}$
(9) $x \in \{-2.73, -0.412\}$
(10) $x \in \{-\frac{\pi}{2}, \frac{3\pi}{2}\}$

Q2. (1) $x \in \{\frac{\pi}{3}, \frac{5\pi}{3}\}$
(2) $x \in \{-\frac{5\pi}{3}, -\frac{\pi}{3}\}$
(3) $x \in \{-\frac{4\pi}{3}, -\frac{2\pi}{3}\}$
(4) $x \in \{\frac{2\pi}{3}, \frac{4\pi}{3}\}$
(5) $x \in \{-3\pi, -\pi, \pi\}$
(6) $x \in \{1.16, 5.12\}$
(7) $x \in \{1.98, 4.30\}$
(8) $x \in \{-5.12, -1.16\}$
(9) $x \in \{-4.30, -1.98\}$
(10) $x \in \{-\pi, \pi\}$

Q3. (1) $x = \pm\frac{2\pi}{3}$ (2) $x \in \{-1.89, 1.25\}$ (3) $x \in \{-2.82, 0.322\}$

Q4. (1) $x \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\}$
(2) $x \in \{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\}$
(3) $x \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, 2.03, 5.18\}$
(4) $x = \frac{\pi}{2}$
(5) $x \in \{60, 180, 300\}$
(6) $x \in \{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\}$
(7) $x \in \{54.7, 125, 235, 305\}$
(8) $x \in \{0, 2\pi\}$
(9) $x \in \{\frac{7\pi}{6}, \frac{11\pi}{6}\}$
(10) $x = \frac{\pi}{2}$
(11) $x \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$
(12) $x \in \{1.19, \frac{3\pi}{4}, 4.33, \frac{7\pi}{4}\}$
(13) $x \in \{0.841, \frac{2\pi}{3}, \frac{4\pi}{3}, 5.44\}$
(14) $x \in \{\frac{\pi}{4}, 2.55, \frac{5\pi}{4}, 5.70\}$
(15) $x \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$
(16) $x \in \{0, 120, 240, 360\}$
(17) $x \in \{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\}$
(18) $x \in \{0, 120, 240, 360\}$
(19) $x \in \{1.16, 1.91, 4.37, 5.12\}$
(20) $x \in \{1.25, 2.68, 4.39, 5.82\}$
(21) $x \in \{\frac{\pi}{3}, \frac{5\pi}{3}, 1.95, 5.09\}$
(22) $x \in \{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}\}$
(23) $x \in \{\frac{\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{3}, \frac{7\pi}{4}\}$

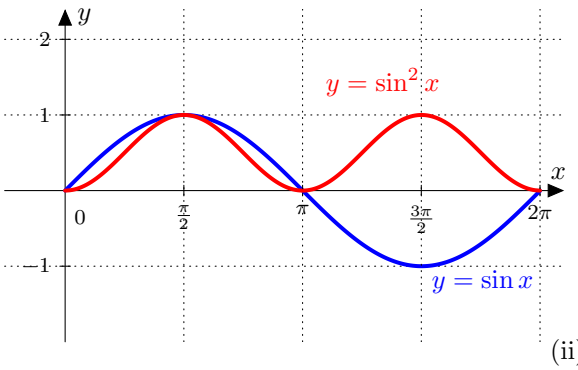
Q5. (1) $x \in \{\frac{\pi}{4}, \frac{7\pi}{4}\}$
(2) $x = \frac{\pi}{3}$
(3) $x = \frac{4\pi}{3}$
(4) $x = \frac{3\pi}{4}$
(5) $x = \frac{k\pi}{18}, k \in \{1, 5, 13, 17, 25, 29\}$
(6) $x = \frac{5\pi}{6}$
(7) $x \in \{0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi\}$
(8) $x \in \{\pi, \frac{5\pi}{3}\}$
(9) $x = \frac{4\pi}{3}$

Q6. (1) $x \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\}$
(2) $x = 30k, k \in \{0, 1, 3, 5, 6, 7, 9, 11, 12\}$
(3) $x \in \{0, \pi, 2\pi\}$
(4) $x \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$
(5) $x = \frac{4\pi}{3}$
(6) $x \in \{0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{3\pi}{2}, 2\pi\}$
(7) $x \in \{0, \pi, 2\pi\}$
(8) $x \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$
(9) $x \in \{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}\}$
(10) $x \in \{0, \pi, 2\pi\}$
(11) $x \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$
(12) $x = k\frac{\pi}{2}, k \in \{0, 1, 2, 3, 4\}$
(13) $x = 120k - 100$ or $120k - 20, k \in \{1, 2, 3\}$ or $x = 60k, k \in \{0, 1, 2, 3, 4, 5, 6\}$
(14) $x \in \{0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi\}$

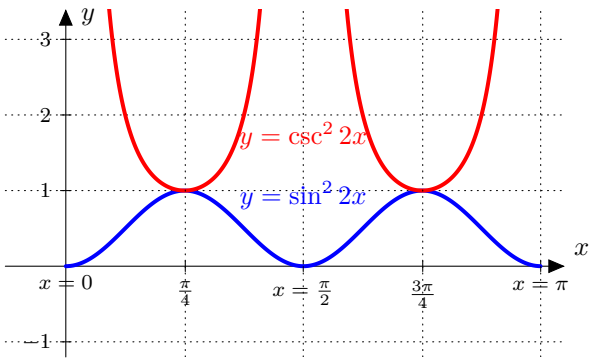
Q7. (1) $x \in]30, 60[\cup]210, 240[$
(2) $x \in [0, 2\pi]$
(3) $\frac{\pi}{6} < x < \frac{5\pi}{6}$
(4) $x \in [0, \frac{\pi}{4}] \cup]\frac{\pi}{2}, \frac{5\pi}{4}] \cup]\frac{3\pi}{2}, 2\pi]$
(5) $x \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$
(6) $x \in [0, \frac{\pi}{3}] \cup [\frac{2\pi}{3}, \frac{4\pi}{3}] \cup [\frac{5\pi}{3}, 2\pi]$
(7) $x \in [0, 2\pi]$
(8) $\frac{2\pi}{3} < x < \frac{4\pi}{3}$
(9) $x \in [0, 0.927] \cup [5.36, 2\pi]$
(10) $x \in [240, 300]$

Q8. (i) $A = B + 2\pi k, k \in \mathbb{Z}$ or $A = \pi - B + 2\pi k, k \in \mathbb{Z}$
(ii) $x \in \{\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}\}$
(iii) hint: multiply both sides by $\frac{\sqrt{2}}{2}$
$x \in \{0, \frac{\pi}{8}, \frac{5\pi}{8}, \pi, \frac{9\pi}{8}, \frac{13\pi}{8}, 2\pi\}$

Q9. (i)



(ii)



(iii) $\frac{\pi}{4}$.
(v) $(\frac{\pi}{8}, 4), (\frac{3\pi}{8}, 4)$

CHAPTER 17

Arcs, sectors, sine rule, cosine rule

Questions 17.1.

Q1. 119°

Q2. 60°

Q3. 70.5°

Q4. $A \approx 64.9°, B \approx 47.1°$

Q5. no such triangle

Q6.

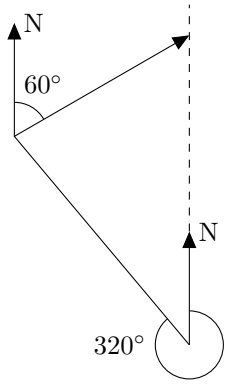
	A	B	C	a	b	c
(1)	48°	42.4°	89.6°	$7.06cm$	$6.4cm$	$9.5cm$
(2)	48°	65°	67°	$7.67cm$	$9.35cm$	$9.5cm$
(3)	45.4°	50.5°	84°	$4.8cm$	$5.2cm$	$6.7cm$
(4)	41.1°	56°	82.9°	$6.5cm$	$8.2cm$	$9.82cm$
(5)	40°	60.4°	79.5°	$3.4cm$	$4.6cm$	$5.2cm$
(6)	51.1°	31°	97.9°	$20.4cm$	$13.5cm$	$26cm$
(7)	54°	47.7°	78.3°	$7.44cm$	$6.8cm$	$9cm$
(8)	54°	61°	65°	$7.94cm$	$8.59cm$	$8.9cm$
(9)	45.4°	50.7°	83.9°	$5.8cm$	$6.3cm$	$8.1cm$
(10)	42.6°	58°	79.4°	$7.5cm$	$9.4cm$	$10.9cm$

Q7. $15.4cm^2$

Q8. $10.6cm^2$

Q9. (a) $1.27cm^2$
(b) $3.67cm$
(c) $9.44cm$

Q10. a) $178km$
b) $273km$



c)

Q11. (i) $4.49km$
(ii) 290°
(iii) 110°

Q12. (i) $5.44km$
(ii) 288°
(iii) 108°

Q13. a) $AB = 4\sqrt{2}cm$
b) $ASB = 0.964(55.2°)$
c) $7.22cm^2$

Q14. $2.63cm^2$

Q15. $B \approx 59.8°$, $C \approx 52.5°$

Q16. $268cm^2$

Q17. 61.7°

Q18. $2.92km$

Q19. (a) 75°
(b) $97.9cm^2$
(c) $7.96cm$
(d) $36.9cm$

Q20. (a) $15\sqrt{2}m$
(b) $5\sqrt{17}m$
(c) 76.0°
(d) 93.4°

17.1 Miscellaneous problems

Q21. 81.9°

Q22. 46.9°

Q23. (i) $\sqrt{2}+1$
(ii) $\frac{1}{2}(2+\sqrt{2}+\sqrt{6}2\sqrt{3})$

Q24. —

Q25. —

Q26. $234m$

Q27. 0.9

PART V

DIFFERENTIAL CALCULUS

CHAPTER 18

EVALUATING DERIVATIVES

18.1 DEFINITION OF A DERIVATIVE

QUESTIONS 18.1.

Q1. —

Q2. (1) $\frac{dy}{dx} = 2x$
(2) $\frac{dy}{dx} = 3x^2$
(3) $\frac{dy}{dx} = 4x^3$
(4) $\frac{dy}{dx} = 6x + 2$
(5) $\frac{dy}{dx} = 6x^2$
(6) $\frac{dy}{dx} = -\frac{1}{x^2}$
(7) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$
(8) $\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$
(9) $\frac{dy}{dx} = \frac{\sqrt{2}}{2\sqrt{x}}$
(10) $\frac{dy}{dx} = -\frac{2}{x^3}$
(11) $\frac{dy}{dx} = 1$
(12) $\frac{dy}{dx} = 0$

Q3. —

18.2 POWER RULE

QUESTIONS 18.2.

Q4. (1) $28x^6 - 12x^3 + 5$
(2) $15x^4 - 6x^2 + 6$
(3) $10x^4 - 15x^2 + 8x + 3$
(4) $-\frac{2}{x^2}$
(5) $\frac{3}{x^3}$
(6) $\frac{4}{x^4}$
(7) $16x^3 - 8x$
(8) $8x^3 - 18x^2 - 2x + 5$
(9) $-9x^{-4} + 6x^3 + 6x^{-5}$
(10) $-\frac{\pi}{2x\sqrt{x}}$
(11) $-\frac{1}{x\sqrt{x}}$
(12) $\frac{3}{x^2} + 3$
(13) $-\frac{1}{2x^2} - \frac{3}{2x^4}$
(14) $-\frac{3\sqrt{2}}{4x^2\sqrt{x}} + \frac{1}{x^2}$
(15) $-\frac{\sqrt{3}}{6x\sqrt{x}} - \frac{2}{x^2}$
(16) $-\frac{5}{4}x^{-\frac{11}{6}} = -\frac{5}{4x\sqrt[6]{x^5}}$
(17) $-\frac{\sqrt{3}}{12}x^{-\frac{7}{6}} - \frac{\sqrt{3x}}{4}$
(18) $-\frac{3\sqrt{2}}{4x\sqrt{x}} + \frac{21}{2x^4\sqrt{x}} - \frac{2}{3x\sqrt[3]{2x}}$

18.3 DERIVATIVES OF ELEMENTARY FUNCTIONS

QUESTIONS 18.3.

Q5. (1) $\frac{dy}{dx} = \sec^2 x$
(2) $\frac{dy}{dx} = -\csc^2 x$
(3) $\frac{\sin x}{\cos^2 x} = \tan x \sec x$
(4) $\frac{\cos x}{\sin^2 x} = -\cot x \csc x$

Q6. (1) $6x\cos x - (3x^2 - 5)\sin x$
(2) $-2\sin x$
(3) $\frac{-4}{(x+1)^3}$
(4) $\frac{\tan x - x\sec^2 x}{\tan^2 x} = \cot x - x\csc^2 x$
(5) $\frac{-2x^2-4}{(x-2)^2(x+1)^2}$
(6) $6x^2t^4 - 2t^3x$
(7) $(10x^4 - 40x^3)\cot x - (2x^5 - 10x^4)\csc^2 x$
(8) $\frac{-2\sin x\tan x - 2\cos x\sec^2 x}{\tan^2 x} = \frac{-2\sin^2 x - 2}{\cos x\tan^2 x}$
(9) $\frac{x^3+3x^2+3x-6}{(x+1)^3} = 1 - \frac{7}{(x+1)^3}$

Q7. (1) $(x^2 + 2x)e^x$
(2) $x^3\cos x + 3x^2\sin x$
(3) $\cos 2x$
(4) $-6x\cos x - (1 - 3x^2)\sin x$
(5) $e^x(\sin x + \cos x)$
(6) $1 + \ln x$
(7) $e^x(\ln x + \frac{1}{x})$
(8) $2\cos 2x$
(9) $-2\sin 2x$
(10) $3\cos 3x$

Q8. $p'(5) = 11$, $q'(5) = -\frac{5}{9}$

18.4 CHAIN RULE

QUESTIONS 18.4.

Q9. (1) $6\sin^2 2x\cos 2x$

(2) $6\cos 3xe^{2\sin 3x}$

(3) $(6x^2-1)(\ln 3)3^{2x^3-x+1}$

(4) $\frac{2\cos x}{\ln 2\sin x}$

(5) $-\frac{(6x^2+1)\sin\ln(2x^3+x-3)}{2x^3+x-3}$

(6) $10(6x+2)(3x^2+2x-1)^9$

(7) $4\sin^3 x\cos^5 x-4\sin^5 x\cos^3 x$

(8) $\frac{\tan^2 x-2x\tan x\sec^2 x}{\tan^4 x}$

(9) $24\pi(2\sin^3\pi x-2)^3\sin^2\pi x\cos\pi x$

(10) $-\frac{2}{5}(1+4x-3x^4)^{-\frac{6}{5}}(4-12x^3)$

(11) $6\sin 3x\cos 3x$

(12) $-2\sin 2xe^{\cos 2x}$

(13) $2x(\ln 3)3^{x^2+1}$

(14) $\frac{2x+2}{(\ln 2)(x^2+2x+3)}$

(15) $\frac{3\cos\ln(2x^3)}{x}$

(16) $30(2x+1)^{14}$

(17) $3\sin^2 x\cos^6 x-5\sin^4 x\cos^4 x$

(18) $36(2\sin^2 3x+1)^2\sin 3x\cos 3x$

(19) $\frac{10}{3}x^4(1-2x^5)^{-\frac{4}{3}}$

(20) $\frac{e^{3x^2+1}(6x-\frac{1}{2\sqrt{x}})}{x}=\frac{e^{3x^2+1}(12x\sqrt{x}-1)}{2x\sqrt{x}}$

(21) $4\left(\frac{2x^2+x-1}{x+2}\right)^3\frac{2x^2+8x+3}{(x+2)^2}$

(22) $\frac{6}{e^{2x}}$

(23) $e^{2x}(2\sin x+\cos x)$

(24) $2\sec^2 x\tan x$

(25) $10x(x+1)(2x-3)^2$

(26) $36x\sin(3x)^2\cos(3x)^2$

(27) $2e^x\cos(2e^x)$

(28) $(\ln\pi)(\pi^{\cos^2 x+1})(-2\cos x\sin x)$

(29) $\frac{(6x-1)(1-\ln(3x^2-x))}{(3x^2-x)^2}$

(30) $\frac{\cos x}{\ln 2\sin x}=\frac{\cot x}{\ln 2}$

(31) 0

(32) $30(\sin 2x)^{14}\cos 2x$

(33) $3\tan^2 x\sec^2 x\cos^5 x-5\tan^3 x\cos^4 x\sin x$
$=3\sin^2 x\cos x-5\sin^4 x\cos x$

(34) $\frac{d}{dx}\frac{x}{\sin x}=\frac{\sin x-x\cos x}{\sin^2 x}$

(35) $3(2e^6x+e^x-1)^2(12e^6x+e^x)$

(36) $\frac{5}{3\sqrt{x}}(1-2\sqrt{x})^{-\frac{8}{3}}$

(37) $2xe^{2x}(x+1)$

(38) $4(\frac{e^x}{\sqrt{x}+1})^3\frac{e^x(\sqrt{x}+1-\frac{1}{2\sqrt{x}})}{(\sqrt{x}+1)^2}=\frac{2e^4x(2x+2\sqrt{x}-1)}{(\sqrt{x}+1)^5\sqrt{x}}$

(39) $\frac{e^{2\ln(4x-2)}}{2x-1}$

(40) $site^2n$

(41) $(12x^3-2)(\ln(3x^4-2x-7)+1)$

(42) $(6x^2+1)(\ln(2x^3+x+1)+1)$

Q10. $h'(1)=-6$, $h'(2)=-2$

Q11. $2\sqrt{3}-\frac{4\pi}{3},-\frac{8\pi}{3}$

Q12. $y=-\frac{1}{24}x+\frac{13}{12}$ or
$y=\frac{1}{24}x+\frac{13}{12}$

18.5 Derivatives of inverse functions

Questions 18.5.

Q13. –

Q14. –

Q15. $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$

Q16. $\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$

Q17. (i) $\frac{1}{9e}$

(ii) $\left(\frac{1}{2}e^{-2.5},\frac{1}{2}\right)$

Q18. (i) —

(ii) $-\frac{1}{2}$

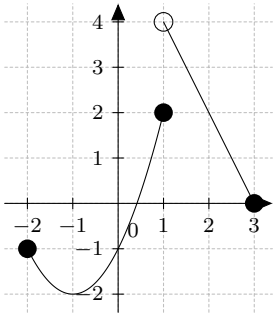
(iii) $(-2\pi,2\pi)$, $(0,0)$, $(2\pi,-2\pi)$

Chapter 19

Piecewise functions

Questions 19.1.

Q1. (i)



(ii) $f(1)=2\neq\lim_{x\to 1^+}f(x)=4$

(iii) $\begin{cases}x^2+2x-1 & \text{for } x\in[-2,1]\\ -2x+4 & \text{for } x\in]1,3]\end{cases}$

(iv) no, it is not continuous at $x=1$, $f'(1)=4\neq\lim_{x\to 1^+}f'(x)=-2$

(v) $y=4x-2$

Q2. (i) $b = 6, c = -3$
(ii) $a = -2, c = -4$
(iii) $a = 2, b = 0$

Q3. (i) —
(ii) $b = 8$, no
(iii) $b = 6, c = -7$

Q4. $b = -12, m = -\frac{3}{2}$

Q5. e.g. $f(x) = \begin{cases} x^2 + 4x + 4 & \text{for } x \in A \\ -x^2 + 2 & \text{for } x \in B \\ x^2 - 4x + 4 & \text{for } x \in C \end{cases}$

CHAPTER 20

LINES TANGENT AND NORMAL TO CURVES

QUESTIONS 20.1.

Q1. (1) $y = 26x - 37$
(2) $y = -\frac{1}{2}x + \frac{1}{2}$
(3) $y = 3ex - 2e$
(4) $y = x$
(5) $y = x$
(6) $y = \frac{3}{2}x + \frac{3}{2} - \frac{\pi}{4}$
(7) $y = -\frac{6}{25}x + \frac{6}{25}\ln 2 + \frac{2}{5}$
(8) $y = 3x - 4$
(9) $y = 2x + 1$

Q2. (1) $y = -\frac{1}{26}x + \frac{196}{13}$
(2) $y = 2x + 3$
(3) $y = -\frac{1}{3e}x + e - \frac{1}{3}e$
(4) $y = -x + \pi$
(5) $y = -x + 2$
(6) $y = -\frac{2}{3}x - \frac{2}{3} + \frac{\pi}{4}$
(7) $y = \frac{25}{6}x - \frac{25}{6}\ln 2 + \frac{2}{5}$
(8) $y = -\frac{1}{3}x + \frac{28}{3}$
(9) $y = -\frac{1}{2}x + 1$

Q3. (a) $y = -4x - 8$ (b) $(-2, 0)$ (c) $y = \frac{1}{4}x - \frac{15}{4}$

Q4. $y = 2x - 3, (2, 1), (3.5, 4)$

Q5. $y = -\frac{1}{2}x + \frac{\pi}{12} + \sqrt{3}$,
$(\frac{\pi}{6}, \sqrt{3}), (1.18, 1.40)$

Q6. Normal at $(0, 0)$: $y = x$
$\frac{dy}{dx} = 0$ at $(\pm 1, \mp\frac{1}{2})$,
$\therefore y = \mp\frac{1}{2}$ respectively
Intersections: $(-\frac{1}{2}, -\frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$

Q7. Normal at $(0, 0)$: $y = 2x$
$\frac{dy}{dx} = 0$ at $(-2, \frac{1}{2}), (2, -\frac{1}{2})$,
$\therefore y = \pm\frac{1}{2}$ respectively
Intersections: $(-\frac{1}{4}, -\frac{1}{2})$ and $(\frac{1}{4}, \frac{1}{2})$.

Q8. $At(2, \sqrt{2}) : y = \frac{\sqrt{2}}{4}x + \frac{\sqrt{2}}{2}$

Q9. $(1, 0), y = 2x - 2$

Q10. $A = B = \pi$

Q11. $A = \frac{1}{3}, y = \frac{1}{3}x + 1$

Q12. $A = 2, B = 4$

Q13. $y = -4x, (-\frac{1}{2}, 2)$

Q14. $c = 4, (1, \frac{1}{2})$

Q15. $(4, -2), y = -0.5x; (3.5, -2.75), y = -x + 0.75$

Q16. $A = \frac{1}{2}, y = -2x + 1$

Q17. $(4, 2), y = -4x + 18$

Q18. $A = \pi, B = -\frac{\pi}{4}$

Q19. $A = 2, (\frac{1}{2}, 0)$

Q20. $y = 4x - 2, (1, 2), (-\frac{9}{8}, -\frac{13}{2})$;
$y = \frac{1}{4}x - \frac{1}{8}, (-\frac{3}{2}, -\frac{1}{2}), (6.78, 1.57)$

Q21. $c = 3, (-1, 1), (4, \frac{8}{3})$

Q22. $a = -4, b = 18$

Q23. $y = ex$

Q24. $y = -\frac{3}{4}x + 1$

CHAPTER 21

FURTHER DERIVATIVES. USE OF DERIVATIVES IN SKETCHING GRAPHS

QUESTIONS 21.1.

Q1. (1) $-\frac{1}{4}e^{-0.5x}(7\cos 2x + 23\sin 2x)$

(2) $(12x+2)\ln(3x+1) + 6x + \frac{3(12x^3+9x^2+2x+9)}{(3x+1)^2}$

(3) $\frac{2(7x^3+6x^2-60x+56)}{(x^2-3x+2)^3}$

Q2. (1) $3^n e^{3x}$

(2) $(-1)^{\lfloor \frac{2n+3}{4} \rfloor} a^n \left((\frac{1+(-1)^{n+1}}{2})\sin(ax+b) + (\frac{1+(-1)^n}{2})\cos(ax+b) \right)$

(3) $\frac{1}{a}(-1)^n(n+1)!x^{-n-2}$

Q3. (i) $\frac{dy}{dx} = e^{-\frac{1}{4}x^2+x}(-\frac{1}{2}x+1)$, $\frac{dy}{dx} = 0$ for $x = 2$ only

(ii) $\frac{dy}{dx} > 0$ for $x < 2$

(iii) $\frac{dy}{dx} < 0$ for $x > 2$

(iv) $\frac{d^2y}{dx^2} = \frac{1}{4}e^{-\frac{1}{4}x^2+x}(x^2-4x+2) = -\frac{1}{2}e$ for $x = 2$

(v) $\frac{d^2y}{dx^2} > 0$ for $x < 2-\sqrt{2}$ or $x > 2+\sqrt{2}$,
$\frac{d^2y}{dx^2} < 0$ for $2-\sqrt{2} < x < 2+\sqrt{2}$

Q4.

		(1)	(2)	(3)	(4)
	$\frac{dy}{dx}$	$3x^2-3$	$e^x(x+1)$	$\ln x + 1$	$\frac{2x}{(x^2+1)^2}$
	$\frac{d^2y}{dx^2}$	$6x$	$e^x(x+2)$	$\frac{1}{x}$	$\frac{2-6x^2}{(x^2+1)^3}$
(i)		$0, \pm\sqrt{3}$	0	1	0
(ii)	$y > 0$	$-\sqrt{3} < x < 0, x > \sqrt{3}$	$x > 0$	$x > 1$	$x \neq 0$
	$y < 0$	$x < -\sqrt{3}, 0 < x < \sqrt{3}$	$x < 0$	$0 < x < 1$	none
(iii)		± 1	-1	$\frac{1}{e}$	0
(iv)	$\frac{dy}{dx} > 0$	$x < -1, x > 1$	$x > -1$	$x > \frac{1}{e}$	$x > 0$
	$\frac{dy}{dx} < 0$	$-1 < x < 1$	$x < -1$	$0 < x < \frac{1}{e}$	$x < 0$
(v)		0	-2	none	$\pm\frac{1}{\sqrt{3}}$
(vi)	$\frac{d^2y}{dx^2} > 0$	$x > 0$	$x > -2$	$x > 0$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$
	$\frac{d^2y}{dx^2} < 0$	$x < 0$	$x < -2$	none	$x < -\frac{1}{\sqrt{3}}, x > \frac{1}{\sqrt{3}}$

21.1 Stationary and inflection points

21.2 Kinds of inflection points

Questions 21.2.

Q5. (1) stationary inflection point

(2) minimum

(3) non-stationary inflection point

(4) minimum

(5) maximum

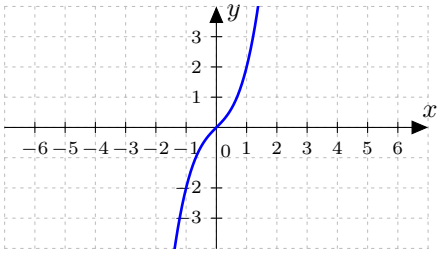
(6) none

(7) non-stationary inflection point

Q6. (i) 0

(ii) $\frac{dy}{dx} = 3x^2 + 1 \neq 0$
$\frac{d^2y}{dx^2} = 6x = 0$ for $x = 0$

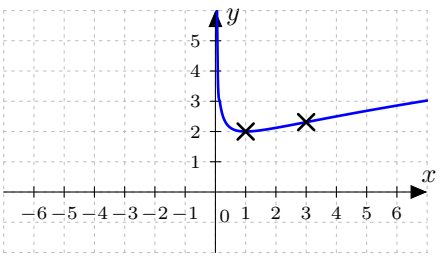
(iii)



Q7. (i) there are no zeroes

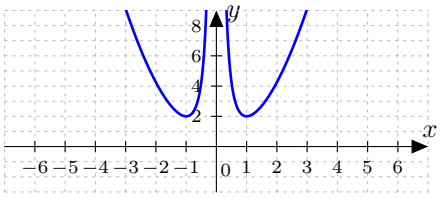
(ii) $\frac{dy}{dx} = \frac{x-1}{2x\sqrt{x}} = 0$ for $x = 1, (1, 2)$
$\frac{d^2y}{dx^2} = \frac{\sqrt{x}(3-x)}{4x^3} = 0$ for $x = 3, (3, \frac{4\sqrt{3}}{3})$

(iii)



Q8. (i) there are no zeroes

(ii) $\frac{dy}{dx} = 2x - \frac{2}{x^3} = 0$
for $x = \pm 1, (1, 2), (-1, 2)$
$\frac{d^2y}{dx^2} = 2 + \frac{6}{x^4} \neq 0$

(iii) 

Q9. $(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e})$

Q10. $(e^{-\frac{1}{3}}, -\frac{1}{3e})$, $\frac{d^2y}{dx^2} > 0$

Q11. *Note that the graphs are not included in the answers to this question*

$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	stationary points	inflection points
(1) $e^{-x}(1-x)$	$e^{-x}(x-2)$	$(1, \frac{1}{e})$ - max	$(2, \frac{2}{e^2})$
(2) $e^{-x}(-x^2+2x)$	$e^{-x}(x^2-4x+2)$	$(0,0)$ - min, $(2, \frac{4}{e^2})$ - max	$(2 \pm \sqrt{2}, (2 \pm 2\sqrt{2})e^{-2\mp\sqrt{2}})$
(3) $-e^{-x}(x^2-4x+3)$	$e^{-x}(x^2-6x+7)$	$(1,0)$ - min, $(3, \frac{4}{e^3})$ - max	$(3 \pm \sqrt{2}, (2 \pm 2\sqrt{2})e^{-3\mp\sqrt{2}})$
(4) $\frac{1-\ln 2x}{x^2}$	$\frac{2\ln 2x-3}{x^3}$	$(\frac{1}{2}e, \frac{2}{e})$ - max	$(\frac{1}{2}e^{1.5}, \frac{3}{e^{1.5}})$
(5) $\frac{1-2\ln x}{x^3}$	$\frac{6\ln x-5}{x^4}$	$(\sqrt{e}, \frac{1}{2e})$ - max	$(e^{\frac{5}{6}}, \frac{5}{6e^{\frac{5}{3}}})$
(6) $\frac{e^x(x-1)}{x^2}$	$\frac{e^x(x^3-2x^2+2x)}{x^4}$	$(1, e)$ - min	none
(7) $\frac{e^x(2x-1)}{2x\sqrt{x}}$	$\frac{e^x\sqrt{x}(4x^2-4x+3)}{4x^3}$	$(\frac{1}{2}, \sqrt{2e})$ - min	none
(8) $27x^2-6x-8$	$54x-6$	$(-\frac{4}{9}, 6\frac{14}{81})$ - max, $(\frac{2}{3}, 0)$ - min	$(\frac{1}{9}, 3\frac{7}{81})$
(9) $12(x^3-x^2-4x+4)$	$12(3x^2-2x-4)$	$(-2,-113), (2,15)$- min,$(1,22)$- max	$(\frac{1-\sqrt{13}}{3}, -56.5)$, $(\frac{1+\sqrt{13}}{3}, 18.3)$
(10) $12(x^3-x^2-16x+16)$	$12(3x^2-2x-16)$	$(-4,-1024), (4,0)$- min, $(1,351)$- max	$(-2,-432), (\frac{8}{3}, 161)$
(11) $4x^3-12x-8$	$12x^2-12$	$(2,0)$ - min	$(-1,27), (1,11)$

Q12. (i) $(-1,2), (2-\sqrt{3}, \frac{1+\sqrt{3}}{2}), (2+\sqrt{3}, \frac{1-\sqrt{3}}{2})$ (ii) At $(-1,2): y = -x+1$

CHAPTER 22

APPLICATIONS OF DIFFERENTIAL CALCULUS

22.1 IMPLICIT DIFFERENTIATION

QUESTIONS 22.1.

Q1. (1) $3x^2 - 2y\frac{dy}{dx} = 4xy + 2x^2\frac{dy}{dx}$

(2) $2x + 4y + 4x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

(3) $\sec^2(x+y)(1+\frac{dy}{dx}) - \sec^2(xy^2)(y^2 + 2xy\frac{dy}{dx}) = 0$

(4) $\frac{2y\frac{dy}{dx}}{1+y^4} = 1 + y + x\frac{dy}{dx}$

(5) $\frac{y+x\frac{dy}{dx}}{\sqrt{xy}(1+4xy)} = \frac{y-\frac{dy}{dx}x}{y^2}$

(6) $\frac{y+x\frac{dy}{dx}}{1+x^2y^2} = 1 + \frac{dy}{dx}$

(7) $\cos(x^2y)(2xy + x^2\frac{dy}{dx}) - 2\cos(xy)\sin(xy)(y + x\frac{dy}{dx}) = 0$

(8) $\frac{y(2xy-x^2\frac{dy}{dx})}{x^2y^2} = 1$

(9) $3e^{xy}(y + x\frac{dy}{dx}) = 0$

(10) $e^{x\cos y}(\cos y - x\sin y\frac{dy}{dx}) = 0$

(11) $\frac{2xy+x^2\frac{dy}{dx}}{x^2y} = 0$

(12) $\cos(x^3y)(3x^2y + x^3\frac{dy}{dx}) - \sin(xy^3)(y^3 + 3xy^2\frac{dy}{dx}) = 0$

(13) $e^{2xy}(2y + 2x\frac{dy}{dx}) = \frac{e^x(e^y - e^y\frac{dy}{dx}+1)}{(e^y+1)^2}$

Q2. $\frac{43}{34}$

Q3. @$(1,-1)$, $\frac{dy}{dx} = \frac{1}{2}$; @$(1,1)$, $\frac{dy}{dx} = -\frac{1}{2}$

Q4. $-\frac{19}{4}$

Q5. (1) $\frac{dy}{dx} = \frac{3x^4y - x^2y^2 - y^2}{x^5}$

(2) $\frac{dy}{dx} = \frac{ye^{xy}}{1 - xe^{xy}}$

Q6. (1) $\frac{dy}{dx} = \frac{y}{1-x}; \frac{d^2y}{dx^2} = \frac{2y}{(1-x)^2}$

(2) $\frac{dy}{dx} = -\frac{y+\sqrt{y}}{x}; \frac{d^2y}{dx^2} = \frac{4y+5\sqrt{y}+1}{2x^2}$

Q7. —

Q8. (i) —

(ii) -3

(iii) 2

(iv) decreasing and concave up

Q9. (i) $-3 \leq x \leq 3, -2 \leq y \leq 2$

(ii) $\frac{dy}{dx} = -\frac{4x}{9y}$

(iii) $\frac{dy}{dx} = -\frac{4x}{9y} = \frac{-4x}{\pm 3\sqrt{36-4x^2}}$

(iv) An ellipse

Q10. (i) $x \geq -1, y \in \mathbb{R}$

(ii) $\frac{dy}{dx} = \frac{1}{2y}$

(iii) $\frac{dy}{dx} = \frac{1}{\pm 2\sqrt{x+1}}$

(iv) parabola symmetrical in x-axis

Q11. $e^{xy^2} = 3x^2y - 6xy^2$
$\therefore e^{xy^2}(y^2 + 2xy\frac{dy}{dx}) =$
$= 6xy + 3x^2\frac{dy}{dx} - 6y^2 - 12xy\frac{dy}{dx}$

Q12. (1) $\pm\frac{4}{5}$

(2) $-\frac{37}{26}$

22.2 Curves defined parametrically

Questions 22.2.

Q13. (i) $-5 \leq x, y \leq 5$

(ii) $\frac{dy}{dt} = 5\cos t, \frac{dx}{dt} = -5\sin t, \frac{dy}{dx} = -\cot t$

(iii) —

(iv) —

(v) $(\frac{5\sqrt{2}}{2}, \pm\frac{5\sqrt{2}}{2}), (-\frac{5\sqrt{2}}{2}, \pm\frac{5\sqrt{2}}{2})$

(vi) circle with centre at origin and radius 5

(vii) $x^2 + y^2 = 25$

Q14. (i) $\frac{dy}{dx} = -\cot t$

(ii) circle radius 1, centre at $(2, -3); (x-2)^2 + (y+3)^2 = 1$

Q15. (i) $\frac{\pi}{3}, \frac{5\pi}{3}$

(ii) $\frac{dy}{dx} = \frac{1}{8\sin^3 t}; P_1(1, \frac{2}{\sqrt{3}}), y = \frac{1}{3\sqrt{3}}x + \frac{5}{3\sqrt{3}}; P_2(1, -\frac{2}{\sqrt{3}}, y = -\frac{1}{3\sqrt{3}}x - \frac{5}{3\sqrt{3}}$

(iii) $x^2 = \frac{4\sqrt{y^2-1}}{y^3}$

Q16. (i) $t = \frac{\pi}{6}, x = \frac{3\sqrt{3}}{2}$ or $t = \frac{5\pi}{6}, x = -\frac{3\sqrt{3}}{2}$

(ii) $\frac{dy}{dx} = -4/3\cot t, y = -\frac{4\sqrt{3}}{3}x + 5$ or $y = \frac{4\sqrt{3}}{3}x + 5$

(iii) Ellipse 6×8, centre $(0, -3)$, $\frac{x^2}{9} + \frac{(y+3)^2}{16} = 1$

Q17. (i) $\pm 2\sqrt{6}$

(ii) $\frac{dy}{dx} = -\frac{\sqrt{2}}{2}\cot t$, normal is $y = \sqrt{6}x$ or $y = -\sqrt{6}x$, so it passes through $(0, 0)$

(iii) Ellipse $16 \times 8\sqrt{2}$, centre $(-2, 0)$, $\frac{(x+2)^2}{64} + \frac{y^2}{32} = 1$

Q18. (i) $-4 \pm \sqrt{2}$

(ii) $\frac{dy}{dx} = \csc t, (-4, 2)$

(iii) Hyperbola; asymptotes $y = \pm(x+4)$; zeroes $-5, -3. y^2 = (x+4)^2 - 1$

Q19. $y = -2x - 6, y = 2x + 8$

Q20. —

Q21. (i) —

(ii) $y = 8x - 7.5$

(iii) $y = \frac{x+1}{3x+1}$

Q22. At $(\pm 4\sqrt{2}, 6)$ the tangent is $y = 6$, at $(\pm 4\sqrt{2}, -6)$ the tangent is $y = -6$.

22.3 Optimisation

Questions 22.3.

Q23. 15×15 **Q25.** $5 \times 20 \times 20$ **Q27.** $(4, 1)$ **Q29.** $(\frac{1}{\sqrt{3}}, -\frac{2}{3})$

Q24. 15×30 **Q26.** $\frac{256}{27}$ **Q28.** $20cm \times 10cm$

22.4 Applying derivatives in kinematics

Questions 22.4.

Q30. (i) 4 seconds
(ii) $-1km$
(iii) $1km$
(iv) $2km$
(v) $0.25\frac{km}{s}$
(vi) $0.25\frac{km}{s}$
(vii) $a = 0$

Q31. (i) $1m$
(ii) $15m$
(iii) $14m$
(iv) (a) $3.5\frac{m}{s}$
(b) $4\frac{m}{s}$
(c) $3.5\frac{m}{s}$
(d) $3\frac{m}{s}$
(e) $2.75\frac{m}{s}$
(f) $2.55\frac{m}{s}$
(g) $2.525\frac{m}{s}$
(h) $2.505\frac{m}{s}$
(v) $2.5\frac{m}{s}$
(vi) constantly $= 1$

Q32. (i) $v = e^{-\frac{t}{2}}(4 - 2t)$
(ii) $t = 2s$
(iii) $8/e \approx 2.94m$
(iv) $\frac{32}{e^4} \approx 0.586m$
(v) $a = e^{-\frac{t}{2}}(t - 4)$

Q33. —

Q34. (i) $1.42 \leq t \leq 2.58$
(ii) $-1ms^{-1}$
(iii) $\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 13.5$

Q35. (i) 4
(ii) (a) 5
(b) 9
(iii) 5
(iv) —
(v) —

Q36. (i) $6m$
(ii) $20m$
(iii) $100\pi mh^{-1}$
(iv) $200mh^{-1}$

22.5 Related rates

Questions 22.5.

Q37. $\frac{3}{121} \approx 0.0248(cms^{-1})$

Q38. $\frac{dz}{dt} = -36.4kmh^{-1}$

Q39. $4\pi \approx 12.6(ms^{-1})$ / $\frac{25}{4}\pi \approx 19.6(ms^{-1})$

Q40. $\frac{dp}{dt} = 100$

Q41. (a) -51.8
(b) -219

Part VI

Integral calculus

Task

(1) $y = x^3 + 2x^2 - 2x + C$

(2) $y = 2e^{2x} + \frac{3}{2}x^2 + C$

(3) $y = \frac{2}{3}x\sqrt{x} + C$

(4) $y = -\frac{3}{x} - \frac{1}{x^2} + C$

(5) $y = 3\ln x + \frac{1}{3}\ln x + C$

Chapter 23

Integrals

23.1 Indefinite integrals

Questions 23.1.

Q1. (all answers should include "$+C$")

(1) x^2
(2) $\frac{x^5}{5}$
(3) $\frac{1}{2}x^4$
(4) $\frac{1}{2}x^6$
(5) $\frac{6}{11}x^{5.5}$
(6) $\frac{6}{7}x^{\frac{7}{3}}$
(7) $\frac{5}{2}x^4 - \frac{5}{3}x^3$
(8) $-\frac{3}{2x^2}$
(9) $-\frac{1}{2x^4}$
(10) $\frac{4}{5}x^2\sqrt{x}$
(11) $-\frac{4}{\sqrt{x}}$
(12) $\frac{2\sqrt{2}}{11}x^5\sqrt{x}$
(13) $\frac{6}{5}x^2\sqrt{x}$
(14) $\frac{1}{2}x^2 + 3x$
(15) $\frac{1}{4}x^4 + \frac{3}{2}x^2 + 3\ln x - \frac{1}{2}x^{-2}$
(16) $\frac{1}{2}x^2 + 6x + 12\ln x - \frac{8}{x}$
(17) $\frac{4}{3}x\sqrt{x} - \frac{3}{2}\ln x$
(18) $\frac{3\times 2^{\frac{11}{6}}}{13}x^{\frac{25}{6}}$
(19) $\frac{1}{8}x^4 + \frac{3}{2}x - \frac{9}{16}x^{-2}$
(20) $-\frac{2}{\sqrt{t}} - 6\sqrt{t}$
(21) $\frac{y^3}{3} - 4y - \frac{4}{y}$
(22) $\frac{2}{5}u^2\sqrt{u} - \frac{4}{3}u\sqrt{u} + 2\sqrt{u}$
(23) $\frac{6}{7}\sqrt[3]{v^7} - 12\sqrt[4]{v}$
(24) $\frac{1}{4}a^4 - \ln a$
(25) $\frac{4}{9}x^4\sqrt{x} - \frac{4}{13}x^6\sqrt{x}$
(26) $\frac{3}{2}x^4 - 2x^3 + \frac{3}{2}x^2 + 4x$
(27) $x^2 - \frac{4}{3}x\sqrt{x} + \frac{1}{2}x$
(28) $-\frac{2}{\sqrt{x}} - \frac{2\sqrt{2}}{3}x\sqrt{x}$
(29) $2\sqrt{3x} - \frac{2}{3\sqrt{3}}x\sqrt{x}$

More basic integrals

Questions 23.2.

Q2. (all answers should include "$+C$")

(1) $\frac{1}{2}\ln x$ or $\frac{1}{2}\ln 2x$
(2) $-\frac{1}{5}\cos 5x$
(3) $\frac{1}{21}(3x-2)^7$
(4) $\frac{1}{12}(2x+5)^6$
(5) $\frac{1}{3}(2x+3)^{\frac{3}{2}}$
(6) $\frac{1}{3}e^{3x}$
(7) $\frac{1}{3}e^{3x+1}$
(8) $-\frac{1}{3}e^{1-3x}$
(9) $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}$
(10) $\frac{1}{\pi}\sin(\pi x)$
(11) $x - \frac{1}{2}\cos 2x$
(12) $\frac{1}{\ln 2}2^x$
(13) $\frac{1}{2\ln 2}2^{2x}$
(14) $\frac{1}{2\ln 2}2^{2x+1}$
(15) $\frac{1}{2}\tan 2x$
(16) $-\frac{1}{2}\cos(2x-\pi)$
(17) $2\sin(\frac{x-\pi}{2})$
(18) $e^x + 4x - 4e^{-x}$
(19) $\frac{2}{\pi}\sin(\frac{\pi x-\pi}{2})$
(20) $\cos(\pi - x)$
(21) $\ln(x+2)$
(22) $\frac{1}{2}\ln(2x+4)$
(23) $-\ln(3-2x)$
(24) $-\frac{1}{2}\tan(\pi - 2x)$
(25) $\frac{1}{2}e^{2x} + \frac{1}{2}x^2$
(26) $-\frac{1}{2}\cos 2x + \frac{1}{3}\tan 3x$
(27) $\frac{1}{2}\ln|2x+1|$
(28) $\frac{1}{2\ln 3}3^{2x}$
(29) $\frac{\sqrt{100x-1}}{75}(50x-2)$
(30) $\frac{1}{4}\sin(4x+2) - e^{-x}$
(31) $\frac{1}{2}(\ln 2)(\ln(2x-1))$
(32) $\frac{1}{3\ln 5}5^{3x+4}$

Integration with use of trigonometric identities

Questions 23.3.

Q3. (1) $\frac{9}{2}x + 4\sin x + \frac{1}{4}\sin 2x$ (2) $\frac{x}{2} - \frac{1}{8}\sin 4x$ (3) $\frac{5}{2}x - \frac{3}{4}\sin 2x + \cos 2x$

Q4. (1) $\frac{1}{10}(5\cos x - \cos 5x) + c$ (2) $-\frac{1}{10}\cos 5x - \frac{1}{6}\cos 3x + c$ (3) $-\frac{1}{10}(5\cos x + \cos 5x) + c$

Q5. (1) $\frac{1}{10}(5\sin x + \sin 5x) + c$
(2) $\frac{1}{30}(5\sin 3x + 3\sin 5x) + c$
(3) $\frac{1}{42}(7\sin 3x + 3\sin 7x) + c$
(4) $\frac{1}{10}(5\sin x - \sin 5x) + c$
(5) $\frac{1}{30}(5\sin 3x - 3\sin 5x) + c$
(6) $\frac{1}{42}(7\sin 3x - 3\sin 7x) + c$

Integrals of inverse trigonometric functions

Questions 23.4.

Q6. (all answers should include "$+C$")

(1) $\frac{1}{2}\arctan 2x$
(2) $\frac{1}{\sqrt{2}}\arctan\sqrt{2}x$
(3) $2\arctan(x/2)$
(4) $\frac{3}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}$
(5) $2\arctan(x-1)$
(6) $\arctan(x+1)$
(7) $\frac{1}{2}\arctan(2x+3)$
(8) $\frac{1}{\sqrt{2}}\arcsin(\sqrt{2}x)$
(9) $\frac{1}{2}\arcsin(\sqrt{2}x)$
(10) $\arcsin(x+1)$
(11) $\arcsin(x-1)$
(12) $\frac{1}{2}\arcsin(2x-1)$

Q7. (all answers should include "$+C$")

(1) $\frac{1}{3}x^3 + 4x\sqrt{x} + 9\ln x$
(2) $\frac{1}{3}(x-2)^{\frac{3}{2}}$
(3) $2\sqrt{2(x-2)}$
(4) $-\frac{1}{3}\cos 3x$
(5) $2\tan 0.5x$
(6) $\frac{1}{12}\sin 6x + \frac{1}{2}x$
(7) $4\arctan x$
(8) $\frac{1}{2\sqrt{2}}\arctan\sqrt{2}x$
(9) $\frac{1}{3\ln 2}2^{3x}$
(10) $\frac{2}{\ln 2}(\sqrt{2})^x$
(11) $\frac{2}{3}\ln(3x-6)$
(12) $\frac{1}{3}e^{3x+2} - 2\sqrt{e^x}$

23.2 Definite integrals

Questions 23.5.

Q1. $y = x^2 + \cos x + 1$

Q2. $y = e^x - x^2 + 2$

Q3. $\frac{2}{3}\sin(3x - \frac{\pi}{2}) + \frac{3x}{\pi}$

Q4. $y = -2\cos(2x - \frac{\pi}{2}) + \frac{16}{\pi}x - 2 + \sqrt{2}$

Q5. $y = -2\sin(\pi x) - 1$

Q6. (1) $\sin x + x\cos x$ (2) $x\sin x + \cos x + C$ (3) -2

Q7. (a) $e^x + xe^x$ (b) $xe^x - e^x + c$ (c) e^2

Q8. (1) 4 (2) 4

Questions 23.6.

Q9. (1) 4
(2) 99
(3) $-\frac{52}{3}$
(4) 18
(5) $\frac{112}{3}$
(6) 96.8
(7) 1
(8) 1
(9) $\frac{3}{8}$
(10) 2
(11) $\frac{7}{6}$
(12) $3 + \ln 2$
(13) $\frac{7}{6}$
(14) $\frac{3}{2} + 2\ln 2$
(15) $\frac{20}{3}$

Questions 23.7.

Q10. (1) 18
(2) -12
(3) -18
(4) 9
(5) -10
(6) $\frac{106}{3}$

Q11. (1) 32
(2) $\frac{14}{3}$
(3) 1
(4) $\frac{1}{2}$
(5) Does not exist
(6) $\frac{1}{2}$
(7) $\frac{1}{2} + \frac{\sqrt{3}}{2}$
(8) $\frac{3}{2}$
(9) $\frac{\pi}{4}$
(10) $\frac{\pi}{8}$
(11) $\frac{\pi}{36}$
(12) $\frac{1}{e}\ln(\frac{3}{2})$
(13) $16 - 2\ln 3$
(14) 63

Q12. $\frac{e^2+3}{2(e^2-1)}$

Q13. (1) $\frac{1}{2}\pi\ln 3$ (2) $155\frac{1}{4}$

Chapter 24

Area and volume

24.1 Area under a curve

Questions 24.1.

Q1. —

Q2. (1) $\frac{64}{3}$ (2) 8 (3) $\frac{937}{12}$

Q3. $\frac{1}{2}(e^2 - 1) \approx 3.19$

Q4. $\frac{7}{3}$

24.2 Area between curves

Questions 24.2.

Q5. $\frac{125}{6}$ **Q6.** $\frac{16}{3}$ **Q7.** $4\sqrt{3}$ **Q8.** 0.201

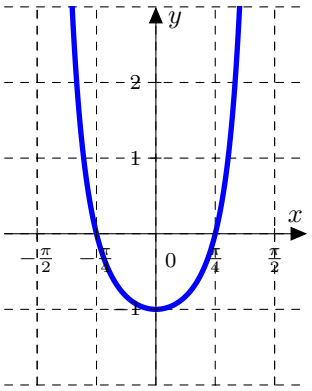
Q9. (i) $f'(x) = -x + 2$
(ii) 2
(iii) $-\frac{1}{2}$
(iv) $y = -\frac{1}{2}x + 1.5$
(v) $(5, -1)$
(vi) $\int_0^5 f(x) + \frac{1}{2}x - 1.5dx =$
$= \int_0^5 (-0.5x^2 + 2.5x)dx$
(vii) $\frac{125}{12} \approx 10.4$

Q10. 1.60

Q11. $\frac{\pi}{2} - \frac{1}{3}$

Q12. (i) $y = \cos 2x$ - lower, $y = \cos^2 x$ - upper
(ii) $A(0,1), B(\frac{\pi}{4}, 0), C(\frac{\pi}{2}, 0)$,
$D(\frac{\pi}{2}, -1), E(\frac{3\pi}{4}, 0), F(\pi, 1)$
(iii) $\int_0^{\pi} (\cos^2 x - \cos 2x)dx$
(iv) $\frac{\pi}{2}$

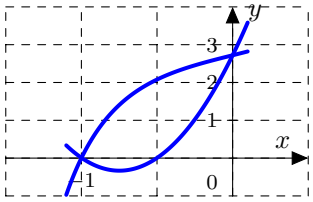
Q13. (i) $k = \frac{\pi}{2}$
(ii)



(iii) $y = \frac{1}{4}x + \frac{\pi}{16}$
(iv) $(0.872, 0.414)$
(v) $\int_{-\frac{\pi}{4}}^{0.872} (\frac{1}{4}x + \frac{\pi}{16} - \sec^2 x + 2)dx$
(vi) 1.4680

Q14. 0.0917

Q15. (i) –



(ii) $g'(x) = e^{x^2}(2x^2 + 1)$;
g is increasing as $g'(x) > 0$
for all x (both factors +ive)
(iii) $g''(x) = e^{x^2}(4x^3 + 6x)$, $Q(0, e)$
(iv)
(v) $\frac{d}{dx}e^{x^2} = 2xe^{x^2}$;
$\int (xe^{x^2} + e)dx =$
$= \frac{1}{2}e^{x^2} + ex + C$
(vi) Area$= \frac{1}{2} + \frac{1}{3}e$

Q16. (i) $y = x - 2$
(ii) $(-2.73, -4.73), (0.732, -1.27)$
(iii) $\int_{-2.73}^{0} (x^3 + 2x^2 - x - 2) - (x - 2)dx + \int_0^{0.732} (x - 2) - (x^3 + 2x^2 - x - 2)dx$
(iv) 7.33

Q17. —

Q18. (a) 2 : 1
(b) n : 1

24.3 Volume of revolution

Questions 24.3.

Q19. $\frac{52}{3}\pi$

Q20. $\frac{16}{15}\pi \approx 3.35$

Q21. π

Q22. (1) $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$
(2) $\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$
(3) $\frac{\pi^2}{2}$

Q23. (1) $-\frac{1}{2}x + \frac{1}{2}\ln 2$
(2) 0.115
(3) 0.0537

Q24. (1) 1
(2) $\frac{\pi^2}{4}$

Questions 24.4.

Q25. (1) $\frac{43}{6}$
(2) $\frac{101}{3}\pi$
(3) $\frac{706}{15}\pi$

Q26. (1) tangent: $y = 2x - 1.5$, area $= \frac{7}{48}$
(2) $\frac{29}{240}\pi \approx 0.380$
(3) 0.949

Q27. (1) $2\ln 2 - 1 \approx 0.386$
(2) $\frac{\pi}{2} \approx 1.57$
(3) 0.592

Q28. 1.47 or $\frac{\pi}{4}(\pi^2 - 8)$

24.4 Kinematics

Questions 24.5.

Q29. (i) after $4.47h$ (ii) $89.4kmh^{-1}$ (iii) after $6.32h$

Q30. (i) after $2.53h$ (ii) $98.0kmh^{-1}$ (iii) after $4.33h$

Q31. (i) after $2.95h$ (ii) $88.3kmh^{-1}$ (iii) after $4.87h$

Q32. (i) after $3.37h$ (ii) $61.0kmh^{-1}$ (iii) after $13.2h$, on the way back

Q33. (i) $-0.04\pi ms^{-2}$ (ii) $\frac{800}{\pi}m \approx 255m$ (iii) $\frac{2400}{\pi}m \approx 764m$

Q34. (i) at $t = 2h$
(ii) at $t = 6h$
(iii) $v = 12kmh^{-1}$ (at $t = 2h$)
(iv) $v = 36kmh^{-1}$ $(-36kmh^{-1})$ (at $t = 10h$)
(v) $64km$
(vi) $0km$
(vii) $118km$

Chapter 25

Further integration

25.1 Integration by parts

(all answers should include "$+C$")

Introducing integration by parts

Questions 25.1.

Q1. (1) $x\sin x + \cos x$
(2) $-x\cos x + \sin x$
(3) $e^x(x-1)$
(4) $\frac{1}{4}(\sin 2x - 2x\cos 2x)$
(5) $\frac{3}{4}e^{2x-1}(2x-1)$
(6) $\frac{1}{4}((2x^2-1)\sin 2x + 2x\cos 2x)$
(7) $-e^{-x}(x+3)$
(8) $(-x^2+x+2)\cos x + (2x-1)\sin x$
(9) $(3x^2-8)\sin x - (x^3-8x+5)\cos x$
(10) $\frac{1}{4}((2x^2-2x+5)\sin 2x + (2x-1)\cos 2x)$

Exceptions to standard approach

Questions 25.2.

Q2. (1) $x \ln x - x$
(2) $2\sqrt{x}(\ln x - 2)$
(3) $\frac{1}{2}((x^2+1)\arctan x - x)$
(4) $\frac{1}{16}x^4(4\ln 2x - 1)$
(5) $x\ln(x+x^2) - 2x + \ln(x+1)$
(6) $\sin x(\ln \sin x - 1)$

Q3. (1) $\frac{1}{4}x^2(2\ln x - 1)$
(2) $\frac{1}{9}x^3(3\ln x - 1)$

Repeated integration by parts

Questions 25.3.

Q4. (1) $\frac{1}{2}e^x(\sin x + \cos x)$
(2) $\frac{1}{5}e^{2x}(2\sin x - \cos x)$
(3) $\frac{1}{10}e^(-x)(3\sin 3x - \cos 3x)$
(4) $\frac{1}{3}(\cos x \cos 2x + 2\sin x \sin 2x)$
(5) $\frac{1}{13}e^{3x}(3\sin 2x - 2\cos 2x)$
(6) $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)$
(7) $\cos x(1 - \ln(\cos x))$
(8) $\frac{1}{4}\sin 2x(2\ln(\sin x) - 1) - \frac{1}{2}x$

Q5. (1) $\frac{7\pi}{24} - \frac{\sqrt{3}}{8}$
(2) $\frac{\pi}{6} - \frac{\pi^2}{24\sqrt{3}} - 2 + \sqrt{3}$
(3) $2\ln 2 - \frac{3}{4}$
(4) $\frac{8}{15}$

25.2 Integration by substitution

Questions 25.4.

(all answers should include "$+C$")

Q6. (1) $\ln(x^3+2)$
(2) $\frac{1}{4}(x^2+1)^6$
(3) $\frac{1}{3}(\ln(2x-3))^{\frac{3}{2}}$
(4) $\frac{2}{3}(\ln(x^2-3x+2))^{\frac{3}{2}}$
(5) $\frac{1}{3}(2x+1)^{\frac{3}{2}}$
(6) $\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1)$
(7) $-\frac{4}{15}(1-x)^{\frac{3}{2}}(3x+2)$
(8) $\frac{4}{3}(3+\sin\frac{1}{2}x)^{\frac{3}{2}}$
(9) $\frac{1}{9}\tan^3 3x$
(10) e^{x^2+1}
(11) $-\frac{1}{3}\cos 3x^2$
(12) $-\cos\ln(x+1)$
(13) $\frac{1}{3}\ln(2-\cos 3x)$
(14) $\frac{2}{3}\sqrt{x^3+2}$

Q7. (1) $\frac{1}{4}\sin^4 x$
(2) $-\ln\cos x = \ln\sec x$
(3) $\frac{1}{2}\tan^2 x$
(4) $\frac{1}{4}\tan^4 x$
(5) $-\frac{1}{5}\cos^5 x$
(6) $-\cos x + \frac{1}{3}\cos^3 x$
(7) $\sin x - \frac{1}{3}\sin^3 x$
(8) $\sin x + \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x$
(9) $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x$
(10) $\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x$
(11) e^{e^x}
(12) $\frac{1}{3}\ln^3 x$
(13) $\frac{1}{3}\sin^3 x$
(14) $\cos\cos x$
(15) $e^{\sqrt{x}}$
(16) $-\sin\frac{1}{x}$
(17) $-2\cos\sqrt{x}$
(18) $\sqrt{x}\ln x - 2\sqrt{x}$

Q8. (1) $\tan x - x$
(2) $\ln|\sec x|$
(3) $\frac{1}{2}\tan^2 x + \ln\cos x$
(4) $\frac{1}{2}\tan^2\ln x + \ln\cos\ln x$
(5) $\frac{1}{3}\tan^3 x$
(6) $\frac{1}{3}\tan^3 x - 2\tan x + 2x$
(7) $-\frac{1}{2}\cot 2x - x$

Definite integration by substitution

Questions 25.5.

Q9. (1) 2
(2) $-\frac{1}{2}\ln 2$
(3) $-\frac{1}{2}\ln 2$
(4) $\frac{1}{2}$
(5) 3333.3
(6) $-\frac{1}{\pi}$
(7) $\frac{e-1}{3e}$
(8) $-\frac{1}{3}$
(9) $\frac{\pi^2}{9}$
(10) $\frac{3\pi}{8}$
(11) $\frac{1}{32}(8\pi - \pi^2 - 16\ln 2)$
(12) $\frac{1}{4}(\pi - \ln 4)$
(13) $\frac{1}{4}\ln 2 - \frac{\pi\sqrt{3}}{12}$
(14) $\frac{1}{108}(8\sqrt{3}-9)\pi^{1.5}$

Trigonometric substitutions

Questions 25.6.

Q10. – **Q11.** $\frac{\pi}{4}$ **Q12.** 2π **Q13.** $-\frac{1}{8\sqrt{2}}(2+\pi)$ **Q14.** $\frac{\pi}{3}$

Q15. (1) $\arcsin x + C$
(2) $-\sqrt{1-x^2} + C$
(3) $\frac{1}{2}(\arcsin x - x\sqrt{1-x^2}) + C$
(4) $-\frac{1}{3}\sqrt{1-x^2}(x^2+2) + C$

Q16. 2π

Q17. (1) $\frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\ln\left(\sqrt{x^2-9}+x\right) + c$
(2) $\frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + c$
(3) $\frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left(\sqrt{x^2+9}+x\right) + c$
(4) $\ln(x+\sqrt{x^2+4}) + c$
(5) $\sqrt{x^2+4} + c$
(6) $\frac{1}{2}x\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right) + c$
(7) $-\frac{1}{\sqrt{x^2+4}} + c$
(8) $\frac{x}{4\sqrt{x^2+4}} + c$
(9) $\frac{1}{\sqrt{4-x^2}} + c$

Q18. (1) $\frac{1}{2}(x\sqrt{x^2-1} - \ln(x+\sqrt{x^2-1})) + C$
(2) $\frac{1}{2}(x\sqrt{x^2+1} + \ln(x+\sqrt{x^2+1})) + C$
(3) $\frac{1}{2}x\sqrt{4-x^2} + 2\arcsin(\frac{x}{2}) + C$
(4) $\frac{1}{2}x\sqrt{x^2-4} - 2\ln(x+\sqrt{x^2-4}) + C$
(5) $\frac{1}{2}x\sqrt{x^2+4} + 2\ln(x+\sqrt{x^2+4}) + C$
(6) $\frac{1}{2}x\sqrt{9-4x^2} + \frac{9}{4}\arcsin(\frac{2x}{3}) + C$
(7) $\frac{1}{2}x\sqrt{4x^2-9} - \frac{9}{4}\ln(2x+\sqrt{4x^2-9})) + C$
(8) $\frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\ln(2x+\sqrt{4x^2+9}) + C$
(9) $\arcsin x + C$
(10) $\ln(x+\sqrt{x^2+1}) + C$
(11) $\ln(x+\sqrt{x^2-1}) + C$
(12) $\arcsin(\frac{x}{2}) + C$
(13) $\ln(x+\sqrt{x^2+4}) + C$
(14) $\ln(x+\sqrt{x^2-4}) + C$
(15) $\frac{1}{2}\arcsin(2x) + C$
(16) $\frac{1}{2}\ln(2x+\sqrt{4x^2+1}) + C$
(17) $\frac{1}{2}\ln(2x+\sqrt{4x^2-1}) + C$

Q19. (1) $\sqrt{1-x^2} + x\arcsin x$
(2) $\frac{1}{4}(x\sqrt{1-x^2} + (2x^2-1)\arcsin x)$
(3) $x\arctan x - \frac{1}{2}\ln(x^2+1)$
(4) $x\arccos x - \sqrt{1-x^2}$

25.3 Integrating rational expressions

Questions 25.7.

Q20. (1) $\frac{1}{4}\arctan(\frac{x}{2})$
(2) $\frac{2}{\sqrt{3}}\arctan(\frac{x}{\sqrt{3}}$
(3) $\arctan(x+3)$
(4) $\frac{1}{4}\arctan(x-0.5)$
(5) $\arctan(x+3)$
(6) $\frac{1}{2}\arctan(\frac{x-4}{2})$
(7) $\frac{2}{\sqrt{3}}\arctan(\frac{2x+3}{\sqrt{3}})$
(8) $-\frac{1}{\sqrt{2}}\arctan(\frac{x-2}{\sqrt{2}})$
(9) $\frac{1}{\sqrt{6}}\arctan(\frac{x\sqrt{3}+2\sqrt{3}}{\sqrt{2}})$
(10) $\frac{1}{3}\arctan(\frac{2x-5}{3})$
(11) $-\frac{1}{3(3x+2)}$
(12) $\frac{1}{2-x}$
(13) $-\frac{1}{x+5}$
(14) $-\frac{1}{3(3x-2)}$
(15) $-\frac{1}{3(x+\sqrt{2})}$
(16) $\frac{1}{4(4x-1)}$
(17) $\ln(\frac{x+1}{x+2})$
(18) $\frac{1}{4}\ln(\frac{x-1}{x+3})$
(19) $\frac{1}{3}\ln(\frac{x-2}{x+1})$
(20) $\frac{1}{3}\ln(\frac{2x+1}{x+2})$
(21) $\frac{1}{5}\ln(\frac{3x-2}{x+1})$

Q21. (1) $\ln(x^2+4)$
(2) $\frac{1}{2}\ln(x^2+3)$
(3) $\frac{1}{2}\ln((x+3)^2+1)$
(4) $\frac{1}{8}\ln((2x-1)^2+4)$
(5) $\ln(x^2+6x+10) - 6\arctan(x+3)$
(6) $\frac{1}{2}\ln(x^2-8x+20) + 2\arctan(x-4)$
(7) $\frac{1}{2}\ln(x^2+3x+3) - \sqrt{3}\arctan(\frac{2x+3}{\sqrt{3}})$
(8) $-\frac{1}{2}\ln(x^2-4x+6) - \sqrt{2}\arctan(\frac{x-2}{\sqrt{2}})$
(9) $\frac{1}{3}\ln(x+2) - 2\sqrt{6}\arctan((\frac{\sqrt{3}x+2\sqrt{3}}{\sqrt{2}})$
(10) $\frac{1}{4}\ln(2x^2-10x+17) + \frac{5}{6}\arctan(\frac{2x-5}{3})$
(11) $\frac{1}{9}(\frac{2}{3x+2} + \ln(3x+2))$
(12) $\ln(x-2) - \frac{2}{x-2}$
(13) $\frac{1}{2}\ln(x^2+10x+25) + \frac{10}{x+5}$
(14) $\frac{1}{18}\ln(9x^2-12x+4) - \frac{4}{3x-2}$
(15) $\frac{1}{3}\ln(x+\sqrt{2}) + \frac{1}{3(x+\sqrt{2})}$
(16) $\frac{1}{16}\left(\frac{1}{4x-1} - \ln(1-4x)\right)$
(17) $\ln\frac{(x+1)^2}{x+2}$
(18) $\frac{1}{7}\ln\frac{(x-1)^2}{x+3}$
(19) $\frac{1}{3}\ln\frac{(x-2)^2}{x+1}$
(20) $\frac{1}{6}\ln\frac{(x+2)^4}{2x+1}$
(21) $\frac{2}{15}\ln(3x-2) + \frac{1}{5}\ln(x+1)$

Q22. (1) $\frac{1}{2}(x - 2\arctan\frac{x}{2})$
(2) $x^2 + \ln(x^2 + 3)$
(3) $\frac{3}{2}x^2 - x + 2\arctan(x + 3)$
(4) $\frac{3}{2}x^2 - x + \ln(x^2 + 6x + 10) + \arctan(x + 3)$
(5) $\frac{x^3}{3} + x + 3\arctan(\frac{x-4}{2})$
(6) $\frac{1}{3}x^3 - 20x - 50\ln(x^2 - 8x + 20)$
(7) $x^2 + 2x$
(8) $-x^2 + 3x - 9\sqrt{2}\arctan(\frac{x-2}{2})$
(9) $x - 2\ln(3x^2 + 12x + 14)$
(10) $x + \arctan(5 - x)$
(11) $\frac{1}{3}\ln(3x + 2) - \frac{1}{9x+6}$
(12) $\frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{x-1}$
(13) $x - \frac{50}{x+5} - 10\ln(x + 5)$
(14) $\frac{1}{9}(3x + \frac{4}{3x-2} - 2)$
(15) $\ln(x + 1) + 2\ln(x - 2)$
(16) $2\ln(10 - 5x) - \frac{1}{2}\ln(10x + 5)$
(17) $\ln(\frac{3(x-2)^2}{x+1})$
(18) $x + \ln\frac{x+1}{(x+2)^2}$
(19) $\frac{2}{3}\ln(x + 2) + \frac{1}{3}\ln(x - 1)$
(20) $\frac{3}{4}x^2 - \frac{9}{4}x - \frac{3}{8}\ln(2x + 1) + 6\ln(x + 2)$
(21) $5x - \frac{2}{3}\ln(2 - 3x) - \ln(x + 1)$

25.4 More trigonometric substitutions

Questions 25.8.

Q23. (1) $\cos(x) + \ln(\tan(\frac{x}{2}))$
(2) $-\arctan(\cos x)$
(3) $\frac{\sqrt{2}}{4}\ln\left|\frac{\cos x-\sqrt{2}}{\cos x+\sqrt{2}}\right|$

Q24. (1) $\ln(\frac{1+\sin x}{\cos x})$
(2) $\ln(\frac{1+\sin x}{\cos x}) - \sin x$
(3) $\frac{\sqrt{2}}{4}\ln\left|\frac{\sin x+\sqrt{2}}{\sqrt{2}-\sin x}\right|$
(4) $\arctan(\sin(x))$

Q25. (1) $\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x)$
(2) $\frac{1}{\sqrt{2}}\arctan(\frac{1}{\sqrt{2}}\tan x)$
(3) $\frac{1}{6}\arctan(\frac{3}{2}\tan x)$
(4) $\frac{1}{6}\arctan(\frac{2}{3}\tan x)$
(5) $-\frac{1}{3}\arctan(3\cot x)$
(6) $\frac{1}{2}\arctan(2\tan x)$
(7) $\frac{1}{2\sqrt{3}}\arctan(\frac{\sqrt{3}}{2}\tan x)$
(8) $\frac{1}{\sqrt{2}}\arctan(\frac{1}{\sqrt{2}}\tan x)$
(9) $\frac{1}{\sqrt{3}}\arctan(\sqrt{3}\tan x)$
(10) $\frac{1}{4\sqrt{3}}\arctan(\frac{4}{\sqrt{3}}\tan x)$

Q26. (1) $\ln\left|\tan\left(\frac{x}{2}\right)\right|$
(2) $-\frac{\arctan(2\cos(\frac{x}{2}))}{\sin(\frac{x}{2})+\cos(\frac{x}{2})}$
(3) $\frac{2\sin(\frac{x}{2})}{9\cos(\frac{x}{2})-3\sin(\frac{x}{2})}$
(4) $\frac{2\sin(\frac{x}{2})}{9\cos(\frac{x}{2})-6\sin(\frac{x}{2})}$
(5) $\ln(2\cot(\frac{x}{2}) - 1)$
(6) $\frac{1}{3}\ln\frac{\sin(\frac{x}{2})}{\sin(\frac{x}{2})+3\cos(\frac{x}{2})}$
(7) $-\frac{2}{3}\arctan(3\cot(\frac{x}{2}))$
(8) $\frac{2\sin(\frac{x}{2})}{15\sin(\frac{x}{2})+9\cos(\frac{x}{2})}$
(9) $\frac{1}{2\sqrt{2}}\arctan(\frac{1}{\sqrt{2}}\tan(\frac{x}{2}))$
(10) $\frac{2}{5}\arctan(5\tan(\frac{x}{2}))$
(11) $\arctan(2\tan(\frac{x}{2}))$
(12) $\frac{1}{1+\cot(\frac{x}{2})}$
(13) $\frac{1}{2}\ln(\frac{2-\tan(\frac{x}{2})}{2+\tan(\frac{x}{2})})$
(14) $\ln(\frac{1-\cos x}{2+2\sin^2 x})$

25.5 Differential equations

Questions 25.9.

Q27. –

Q28. yes

Q29. (1) $y = \frac{1}{2}\arccos(C - 2\ln x)$
(2) $y = \ln(C - 3e^{-x})$
(3) $y = (\frac{1}{6}x^3 + C)^2$
(4) $y = \arccos k \csc x$
(5) $y = 3 - \frac{k}{\sqrt{2x+1}}$
(6) $y = \sqrt{\frac{1}{C-\arcsin x}}$
(7) $y = \tan(\frac{1}{2}x + \frac{1}{4}\sin 2x + C)$
(8) $y = \arctan(x + \frac{1}{3}x^3 + C)$

Q30. (1) $\ln y = \ln x^2$
(2) $y = \arctan(\frac{1}{2\cos x-1})$
(3) $y = \frac{1}{3-2\sqrt{x}}$
(4) $y = \ln(\frac{1}{1+e^{-2x}})$; $y(0) = -\ln 2$

Q31. $y^2 = x$

25.6 Mixed exercise

Questions 25.10.

Q32. (1) by parts
(2) by parts
(3) $\sin 6x$ formula
(4) $\cos 6x$ formula

Q33. $\frac{26}{15}$

Q34. $A=-1, B=2$,
$p=8, q=9$

Q35. (1) $A=-1, B=1, C=-2$
(2) $-3+6\ln 2$

Q36. (1) $\frac{4}{15}$
(2) $3-\ln 4$
(3) $\frac{1}{2}(4-\pi)$
(4) $\frac{15}{4}$
(5) $\pi+\frac{2}{\sqrt{3}}$
(6) $\frac{\pi}{3}$
(7) $\sqrt{3}(1-\ln 2)-\frac{\pi}{3}$

Q37. $2(1-\sqrt{x})-2\ln|1-\sqrt{x}|+c$

Q38. $\frac{1}{13}e^{2x}(2\sin 3x-3\cos 3x)+c$

Q39. (1) $\frac{1}{3}\ln(x^3-3x^2+6)+C$
(2) $\frac{1}{4}\tan^4 x+C$
(3) $-\frac{2}{3}(1-x)^{\frac{3}{2}}+\frac{4}{5}(1-x)^{\frac{5}{2}}-\frac{2}{7}(1-x)^{\frac{7}{2}}+C$
(4) $\frac{11}{24}$
(5) $\frac{1}{4}(\sin 2x-2(x+1)\cos 2x)+C$
(6) $2\ln^2 2-4\ln 2+2$
(7) $\frac{1}{2}e^x(\sin x+\cos x)+C$
(8) $\frac{\pi}{4}$
(9) $x\tan x+\ln\cos x$

Q40. 0

Q41. $\frac{26}{15}$

Q42. $\frac{1}{3}(2-\sqrt{2})$

PART VII

VECTORS

CHAPTER 26

INTRODUCTION TO VECTORS

QUESTIONS 26.1.

26.1 BASICS

Q1. (1) $\begin{pmatrix}10\\-11\\2\end{pmatrix}$ (2) $\pm\begin{pmatrix}\frac{2}{3}\\-\frac{11}{15}\\\frac{2}{15}\end{pmatrix}$ (3) $\pm\begin{pmatrix}\frac{14}{3}\\-\frac{77}{15}\\\frac{14}{15}\end{pmatrix}$

Q2. (1) $\begin{pmatrix}12\\-4\\3\end{pmatrix}$, (2) $\pm\begin{pmatrix}\frac{12}{13}\\-\frac{4}{13}\\\frac{3}{13}\end{pmatrix}$ (3) $\pm\begin{pmatrix}\frac{60}{13}\\-\frac{20}{13}\\\frac{15}{13}\end{pmatrix}$

Q3. (1) $(-5,6,-6)$ (2) $(1.5,2.5,2.5)$

Q4. (1) $\begin{pmatrix}12\\-12\\21\end{pmatrix}$ (2) $(6,-5,11)$ (4) $33\sqrt{2}$
(3) $(5,-4,12)$ (5) no

26.2 SCALAR PRODUCT

QUESTIONS 26.2.

Q5. (1) $\vec{AB}=\begin{pmatrix}2\\4\\4\end{pmatrix}$, $\vec{AC}=\begin{pmatrix}-2\\9\\-6\end{pmatrix}$ (2) $83.0°(1.45)$
(3) 32.8

Q6. (1) $\vec{AB} = \begin{pmatrix} -10 \\ 5 \\ -10 \end{pmatrix}$, $\vec{AC} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$
(2) $79.0°(1.38)$
(3) 51.5

Q7. (1) $\vec{AB} = \begin{pmatrix} -6 \\ 10 \\ 4 \end{pmatrix}$, $\vec{AC} = \begin{pmatrix} 9 \\ -15 \\ -6 \end{pmatrix}$
(2) $180°(\pi)$
(3) 0

Q8. $A \approx 78.2°(1.36), B \approx 63.2°(1.10), C \approx 38.6°(0.673)$

Q9. 1.5

Q10. 1.5 or -1

Q11. $c = \frac{1}{6}, d = \frac{4}{3}$

Q12. $\vec{u} = \pm(\frac{8}{3}\vec{i} + \frac{4}{3}\vec{j} - \frac{8}{3}\vec{k})$

Q13. ± 5

Q14. (1) 15
(2) $72.9°(1.27)$

Q15. $x = -1, y = 2$

CHAPTER 27

STRAIGHT LINES

Task

- 1 mile West, 7 miles South
- 21 miles East, 51 miles South
- 45 miles East, 100 miles South
- 15 miles West, 21 miles North
- 1^{st} of June at 02.30hrs, 4.5 miles South

27.1 EQUATIONS OF STRAIGHT LINES

QUESTIONS 27.1.

Q1. (i) $(-12, 4, 22)$
(ii) 2
(iii) (1) yes
(2) no
(3) no
(iv) $\lambda = 1, p = 0, q = 6$
(v) (1) $\begin{cases} x = 0 + 3\lambda \\ y = 0 - 1\lambda \\ z = 6 - 4\lambda \end{cases}$
(2) $\begin{cases} x = 9 + 6\lambda \\ y = -3 - 2\lambda \\ z = -6 - 8\lambda \end{cases}$
(3) $\begin{cases} x = 6 + 2\lambda \\ y = -2 - \frac{2}{3}\lambda \\ z = -2 - \frac{8}{3}\lambda \end{cases}$

Q2. (i) $y = \frac{1}{2}x + 1$ (ii) $y = -\frac{3}{2}x + \frac{7}{2}$ (iii) $y = -\frac{2}{3}x - \frac{5}{3}$ (iv) $y = -\frac{2}{3}x - \frac{8}{3}$

Q3. (i) $\begin{cases} x = 0 + \lambda \\ y = -3 + 2\lambda \end{cases}$ (ii) $\begin{cases} x = 0 + 4\lambda \\ y = -0.5 + 3\lambda \end{cases}$ (iii) $\begin{cases} x = 0 + 3\lambda \\ y = 1 - 2\lambda \end{cases}$ (iv) $\begin{cases} x = -2 + 5\lambda \\ y = 0 + 2\lambda \end{cases}$

Q4. (i)

λ	-2	-1	0	1	2
point	$(-4, -1)$	$(-2, 0)$	$(0, 1)$	$(2, 2)$	$(4, 3)$

(ii) *(sketch of $y = \frac{1}{2}x + 1$)*
(iii) $2\vec{i} + \vec{j}$

Q5. (i) e.g.$(5, -1, 2), (3, 1, 3), (1, 3, 4), (-1, 5, 5), (-3, 7, 6)$
(ii) $\begin{cases} x = 5 - 2t \\ y = -1 + 2t \\ z = 2 + t \end{cases}$
(iii) $t = \frac{x-5}{-2} = \frac{y+1}{2} = \frac{z-2}{1}$
(iv) $\frac{x-(5)}{(-2)} = \frac{y-(-1)}{(2)} = \frac{z-(2)}{(1)}$

Questions 27.2.

Q6. (1) $\frac{x-1}{1} = \frac{y+3}{2} = \frac{z+6}{5}$

(2) $\frac{x+3}{-2} = \frac{y-2}{1} = \frac{z-1}{-1}$

(3) $\frac{x-2}{-1} = \frac{y+1}{2} = \frac{z+5}{-3}$

(4) $\frac{x-3}{3} = \frac{z-2}{-1}, y = -1$

(5) $x - 3 = y + 1 = 1 - z$

(6) $\frac{x-3}{2} = \frac{y}{5}, z = -2$

(7) $x = 0, z = 1, y \in \mathbb{R}$

Q7. (1) $\vec{r} = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

(2) $\vec{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

(3) $\vec{r} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(4) $\vec{r} = \begin{pmatrix} -1.5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

(5) $\vec{r} = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$

Q8. (i) $\frac{x-1}{4} = \frac{y-9}{5}, z = 5$

(ii) $\frac{x-3}{4} = \frac{y-5}{5}, z = 1$

Q9. (i) $\frac{x-6}{1} = \frac{y-2}{3} = \frac{z-8}{-4}$

(ii) $\frac{x-4}{1} = \frac{y-3}{3} = \frac{z-2}{-4}$

Q10. (i) $\frac{x-5}{5} = \frac{y-8}{9} = \frac{z-2}{6}$

(ii) $\frac{x-1}{5} = \frac{y-2}{9} = \frac{z-2}{6}$

27.2 Intersections of straight lines

Questions 27.3.

Q11. (i)

$$l_1 : \begin{cases} x = 10 - 6t \\ y = -7 + 3t \\ z = -21 + 12t \end{cases} \qquad l_4 : \begin{cases} x = -1 - \mu \\ y = 3 + 2\mu \\ z = -14 - 3\mu \end{cases}$$

$$l_2 : \begin{cases} x = -4 + 6s \\ y = -3s \\ z = 7 - 12s \end{cases} \qquad l_5 : \begin{cases} x = -4 - 2p \\ y = p \\ z = 7 \end{cases}$$

$$l_3 : \begin{cases} x = -10 + 4\lambda \\ y = 3 - 2\lambda \\ z = -19 - 8\lambda \end{cases} \qquad l_6 : \begin{cases} x = -11 + 7q \\ y = 5 - 5q \\ z = 16 - 9q \end{cases}$$

(ii)

?	l_1	l_2	l_3	l_4	l_5
l_6	$(-4, 0, 7)$	$(-4, 0, 7)$	skew	$(3, -5, -2)$	$(-4, 0, 7)$
l_5	$(-4, 0, 7)$	$(-4, 0, 7)$	$(-23, 9.5, 7)$	skew	
l_4	$(2, -3, -5)$	$(2, -3, -5)$	skew		
l_3	parallel	parallel			
l_2	coincident				

Q12. (1) $(-7, 4, 1)$, angle: $19.4°(0.339)$

(2) no intersection, (angle: $68.1°(1.19)$)

(3) no intersection, (angle: $19.5°(0.340)$)

(4) no intersection, (angle: $90°(\pi)$)

(5) $(-5, 3, -2)$, angle: $40.2°(0.702)$

chapter 28

Matrices

28.1 Basics

28.2 Multiplication of matrices

Questions 28.1.

Q1. $AB = \begin{pmatrix} -6 & -1 & -8 \\ -9 & 4 & -10 \end{pmatrix}$

Q2. (i) $M^2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, $M^3 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$, $M^4 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$,

(ii) $M^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$

(iii) —

Q3. (i) $\begin{pmatrix} 14 & 3 \\ 2 & 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 14 & -1 \\ -6 & 2 \end{pmatrix}$

(iii) $\begin{pmatrix} -7 & 4 \\ 4 & -1 \end{pmatrix}$

(iv) $\begin{pmatrix} -9 & -2 \\ 0 & 1 \end{pmatrix}$

(v) $\begin{pmatrix} -25 & 13 \\ -2 & 5 \end{pmatrix}$

(vi) $\begin{pmatrix} -25 & 13 \\ -2 & 5 \end{pmatrix}$

Q4. (i) $\begin{pmatrix} 19 & 1 \\ -9 & -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 16 & 12 \\ 4 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 9 & 16 \\ 18 & 28 \end{pmatrix}$

(iv) $\begin{pmatrix} 0 & 0 & -10 \\ -4 & 0 & 3 \\ 5 & 0 & -5 \end{pmatrix}$

(v) $\begin{pmatrix} 12 & -6 & 9 \\ -6 & 0 & 3 \\ 9 & 3 & -6 \end{pmatrix}$

(vi) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

(vii) $\begin{pmatrix} -2 & 3 \\ -7 & 5 \end{pmatrix}$

(viii) $\begin{pmatrix} -1 & 5 & -1 \\ -73 & 8 & 80 \\ 2 & 4 & -4 \end{pmatrix}$

Q5. A - anticlockwise rotation by 90° about the origin,
B - anticlockwise rotation by 45° about the origin,
C - reflection in the x-axis,
D - reflection in the y-axis,
E - reflection in the line $y = x$

Q6. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Q7. $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

28.3 Determinant of a matrix, inverse matrix

Questions 28.2.

Q8. (i) $AB = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(ii) $A^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{3}{2} \end{pmatrix}$

Q9. (i) $AB = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(ii) $A^{-1} = \begin{pmatrix} 0.6 & 0.4 & -0.2 \\ -0.8 & -0.2 & 0.6 \\ -1 & -1 & 1 \end{pmatrix}$.

Q10. —

Q11. —

Q12. —

Questions 28.3.

Q13. (i) (a) $\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$

(b) $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} -0.25 & 1.25 \\ 0.5 & -1.5 \end{pmatrix}$

(d) $\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$

(e) $\frac{1}{6}\begin{pmatrix} -4 & 3 \\ 2 & -3 \end{pmatrix}$

(f) $\begin{pmatrix} -2 & -3 \\ 3 & 5 \end{pmatrix}$

(g) $\frac{1}{4}\begin{pmatrix} 3 & -1 \\ -5 & 3 \end{pmatrix}$

(h) $\frac{1}{5}\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$

(i) $\frac{1}{5}\begin{pmatrix} -3 & -7 \\ -2 & -3 \end{pmatrix}$

(j) no inverse

(k) $\frac{1}{3}\begin{pmatrix} -6 & -3 \\ -5 & -2 \end{pmatrix}$

(l) $\frac{1}{15}\begin{pmatrix} 9 & -12 \\ 6 & 2 \end{pmatrix}$

(ii) —

Questions 28.4.

Q14. —

Q15. (i) -2

(ii) -6

(iii) -8

(iv) 0 (singular)

(v) -23

(vi) 0 (singular)

Q16. (i) $\det A = 4, \det B = -2, \det C = -2$
(ii) $\det(AC^{-1}B) = (4)(-\frac{1}{2})(-2) = 4.$
(iii) $\frac{1}{4}\begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix}$
(iv) $\frac{1}{-2}\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$
(v) $\frac{1}{8}\begin{pmatrix} 15 & -29 \\ 23 & -45 \end{pmatrix}$
(vi) $\frac{1}{-4}\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}$

Questions 28.5.

Q17. -2

Q18. (i) — (ii) $\begin{pmatrix} a^2 - b^2 & 2ab \\ -2ab & a^2 - b^2 \end{pmatrix}$ (iii) —

Q19. 24 **Q20.** 5 **Q21.** -41

Q22. (i) $k = -4,\ s = 3.$
(ii) $m = 40, t = -39$
(iii) $k = -6, s = 5, m = 156, t = -155$

Q23. (i) $-1, 1, 2$
(ii) $\lambda = -1 \Rightarrow \vec{v} = \begin{pmatrix} t \\ -2.5t \\ 0.5t \end{pmatrix}, t \in \mathbb{R};$
$\lambda = 1 \Rightarrow \vec{v} = \begin{pmatrix} t \\ -2t \\ -t \end{pmatrix}, t \in \mathbb{R};$
$\lambda = 2 \Rightarrow \vec{v} = \begin{pmatrix} t \\ -t \\ -t \end{pmatrix}, t \in \mathbb{R}$

Q24. (i) —
(ii) 1

Q25. (i) incorrect; $(A + B)^2 = A^2 + B^2 + AB + BA$
(ii) incorrect; $ADB = C \Leftrightarrow D = A^{-1}CB^{-1}$
(iii) correct
(iv) incorrect; $AB = I \Rightarrow B = A^{-1}$
(v) incorrect; $(A + 2I)^2 = A^2 + 4A + 4I$
(vi) correct
(vii) correct
(viii) correct when order of A is odd, incorrect otherwise
(ix) correct when order of A is even, incorrect otherwise

28.4 Solving simultaneous equations - intersections of the planes

Questions 28.6.

Q26. $x = \frac{1}{3}, y = -\frac{4}{3}, z = 0$

Q27. $x = -3, y = 18, z = 31$

chapter 29

Planes in three-dimensional space

29.1 Intersections of the planes

Gaussian elimination (row reduction)

Questions 29.1.

Q1. (1) line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\frac{4}{7}\\\frac{1}{7}\\0\end{pmatrix} + t\begin{pmatrix}-2\\17\\7\end{pmatrix}$

(2) line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\1\end{pmatrix}$

(3) line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-\frac{10}{7}\\-\frac{2}{7}\\0\end{pmatrix} + t\begin{pmatrix}1\\3\\7\end{pmatrix}$

(4) no intersection

(5) point $(2, 2, -1)$

(6) point $(2.5, -1, 0.5)$

(7) point $(-2, -5.5, -2.5)$

(8) line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\3\\0\end{pmatrix} + t\begin{pmatrix}3\\1\\1\end{pmatrix}$

(9) line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\-4\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix}$

(10) line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-3\\0.5\\0\end{pmatrix} + t\begin{pmatrix}-5\\6\\2\end{pmatrix}$

(11) line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\0\\-7\end{pmatrix} + t\begin{pmatrix}1\\1\\0\end{pmatrix}$

(12) point $(-6, -10, 3)$

(13) point $(2, 1, -1)$

(14) point $(-1, -4, 2)$

(15) point $(-2, 1, -1)$

(16) point $(1, 1, -2)$

(17) point $(5, -2, 1)$

(18) point $(4, 3, -2)$

(19) plane $x - 2y - 3z = 2$

(20) no intersection

(21) line $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\frac{13}{29}\\\frac{1}{29}\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$

(22) no intersection

Q2. (1) $(2, -1, -1)$ (2) $(-2, -1, 5)$ (3) $(-4, 3, 5)$ (4) whole line

29.2 Vector product

29.3 Properties of scalar and vector product

Questions 29.2.

Q3. (i) $-60\vec{i} + 4\vec{j} + 26\vec{k}$ (ii) $-30x + 2y + 13z = 113$ (iii) $\sqrt{1073}$

Q4. (i) $-50\vec{i} - 90\vec{j} + 5\vec{k}$ (ii) $-10x - 18y + z = 16$ (iii) $\frac{25}{2}\sqrt{17}$

Q5. (i) $0\vec{i} + 0\vec{j} + 0\vec{k}$ (ii) no unique plane (iii) 0

Q6. (i) $(3, 1, -1)$ (ii) $9\vec{i} + 23\vec{j} + 1\vec{k}$ (iii) $\sqrt{611}$

Q7. (i) $(2, 7, 6)$ (ii) $-9\vec{i} - 5\vec{j} + 17\vec{k}$ (iii) $\sqrt{395}$

Q8. $7x + 4y + 5z = -3$

Q9. $-2x - 11y + 7z = 16$

Q10. $4x - 3z = 11$

Q11. (i) $x + 4y - 4z = 1$

(ii) $\vec{i} + 4\vec{j} - 4\vec{k}$

(iii) $\vec{r} = \begin{pmatrix}5\\-4\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\4\\-4\end{pmatrix}$

(iv) —

(v) $\sqrt{33}$

(vi) 33

Q12. (i) —

(ii) $3x + 2y - z = 13$

(iii) $3\vec{i} + 2\vec{j} - \vec{k}$

(iv) $(3, 1, -2)$

(v) $\vec{r} = \begin{pmatrix}3\\1\\-2\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-1\end{pmatrix}$

(vi) $(2 - 3t, -1 - 2t, 4 + t)$

(vii) $\pm\sqrt{2}$

(viii) $28\sqrt{2}$

Q13. (1) — (2) — (3) 0 (4) 90° (5) — (6) — (7) —

Q14. whole line

Q15. $9x + 21y + 2z = 33$

Q16. $x + z + 3 = 0$

Q17. $25.1°(0.438), (0, -\frac{11}{3}, 0)$

29.4 Alternate forms of equations of planes

Questions 29.3.

Q18. (1) $-4x + 6y + 5z + 12 = 0$ (2) $x - y + 3z + 5 = 0$ (3) $y + z = 2$

Q19. a number of possibilities

Q20. a number of possibilities

29.5 3D distances

Distance between a point and a plane

Questions 29.4.

Q21. (i) $2\vec{i} - 5\vec{j} + \vec{k}$

(ii) $\vec{r} = \begin{pmatrix} -4 \\ 9 \\ -4 \end{pmatrix} + t \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$

(iii) $(0, -1, -2)$

(iv) $2\sqrt{30}$

Q22. $\frac{5}{2}\sqrt{62}$

Q23. 8

Q24. $2\sqrt{10}$

Distance between two (parallel) planes

Q25. (i) lots of them

(ii) $2\sqrt{14}$

(iii) $2\sqrt{14}$

Q26. $3\sqrt{51}$

Q27. 13

Q28. $\frac{6}{5}\sqrt{2}$

Distance between a point and a line

Q29. (1st method)

(i) $\begin{cases} x = 3t \\ y = 7 - 5t \\ z = -3.5 + 2t \end{cases}$

(ii) $\begin{pmatrix} 3t \\ 7 - 5t \\ -3.5 + 2t \end{pmatrix}$

(iii) $\begin{pmatrix} 3t - 1 \\ 6 - 5t \\ z = -2.5 + 2t \end{pmatrix}$

(iv) $0, (3, 2, -1.5)$

(v) $\frac{1}{2}\sqrt{21}$

Q29. (2nd method)

(i) $3\vec{i} - 5\vec{j} + 2\vec{k}$

(ii) $3x - 5y + 2z = -4$

(iii) $(3, 2, -1.5)$

(iv) $\frac{1}{2}\sqrt{21}$

Q29. (3rd method)

(i) $3\vec{i} - 5\vec{j} + 2\vec{k}$

(ii) e.g. $(0, 7, -3.5)$

(iii) $3\vec{i} - 5\vec{j} + 2\vec{k}$

(iv) $\frac{1}{2}\sqrt{21}$

(v) $\frac{1}{2}\sqrt{21}$

Q30. $\frac{1}{2}\sqrt{11}$

Q31. $\sqrt{26}$

Distance between parallel lines

Q32. (i) $\begin{cases} x = 2 - 2t \\ y = 5 + 3t \\ z = 6 - t \end{cases}$

(ii) e.g. $(1, -1, -10)$

(iii) $\sqrt{293}$

(iv) $\sqrt{293}$

Q33. $\frac{1}{2}\sqrt{145}$

Q34. $\sqrt{30}$

Q35. $\sqrt{29}$

Distance between skew lines

Questions 29.5.

Q36. (1st method)

(i) $\begin{cases} x = -2 + 3t \\ y = 8 + 3t \\ z = 3 + 4t \end{cases}$

(ii) $\begin{cases} x = -4 + s \\ y = -7 + 2s \\ z = 2 + 3s \end{cases}$

(iii) $\begin{pmatrix} -2 + 3t \\ 8 + 3t \\ 3 + 4t \end{pmatrix}$

(iv) $\begin{pmatrix} -4 + s \\ -7 + 2s \\ 2 + 3s \end{pmatrix}$

(v) $\begin{pmatrix} -2 + 1s + -3t \\ -15 + 2s + -3t \\ -1 + 3s + -4t \end{pmatrix}$

(vi) $s = 1, t = -1$

(vii) $2\sqrt{35}$

Q36. (2nd method)

(i) $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(iii) $(-2, 8, 3)$

(iv) $(-4, -7, 2)$

(v) $\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$

(vi) $x - 5y + 3z = -33$
(vii) —
(viii) —
(ix) $2\sqrt{35}$
(x) $2\sqrt{35}$

Q36. (3rd method)

(i) $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$
(ii) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
(iii) $(-2, 8, 3)$
(iv) $(-4, -7, 2)$
(v) $\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$
(vi) $\begin{pmatrix} 2 \\ -10 \\ 6 \end{pmatrix}$
(vii) $2\sqrt{35}$

Q37. 4.04

Q38. $3\sqrt{11}$

29.6 Angles between lines, planes...

Questions 29.6.

Q39. 0° **Q40.** 72° **Q41.** 41° **Q42.** 65° **Q43.** 50.0° **Q44.** 36.2°

part VIII

Complex numbers

chapter 30

Introduction to complex numbers

30.1 Polar coordinates

Questions 30.1.

Q1. (1) $(4, \frac{\pi}{3})$
(2) $(2\sqrt{3}, \frac{\pi}{6})$
(3) $(3, \pi)$
(4) $(2, -\frac{\pi}{2})$
(5) $(5\sqrt{2}, \frac{3\pi}{4})$
(6) $(4\sqrt{3}, -\frac{2\pi}{3})$
(7) $(8, -\frac{\pi}{6})$
(8) $(3\sqrt{2}, -\frac{\pi}{4})$
(9) $(2.5\sqrt{2}, \frac{3\pi}{4})$
(10) $(6\sqrt{3}, \frac{5\pi}{6})$
(11) $(10/3, 0)$
(12) $(8\sqrt{2}, -\frac{3\pi}{4})$
(13) $(2\sqrt{3}, \frac{2\pi}{3})$
(14) $(4\sqrt{6}, -\frac{5\pi}{6})$
(15) $(\sqrt{5}, \frac{\pi}{2})$
(16) $(5, 0.927)$
(17) $(6.08, 0.165)$
(18) $(8.54, 2.78)$
(19) $(6.16, -0.409)$
(20) $(8.6, -2.19)$
(21) $(6.02, 1.65)$
(22) $(7.01, -0.0485)$
(23) $(5.73, -2.72)$
(24) $(0.734, 1.86)$
(25) $(13, 2.75)$
(26) $(11.8, -1.20)$
(27) $(17, -2.65)$

Q2. (1) $(3, 3\sqrt{3})$
(2) $(4, 4)$
(3) $(2\sqrt{3}, 2)$
(4) $(-3\sqrt{3}, 3)$
(5) $(-1, -\sqrt{3})$
(6) $(-1.2\sqrt{2}, 1.2\sqrt{2})$
(7) $(\frac{3}{2}, -\frac{3\sqrt{3}}{2})$
(8) $(-\frac{9}{5}, -\frac{9}{5})$
(9) $(-\sqrt{3}, 3)$
(10) $(3, -\sqrt{3})$
(11) $(\frac{3}{2}, -\frac{3}{2})$
(12) $(-6\sqrt{3}, -6)$
(13) $(-2.97, 6.07)$
(14) $(2, 16, 3.37)$
(15) $(-1.04, -2.27)$
(16) $(-0.801, -0.598)$
(17) $(7.17, 3.55)$
(18) $(1.70, 2.64)$
(19) $(8.86, -8.09)$
(20) $(-4.70, -8.03)$

30.2 Introducing complex numbers

Questions 30.2.

Q3. e.g. $a = b = -1$

Q4. e.g. $a = 1, b = -1$

Questions 30.3.

Q5. (1) $\pm 3i$
(2) $1 \pm 2i$
(3) $1 \pm 3i$
(4) $1 \pm i\sqrt{2}$
(5) $-3 \pm i\sqrt{2}$
(6) $\frac{1}{2} \pm \frac{3}{2}i\sqrt{3}$
(7) $-1 \pm i\sqrt{6}$
(8) $\pm\frac{4}{3}i$
(9) $\pm i\sqrt{3}$
(10) $-\frac{1}{2} \pm i\frac{\sqrt{5}}{2}$
(11) $\frac{1}{3} \pm \frac{2}{3}i\sqrt{5}$
(12) $-\frac{1}{2} \pm \frac{1}{2}i$
(13) $\frac{1}{3} \pm i\frac{\sqrt{2}}{3}$
(14) $-1, \pm 2i$
(15) $3, \pm\frac{1}{\sqrt{2}}i$
(16) $\pm i, \pm 2i$
(17) $\pm 1, \pm i$
(18) $-1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

30.3 Algebra of complex numbers

Questions 30.4.

Q6. (i) $9 + 7i$
(ii) $-1 + 7i$
(iii) $-8 + i$
(iv) $8 + 14i$
(v) $-26 + 7i$
(vi) $-15 - 8i$

Q7. (i) $-i$
(ii) 1
(iii) i
(iv) -1
(v) $-i$
(vi) 1
(vii) $2i$
(viii) $-2 + 2i$
(ix) -4
(x) 16
(xi) $16 + 16i$
(xii) 0

Q8. $a = -3, b = 2$

30.4 Conjugate numbers. Polar form of a complex number

Questions 30.5.

Q9.

z	$\lvert z\rvert$	z^*	zz^*
$3 - 4i$	5	$3 + 4i$	25
$12i - 5$	13	$-12i - 5$	169
$2\sqrt{3} + 6i$	$4\sqrt{3}$	$2\sqrt{3} - 6i$	48
$4 - 4i$	$4\sqrt{2}$	$4 + 4i$	32
$-1 + 3i$	$\sqrt{10}$	$-1 - 3i$	10

Questions 30.6.

Q10. (i) $0.3 + 1.1i$
(ii) $-1 - i$
(iii) $\frac{4}{13} + \frac{7}{13}i$
(iv) $\frac{3}{13} - \frac{11}{13}i$
(v) $-\frac{1}{2} + \frac{1}{2}i$
(vi) $0.8 - 1.4i$

Q11.

z in cartesian form	z in polar form	z^* in cartesian form	z^* in polar form
$4 + 4i$	$4\sqrt{2}cis\frac{\pi}{4}$	$4 - 4i$	$4\sqrt{2}cis(-\frac{\pi}{4})$
$-\sqrt{3} + 3i$	$2\sqrt{3}cis\frac{2\pi}{3}$	$-\sqrt{3} - 3i$	$2\sqrt{3}cis(-\frac{2\pi}{3})$
$\sqrt{3} - i$	$2cis(-\frac{\pi}{6})$	$\sqrt{3} + i$	$2cis\frac{\pi}{6}$
$-\sqrt{3} + i$	$2cis\frac{5\pi}{6}$	$-\sqrt{3} - i$	$2cis(-\frac{5\pi}{6})$
$-2 - 2i\sqrt{3}$	$4cis(-\frac{2\pi}{3})$	$-2 + 2i\sqrt{3}$	$4cis\frac{2\pi}{3}$
$-3i$	$3cis(-\frac{\pi}{2})$	$3i$	$3cis\frac{\pi}{2}$

Questions 30.7.

Q12.

	z (polar)	v (polar)	zv (cartesian)	zv (polar)
1	$4cis\frac{\pi}{6}$	$3cis\frac{\pi}{2}$	$-6+6i\sqrt{3}$	$12cis\frac{2\pi}{3}$
2	$4\sqrt{3}cis\frac{\pi}{6}$	$2cis\frac{2\pi}{3}$	$-12+4i\sqrt{3}$	$8\sqrt{3}cis\frac{5\pi}{6}$
3	$5\sqrt{2}cis\frac{3\pi}{4}$	$2cis(-\frac{\pi}{2})$	$10+10i$	$10\sqrt{2}cis\frac{\pi}{4}$
4	$2\sqrt{3}cis(-\frac{5\pi}{6})$	$4cis\frac{2\pi}{3}$	$12-4i\sqrt{3}$	$8\sqrt{3}cis(-\frac{\pi}{6})$
5	$2\sqrt{2}cis(-\frac{3\pi}{4})$	$3\sqrt{2}cis(-\frac{3\pi}{4})$	$12i$	$12cis\frac{\pi}{2}$
6	$3\sqrt{2}cis\frac{\pi}{4}$	$\sqrt{2}cis\frac{3\pi}{4}$	-6	$6cis\pi$
7	$2cis(-\frac{2\pi}{3})$	$2cis(-\frac{\pi}{3})$	-4	$4cis\pi$
8	$\sqrt{2}cis\frac{\pi}{4}$	$2cis(-\frac{\pi}{6})$	$(\sqrt{3}+1)+i(\sqrt{3}-1)$	$2\sqrt{2}cis\frac{\pi}{12}$

Questions 30.8.

Q13.

	z (polar)	v (polar)	$\frac{z}{v}$ (cartesian)	$\frac{z}{v}$ (polar)
1	$4cis\frac{\pi}{6}$	$3cis\frac{\pi}{2}$	$\frac{2}{3}-(2\sqrt{3})/3i$	$\frac{4}{3}cis(-\frac{\pi}{3})$
2	$4\sqrt{3}cis\frac{\pi}{6}$	$2cis\frac{2\pi}{3}$	$-2i\sqrt{3}$	$2\sqrt{3}cis(-\frac{\pi}{2})$
3	$5\sqrt{2}cis\frac{3\pi}{4}$	$2cis(-\frac{\pi}{2})$	$-\frac{5}{2}-\frac{5}{2}i$	$(5\sqrt{2})/2cis(-\frac{3\pi}{4})$
4	$2\sqrt{3}cis(-\frac{5\pi}{6})$	$4cis\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}i$	$\frac{\sqrt{3}}{2}cis(\frac{\pi}{2})$
5	$2\sqrt{2}cis(-\frac{3\pi}{4})$	$3\sqrt{2}cis(-\frac{3\pi}{4})$	$\frac{2}{3}$	$\frac{2}{3}cis0$
6	$3\sqrt{2}cis\frac{\pi}{4}$	$\sqrt{2}cis\frac{3\pi}{4}$	$-3i$	$3cis(-\frac{\pi}{2})$
7	$2cis(-\frac{2\pi}{3})$	$2cis(-\frac{\pi}{3})$	$1/2+i\frac{\sqrt{3}}{2}$	$cis(-\frac{\pi}{3})$
8	$\sqrt{2}cis\frac{\pi}{4}$	$2cis(-\frac{\pi}{6})$	$(\sqrt{3}-1)/2+i(\sqrt{3}+1)/2$	$1/\sqrt{2}cis\frac{5\pi}{12}$

chapter 31

De Moivre's theorem

Questions 31.1.

Q1.

	z^2	z^3	z (polar)	z^2 (polar)	z^3 (polar)
1	$8+8i\sqrt{3}$	$64i$	$4cis\frac{\pi}{6}$	$16cis\frac{\pi}{3}$	$64cis\frac{\pi}{2}$
2	-9	$-27i$	$3cis\frac{\pi}{2}$	$9cis\pi$	$27cis(-\frac{\pi}{2})$
3	$24+24i\sqrt{3}$	$192\sqrt{3}i$	$4\sqrt{3}cis\frac{\pi}{6}$	$48cis\frac{\pi}{3}$	$192\sqrt{3}cis\frac{\pi}{2}$
4	$-2-2i\sqrt{3}$	8	$2cis\frac{2\pi}{3}$	$4cis(-\frac{2\pi}{3})$	$8cis0$
5	$-50i$	$250+250i$	$5\sqrt{2}cis\frac{3\pi}{4}$	$50cis(-\frac{\pi}{2})$	$250\sqrt{2}cis\frac{\pi}{4}$
6	-4	$8i$	$2cis(-\frac{\pi}{2})$	$4cis()$	$8cis(\frac{\pi}{2})$
7	$6+6i\sqrt{3}$	$-24\sqrt{3}i$	$2\sqrt{3}cis(-\frac{5\pi}{6})$	$12cis(\frac{\pi}{3})$	$24\sqrt{3}cis(-\frac{\pi}{2})$
8	$-8-8i\sqrt{3}$	64	$4cis\frac{2\pi}{3}$	$16cis(-\frac{2\pi}{3})$	$64cis0$
9	$8i$	$16-16i$	$2\sqrt{2}cis(-\frac{3\pi}{4})$	$8cis\frac{\pi}{2}$	$16\sqrt{2}cis(-\frac{\pi}{4})$
10	$18i$	$-54-54i$	$3\sqrt{2}cis\frac{\pi}{4}$	$18cis\frac{\pi}{2}$	$54\sqrt{2}cis\frac{3\pi}{4}$
11	$-2i$	$2+2i$	$\sqrt{2}cis\frac{3\pi}{4}$	$2cis(-\frac{\pi}{2})$	$2\sqrt{2}cis\frac{\pi}{4}$
12	$-2+2i\sqrt{3}$	8	$2cis(-\frac{2\pi}{3})$	$4cis\frac{2\pi}{3}$	$8cis0$
13	$-2-2i\sqrt{3}$	-8	$2cis(-\frac{\pi}{3})$	$4cis(-\frac{2\pi}{3})$	$8cis\pi$
14	$2-2i\sqrt{3}$	-8	$2cis(-\frac{\pi}{6})$	$4cis(-\frac{\pi}{3})$	$8cis\pi$

Q2.

	z^4	z (polar)	z^4 (polar)
1	$-8+8i\sqrt{3}$	$2cis\frac{\pi}{6}$	$16cis\frac{2\pi}{3}$
2	16	$2cis\frac{\pi}{2}$	$16cis0$
3	$-8+8i\sqrt{3}$	$2cis\frac{2\pi}{3}$	$16cis\frac{2\pi}{3}$
4	-2500	$5\sqrt{2}cis\frac{3\pi}{4}$	$2500cis\pi$
5	81	$3cis(-\frac{\pi}{2})$	$81cis0$
6	$-72+72i\sqrt{3}$	$2\sqrt{3}cis(-\frac{5\pi}{6})$	$144cis\frac{2\pi}{3}$
7	-64	$2\sqrt{2}cis(-\frac{3\pi}{4})$	$64cis\pi$
8	-4	$\sqrt{2}cis\frac{\pi}{4}$	$4cis\pi$
9	-4	$\sqrt{2}cis\frac{3\pi}{4}$	$4cis\pi$
10	$-8-8i\sqrt{3}$	$2cis(-\frac{2\pi}{3})$	$16cis(-\frac{2\pi}{3})$
11	$-8+8i\sqrt{3}$	$2cis(-\frac{\pi}{3})$	$16cis\frac{2\pi}{3}$
12	$-8-8i\sqrt{3}$	$2cis(-\frac{\pi}{6})$	$16cis(-\frac{2\pi}{3})$

Questions 31.2.

Q3. (1) circle centre 0, radius 2

(2) circle centre 0, radius 5

(3) line $y = \sqrt{3}x, x > 0$

(4) line $y = \frac{x}{\sqrt{3}}, x < 0$

(5) circle centre 2, radius 2

(6) circle centre $-3i$, radius 2

(7) circle centre $2 - 3i$, radius 2

(8) line $y = \sqrt{3}(x-2), x > 2$

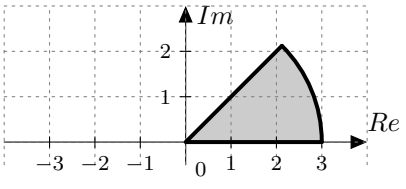
(9) line $y = 1, x < 0$

(10) line $x = 1$

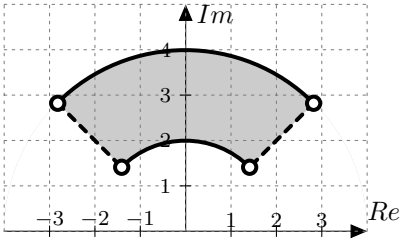
(11) line $y = -\frac{1}{2}x + \frac{3}{2}$

(12) line $y = \frac{4}{3}x - \frac{2}{3}$

(13)



(14)



(15) line $y = \frac{x}{\sqrt{3}}, x \neq 0$

(16) line $y = \sqrt{3}(x+1), x \neq -1$

(17) circle centre 4, radius 2

(18) circle centre $\frac{5}{3}$, radius $\frac{4}{3}$

Q4. $\pm(2+i)$

Q5. $\pm(3-i)$

Q6. (1) $|3z| \leq 9, \frac{\pi}{3} \leq \arg 3z < \frac{\pi}{2}$

(2) $|\frac{z}{2}| \leq \frac{3}{2}, \frac{\pi}{3} \leq \arg \frac{z}{2} < \frac{\pi}{2}$

(3) $|-z| \leq 3, -\frac{2\pi}{3} \leq \arg(-z) < -\frac{\pi}{2}$

(4) $|z^2| \leq 9, \frac{2\pi}{3} \leq \arg z^2 < \pi$

(5) $|z^3| \leq 27, \pi \leq \arg z < \frac{3\pi}{2}$

(6) $|\sqrt{z}| \leq \sqrt{3}, \frac{\pi}{6} \leq \arg z < \frac{\pi}{4}$

Q7. (1) $3 < |3z| \leq 6, -\frac{\pi}{3} \leq \arg 3z < \frac{\pi}{2}$

(2) $\frac{1}{2} < |\frac{z}{2}| \leq 1, -\frac{\pi}{3} \leq \arg \frac{z}{2} < \frac{\pi}{2}$

(3) $1 < |z| \leq 2, \frac{2\pi}{3} \leq \arg(-z) < \frac{3\pi}{2}$

(4) $1 < |z^2| \leq 4, -\frac{2\pi}{3} \leq \arg z^2 < \pi$

(5) $1 < |z^3| \leq 8, -\pi \leq \arg z^3 < \frac{3\pi}{2}$
i.e. no restrictions

(6) $1 < |\sqrt{z}| \leq \sqrt{2}, -\frac{\pi}{6} \leq \arg \sqrt{z} < \frac{\pi}{4}$

(7) $\frac{1}{2} \leq |\frac{1}{z}| < 1, -\frac{\pi}{2} < \arg \frac{1}{z} \leq \frac{\pi}{3}$

(8) $2 < |2z^2| \leq 8, -\frac{2\pi}{3} \leq \arg 2z^2 < \pi$

Q8. (1) $3 \leq |3z|, -\frac{\pi}{2} \leq \arg 3z < -\frac{\pi}{6}$

(2) $\frac{1}{2} \leq |\frac{z}{2}|, -\frac{\pi}{2} \leq \arg \frac{z}{2} < -\frac{\pi}{6}$

(3) $1 \leq |z|, \frac{\pi}{2} \leq \arg(-z) < \frac{5\pi}{6}$

(4) $1 \leq |z^2|, -\pi \leq \arg z^2 < -\frac{\pi}{3}$

(5) $1 \leq |z^3|, -3\frac{\pi}{2} \leq \arg z^3 < -\frac{\pi}{2}$
i.e. no restrictions

(6) $1 \leq |\sqrt{z}|, -\frac{\pi}{4} \leq \arg \sqrt{z} < -\frac{\pi}{12}$

(7) $1 \geq |\frac{1}{z}|, \frac{\pi}{6} < \arg \frac{1}{z} \leq \frac{\pi}{2}$

(8) $2 \leq |2z^2|, -\pi \leq \arg z < -\frac{\pi}{3}$

Q9. —

31.1 De Moivre's theorem and its generalization

Questions 31.3.

Q10. (i) $r = 2$

(ii) $\theta \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\}$

(iii) $z \in \{\sqrt{3}+i, -\sqrt{3}+i, -2i\}$

Q11. (i) ± 1 (ii) $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$ (iii) $\pm 1, \pm i$ (iv) $\pm 1, \pm(\frac{1}{2} \pm i\frac{\sqrt{3}}{2})$

Q12. (i) $z \in \{2, -1 \pm i\sqrt{3}\}$

(ii) $z \in \{-3, \frac{3}{2} \pm i\frac{3\sqrt{3}}{2}\}$

(iii) $z = \pm(\sqrt{2} \pm i\sqrt{2})$

(iv) $z \in \{-3\sqrt{2}, \frac{3\sqrt{2}}{2} \pm i\frac{3\sqrt{6}}{2}\}$

(v) $z \in \{\pm(\sqrt{3}+i), \pm(-1+i\sqrt{3})\}$

(vi) $z \in \{\pm(\frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}), \pm(\frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2})\}$

(vii) $z \in \{\pm\frac{3}{2} + i\frac{\sqrt{3}}{2}, -i\sqrt{3}\}$

Q13. $\cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}, \sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}, \tan\frac{\pi}{12}=2-\sqrt{3}$

Q14. $\cos\frac{5\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}, \sin\frac{5\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}, \tan\frac{5\pi}{12}=2+\sqrt{3}$
$\cos\frac{7\pi}{12}=\frac{\sqrt{2}-\sqrt{6}}{4}, \sin\frac{7\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}, \tan\frac{7\pi}{12}=-2-\sqrt{3}$

Q15. (1) $a=5$ (2) $b=6$ (3) $c=5, d=2$

Q16. (1) $a=0, b=24\sqrt{3}$
(2) $-3+i\sqrt{3}, -2\sqrt{3}i$
(3) $\frac{\sqrt{3}}{2}+\frac{1}{2}i, \frac{\sqrt{3}}{2}+\frac{7}{2}i, -\sqrt{3}+2i$

Q17. modulus, argument:

(1) 5, $\frac{\pi}{2}$
(2) 3, 2
(3) e^2, 2
(4) $4e^2$, -1
(5) πe^2, π
(6) $\sqrt{3}$, 3
(7) 1, $\ln 2$
(8) 3, $\ln 3$
(9) 25, $-3\ln 5$

CHAPTER 32

CONJUGATE ROOTS

32.1 FUNDAMENTAL THEOREM OF ALGEBRA

QUESTIONS 32.1.

Q1. (1) $x=\pm 2i$
(2) $x\in\{-1, \pm i\}$
(3) $x\in\{-1, 3\pm 2i\}$
(4) $x\in\{3, \frac{-1\pm i\sqrt{3}}{2}\}$

32.2 CONJUGATE ROOT THEOREM

QUESTIONS 32.2.

Q2. (1) x^2+4
(2) $x^2-2x+5i$
(3) x^4+13x^2+36
(4) $x^4-2x^3+2x^2-10x+25$

Q3. $2\pm i, \pm 3i$

Q4. $-1\pm 2i, -1\pm 2i\sqrt{6}$

Q5. $-3\pm i\sqrt{3}, 1\pm i\sqrt{3}$

Q6. $2\pm i\sqrt{2}, \frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

Q7. $-3\pm 2i, -\frac{1}{2}\pm i$

Q8. $\pm i, 2\pm i, -2\pm i$

Q9. $\pm i, 1\pm i$

Q10. $2\pm i, 1\pm 2i$

Q11. $\pm 3i, 1\pm i\sqrt{2}$

Q12. (i) —
(ii) —
(iii) $z^4+4z^2+6+4z^{-2}+z^{-4}$
(iv) $\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$
(v) $\frac{1}{4}+\frac{3\pi}{32}$

Q13. (i) —
(ii) —
(iii) $z^5-5z^3+10z-\frac{10}{z}+\frac{5}{z^3}-\frac{1}{z^5}$
(iv) $\frac{1}{16}\sin 5\theta-\frac{5}{16}\sin 3\theta+\frac{5}{8}\sin\theta$
(v) $\frac{8}{15}$

Q14. —

PART IX

STATISTICS

CHAPTER 33

POPULATION, SAMPLE, TYPES OF DATA

CHAPTER 34

PRESENTATION OF DATA

QUESTIONS 34.1.

Q1. mean height $\approx 160cm$

h	f	$h \times f$
152	4	608
153	2	306
157	3	471
160	3	480
163	6	978
165	3	495
168	3	504

Q2. (i) mode is $163cm$

(ii) $Q_0 = 152cm$

(iii) $Q_4 = 168cm$

(iv) range is $16cm$

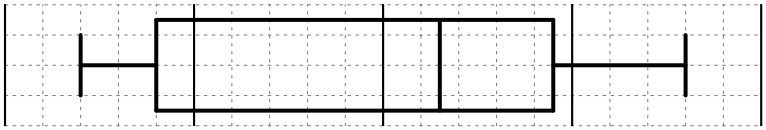
(v) median is $Q_2 = 161.5cm$

(vi) $Q_1 = 154cm$

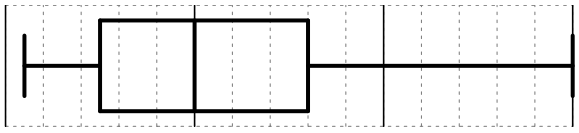
(vii) $Q_3 = 164.5cm$

(viii) I.Q.R. $= 10.5cm$

Q_0	Q_1	Q_2	Q_3	Q_4
152	154	161.5	164.5	168



Q3. (i) 70 (ii) 65 (iii) 76 (iv)



Q4. (i)

no of sand.	freq.	cum.freq.
0	6	6
1	14	20
2	48	68
3	17	85
4	9	94
5	6	100

(ii) median $= 2$, $Q_1 = 2, Q_3 = 3$

(iii)



(iv) TRUE, FALSE, FALSE, FALSE, TRUE

Q5. FALSE, FALSE, FALSE, FALSE

CHAPTER 35

MEASURES OF TENDENCY, MEASURES OF SPREAD

QUESTIONS 35.1.

Q1. (i) mean = 172.5 (ii) range = 22

Q2. 164, 178

Q3. **SET A**: mean= 24.5, st.dev.= 3.64, **SET B**: mean= 24.5, st.dev.= 1.71

Q4. (i) — (ii) A: 46.2, B: 15.8

Q5.

$\mu-2\sigma$	$\mu-\sigma$	μ	$\mu+\sigma$	$\mu+2\sigma$
397	425	453	481	509

(i) 68% (ii) 95% (iii) 2.5% (iv) 16% (v) 97.5% (vi) 97.5%

Q6.

$\mu-2\sigma$	$\mu-\sigma$	μ	$\mu+\sigma$	$\mu+2\sigma$
20	24	28	32	36

(i) 68% (ii) 95% (iii) 16% (iv) 2.5% (v) 84% (vi) 97.5%

CHAPTER 36

CUMULATIVE FREQUENCY

QUESTIONS 36.1.

Q1.

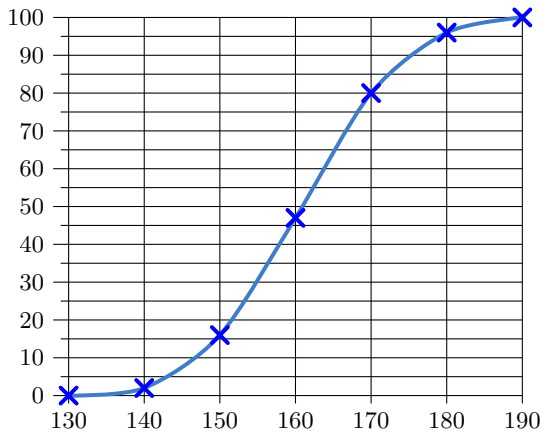
h	f	x	xf
$130 < h \leq 140$	2	135	270
$140 < h \leq 150$	14	145	2030
$150 < h \leq 160$	31	155	4805
$160 < h \leq 170$	33	165	5445
$170 < h \leq 180$	16	175	2800
$180 < h \leq 190$	4	185	740

mean height $\approx 160.9cm$

Q2.

height	freq.	cum.freq.
$h \leq 140$	2	2
$h \leq 150$	14	16
$h \leq 160$	31	47
$h \leq 170$	33	80
$h \leq 180$	16	96
$h \leq 190$	4	100

Q3.



(i) median ≈ 162 (ii) $Q_1 \approx 154$ (iii) $Q_3 \approx 168$ (iv) $I.Q.R. \approx 14$

Q4. (i) median: $18 - 19cm$ (iii) $I.Q.R. = 8$

(ii) $Q_1 = 15, Q_3 = 23$ (iv) $a = 12, b = 28$

PART X

PROBABILITY

CHAPTER 37

COUNTING PRINCIPLES

QUESTIONS 37.1.

Q1. (1) 45
(2) 650
(3) 1640
(4) 560
(5) 1320
(6) 650
(7) 84
(8) 1001

Q2. 4

Q3. (1) 4
(2) 3
(3) 6
(4) 6

Q4. —

Q5. —

Q6. (i) Use $(1+1)^n$
(ii) 2^n

Q7. (1) 120
(2) 3125

Q8. (1) 729
(2) 90
(3) $3^6-(3\times 2^6-3)=540$

Q9. (1) 60
(2) 720

Q10. (1) 16807
(2) 2520
(3) 9604
(4) 1440

Q11. (1) 14406
(2) 2160
(3) 8232
(4) 1260

Q12. 240

Q13. 480

Q14. 288

Q15. (1) 288
(2) 576
(3) 144

Q16. (1) 1287
(2) 1266
(3) 1260
(4) 945

Q17. (1) 5040
(2) 1440
(3) 3600

Q18. (1) 120
(2) 12
(3) 36

Q19. 720

Q20. 210

Q21. 34650

Q22. (1) 495
(2) 210
(3) 285

Q23. (1) 362880
(2) 282240
(3) 1728

Q24. (1) 15120
(2) 4200
(3) 720
(4) 480
(5) 120
(6) 3024
(7) 504
(8) 360
(9) 144
(10) 72

Q25. (1) 2598960

(2)
- **Straight flush** - 36
- **Four of a Kind** - 624
- **Full House** - 3744
- **Flush** - 5112
- **Straight** - 10200
- **3 of a Kind** - 54912
- **Two Pair** - 123552
- **One Pair** - 1098240
- **High Cards** - 1302540

Q26. (1) 1677106640
(2) 1086008
(3) 40
(4) $\approx 4.42\times 10^{11}$
(5) $\approx 2.32\times 10^{11}$
(6) $\approx 1.07\times 10^{11}$

Q27. (1) $4^4 = 256$ (2) $5^5 = 3125$ (3) $5^4 = 625$ (4) $4^5 = 1024$ (5) 24 (6) 24 (7) 120 (8) 0 (9) 0 (10) 240

CHAPTER 38

INTRODUCTION TO PROBABILITY

38.1 CONCEPT OF PROBABILITY

38.2 FORMAL APPROACH

38.3 AXIOMS OF PROBABILITY

38.4 EXEMPLARY SIMPLE PROBABILITY PROBLEMS

38.5 COUNTING PRINCIPLES

QUESTIONS 38.1.

Q1. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) $\frac{1}{3}$

Q2. 0.3

Q3. (i) $\frac{1}{7}$ (ii) $\frac{4}{7}$ (iii) $\frac{5}{7}$

Q4. (i) 0.678 (ii) 0.283 (iii) 0.0391 (iv) 0.407 (v) 0.497 (vi) 0.0957 (vii) 0.22 (viii) 0.26 (ix) 0.52

CHAPTER 39

CONDITIONAL PROBABILITY

39.1 COMBINED EVENTS

QUESTIONS 39.1.

Q1. all 4 are true

Q2. (i) 0.61 (ii) 0.11 (iii) 0.22 (iv) 0.39

Q3. (i) 0.9 (ii) 0.2 (iii) 0.2 (iv) 0.9

Q4. (i) 0.85 (ii) 0.25 (iii) 0.6 (iv) 0.2 (v) 0.8 (vi) 0.6

Q5. (i) 0.35 (ii) 0.15 (iii) 0.85

Q6. (i) 0.35 (ii) 0.9 (iii) 0.35

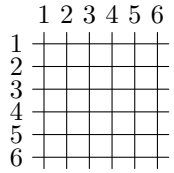
Q7. (i) p^2 (ii) $p - p^2$ (iii) $1 - p^2$

Q8. (i) $\frac{1}{7}$ (ii) $\frac{4}{7}$ (iii) $\frac{3}{7}$

Q9. (i) $\frac{2}{13}$ (ii) $\frac{4}{13}$

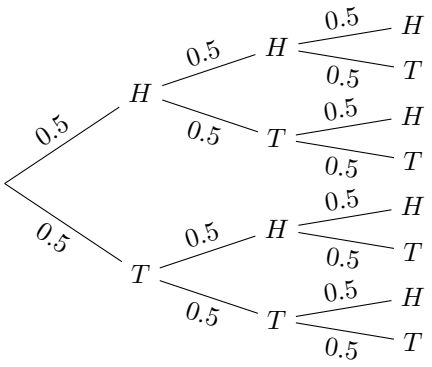
Q10. (i) $\frac{9}{25}$ (ii) $\frac{13}{25}$

Q11. (i)



(ii) $\frac{1}{36}$
(iii) $\frac{1}{6}$
(iv) $\frac{5}{12}$
(v) $\frac{1}{6}$
(vi) $\frac{7}{12}$

Q12. (i)



$U = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(ii) $\frac{3}{8}$
(iii) $\frac{3}{8}$
(iv) $\frac{1}{2}$
(v) $\frac{1}{4}$

39.2 Conditional probability

39.3 Independent events

39.4 The law of total probability

39.5 Bayes' theorem

Questions 39.2.

Q13. (i) 0.2 (ii) $\frac{2}{3}$ (iii) $\frac{2}{7}$ (iv) $\frac{5}{7}$ (v) NO

Q14. (i) 0.8 (ii) 0.6 (iii) 0.5 (iv) 0.5 (v) YES

Q15. (i) 0.3 (ii) 0.4 (iii) 0.75 (iv) 0.6 (v) YES

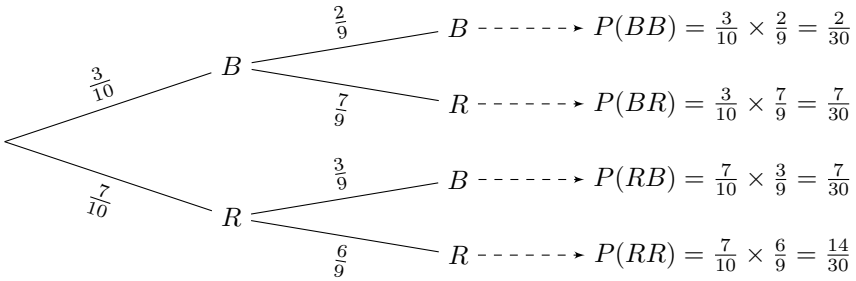
Q16. (i) 0.7 (ii) $\frac{6}{7}$ (iii) $\frac{6}{13}$ (iv) $\frac{7}{13}$ (v) NO

Q17. (i) 0 (ii) 0 (iii) 0 (iv) 0.5

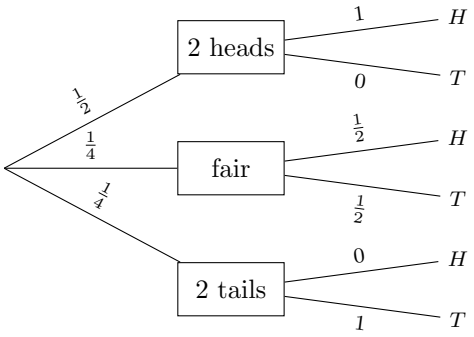
Q18. (i) 0.89 (ii) 0.2 (iii) 0.45 (iv) 0.55

Q19. (i) 0.24 (ii) 0.5 (iii) 0.52 (iv) 0.5

Q20. (i)



(ii) $\frac{7}{15}$
(iii) $\frac{7}{10}$
(iv) $\frac{7}{30}$
(v) $\frac{14}{15}$
(vi) $\frac{7}{15}$
(vii) $\frac{7}{9}$
(viii) $\frac{6}{9}$
(ix) $\frac{7}{10}$
(x) $\frac{1}{2}$



Q21. (i) (ii) $\frac{5}{8}$ (iii) $\frac{1}{5}$

Q22. (i) $\frac{4}{9}$ (ii) $\frac{4}{9}$

Q23. (i) $\frac{1}{3}$ (ii) $\frac{4}{9}$

Q24. (i) $\frac{31}{45}$ (ii) $\frac{1}{5}$ (iii) $\frac{5}{9}$ (iv) $\frac{2}{9}$

Q25. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{4}$ (iv) 1 (v) $\frac{1}{3}$

Q26. (i) $\frac{5}{36}$ (ii) $\frac{1}{5}$ (iii) $\frac{2}{5}$ (iv) $\frac{4}{5}$ (v) $\frac{2}{15}$

Q27. (i) 0.0382 (ii) 0.962 (iii) 2.11×10^{-5}

Q28. —

Q29. (i) $\frac{1}{20}$ (ii) $\frac{7}{20}$ (iii) $\frac{4}{7}$

Q30. (i) 0.48 (ii) 0.1 (iii) 0.64

Q31. (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{7}{8}$ (iv) $\frac{3}{7}$

Q32. (i) $\frac{8}{27}$ (ii) $\frac{4}{9}$ (iii) $\frac{26}{27}$ (iv) $\frac{6}{13}$

Q33. (i) 0.45 (ii) 0.25 (iii) $\frac{1}{3}$ (iv) $\frac{2}{11}$

Q34. (i) 0.041 (ii) $\frac{6}{41}$ (iii) $\frac{35}{41}$

Q35. (i) $\frac{6}{35}$ (ii) $\frac{23}{35}$ (iii) $\frac{18}{35}$ (iv) $\frac{14}{23}$ (v) $\frac{6}{23}$

Q36. (i) $\frac{9}{35} \approx 0.257$, (ii) $\frac{19}{210} \approx 0.0905$, (iii) $\frac{13}{35} \approx 0.371$, (iv) $\frac{5}{26} \approx 0.192$, (v) $\frac{35}{66} \approx 0.53$.

Q37. (i) 0.09, (ii) $\frac{2}{9} \approx 0.222$, (iii) $\frac{17}{91} \approx 0.187$.

Questions 39.3.

Q1. It is twice better to switch.

Q2. $\frac{1}{5}$

Q3. $\frac{3}{8}$

Q4. 0.179

Q5. (i) $\frac{1}{18}$ (ii) $\frac{1}{36}$

Q6. 5

Q7. $\frac{13}{20}$

Q8. (i) $\frac{16}{55}$ (ii) $\frac{4}{11}$ (iii) $\frac{19}{55}$

CHAPTER 40

Discrete random variables

40.1 Discrete random variables

Questions 40.1.

Q1. 0.45

Q2. 0.1

Q3. (i) 0.5 (iii) 0.75 (v) 0.75
(ii) 0 (iv) 1

Q4. (i) $\{0, 1, 2\}$ (ii)

d	0	1	2
$P(D = d)$	0.1	0.6	0.3

(iii) $\frac{1}{3}$

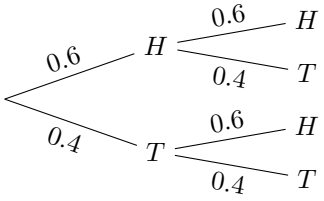
Q5. (i) $\frac{7}{12}$ (ii) $\frac{1}{2}$ (iii) $\frac{3}{7}$

Q6. (i)

x	2	3	4	5	6
$P(X = x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

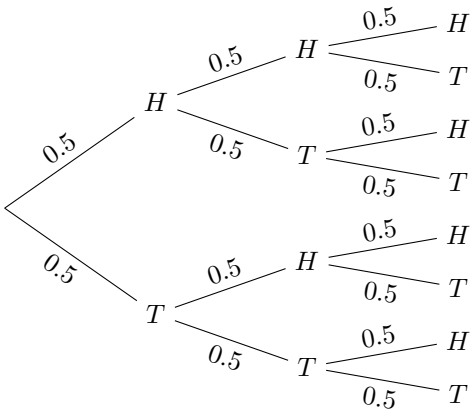
(ii) $\frac{5}{8}$ (iv) $\frac{1}{3}$ both
(iii) $\frac{2}{3}$

Q7. (i)



(ii) 0.64
(iii) $\frac{3}{7}$
(iv) (a) 0.4
(b) 0.24
(c) 0.144
(v) 0.0187
(vi) 0.36
(vii) 0.4704
(viii) 0.64
(ix) 10

Q8. (i)



(ii)

t	0	1	2	3
$P(T = t)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(iii) $\frac{1}{2}$
(iv) $\frac{3}{4}$

Q9. (i)

y	0	2
$P(Y = y)$	$\frac{7}{12}$	$\frac{5}{12}$

(ii) $\frac{7}{12}$

Q10. (i)

b	0	1	2
$P(B = b)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

(ii) $\frac{2}{3}$
(iii) $\frac{1}{2}$

Q11.

x	0	1	2
$P(X = x)$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

Q12. (i)

x	0	1	2	3	4	5
$P(X = x)$	0.0298	0.192	0.383	0.298	0.0894	0.00813

(ii) 0.970
(iii) 0.492

Q13. (i)

l	0	1	2	3
$P(L = l)$	0.399	0.461	0.132	0.00877

(ii) 0.601
(iii) 0.219

Q14. (i) 10 (ii) (a) $\frac{7}{10}$ (b) $\frac{7}{9}$

Q15. (i) $\frac{1}{26}$ (ii) (a) $\frac{25}{26}$ (b) $\frac{10}{11}$

Q16. (i)

x	0	0.5	1	1.5	2	2.5	3	5
$F(x)$	0	0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$	1	1

(ii)

x	0	0.5	1	1.5	2	2.5	3	5
$P(X = x)$	0	0	$\frac{1}{9}$	0	$\frac{1}{3}$	0	$\frac{5}{9}$	0

40.2 Expected value of a discrete random variable
40.3 The meaning of expected value
40.4 Properties of expected value of a discrete random variable
40.5 Variance of a discrete random variable
40.6 Properties of variance of a discrete random variable

Questions 40.2.

Q17. (i) 2.85
(ii) 1.43

Q18. (i) 4.65
(ii) 19.4
(iii) -6.3
(iv) 8.82
(v) 19.85
(vi) 1740

Q19. (i) 0.5
(ii) 1.75
(iii) 2
(iv) 7
(v) 2
(vi) 2.5
(vii) 1.75
(viii) $-\frac{1}{3}$
(ix) $\frac{25}{36}$

Q20. $E(B) = 1, Var(B) = \frac{1}{3}$

Q21. $E(X) = \frac{2}{3}, Var(X) = \frac{16}{45}$

Q22. $E(X) = 2.25, Var(X) \approx 0.977$

Q23. (i) 10
(ii) $E(X) = 3, Var(X) = 1$

Q24. (i) $\frac{1}{26}$
(ii) $E(X) \approx 11.9, Var(X) \approx 24.8$

Q25. $E(X) = 7, Var(X) = \frac{35}{6}$

Q26. 12.25

Q27. $E(X) = 1.5, Var(X) \approx 0.866$

Q28. $E(X) = 0.6, Var(X) = \frac{28}{75}$

Q29. $E(X) = 1, Var(X) = 0.75$

Q30. $E(X) = \frac{2}{3}, Var(X) = \frac{4}{9}$

Q31. $E(X) = \frac{8}{9}, Var(X) = \frac{44}{81}$

Q32. No, it's a loosing game.

Q33. should win $\$\frac{8}{3}$,
i.e. the game cannot be fair

Q34. \$1.50

Q35. (i) 1
(ii) 4
(iii) 1.25

Q36. (i) 2.68
(ii) 8.91
(iii) 1.68

chapter 41

Binomial distribution

Questions 41.1.

Q1. (i) 0.118
(ii) 7.29×10^{-4}
(iii) 0.324
(iv) 0.580
(v) 0.579
(vi) 1.8
(vii) 1.12

Q2. (i) 5.08×10^{-10}
(ii) 4.24×10^{-32}
(iii) 5.49×10^{-4}
(iv) 0.331
(v) 0.838
(vi) 18
(vii) 3.55

Q3. (i) 0.0000276
(ii) 0.154
(iii) 0.249

Q4. (i) 9.09×10^{-13}
(ii) 0.00992
(iii) 0.0409
(iv) 0.00317
(v) 5
(vi) 15

Q5. (i) 0.282
(ii) 10^{-12}
(iii) 0.111
(iv) 20

Q6. (i) 0.329
(ii) 0.649
(iii) 0.351
(iv) 0.320
(v) 0.338

(vi) 0.493
(vii) 0.610
(viii) 0.307

Q7. (i) $\frac{1}{3}$
(ii) 0.273
(iii) 0.961
(iv) 0.0658
(v) at least 6

Q8. (i) $p = 0.3$, $n = 20$
(ii) 0
(iii) 0.107
(iv) 0.238
(v) 0.113
(vi) 0.392
(vii) 0.346
(viii) 0.209

Q9. (i) 0.100
(ii) 0.0865
(iii) 0.973
(iv) 0.433
(v) 0.00317
(vi) 0.320
(vii) 0.505

CHAPTER 42

POISSON DISTRIBUTION

QUESTIONS 42.1.

Q1. (i) 0.223
(ii) 0.603
(iii) 0.380
(iv) 0.959
(v) 0.829
(vi) 0.381
(vii) 0.614
(viii) 0.359
(ix) 0.830
(x) 3
(xi) 3
(xii) 3.2

Q2. (i) 8.78
(ii) 0.00592
(iii) 0.130
(iv) 0.993
(v) 0.128
(vi) 0.993
(vii) 9
(viii) 8
(ix) 2.96

Q3. (i) 4.20
(ii) 0.589
(iii) 0.247
(iv) 0.395
(v) 0.789
(vi) 0.679

Q4. (i) 2.80
(ii) 0.222
(iii) 0.308
(iv) 0.935
(v) 0.300
(vi) 0.0264

Q5. (i) 9.12×10^{-4}
(ii) 2.75×10^{-5}
(iii) 0.497
(iv) 0.966
(v) 0.156
(vi) 0.691
(vii) 0.0114
(viii) 0.514
(ix) 0.126
(x) 0.486

Q6. (i) 0.670
(ii) 0.551
(iii) 0.813

Q7. (i) 7.35×10^{-11}
(ii) 1
(iii) 0.813

Q8. (i) 0.247
(ii) 0.308
(iii) 0.779
(iv) 0.458

Q9. (i) 0.435
(ii) 0.203
(iii) 0.137
(iv) 0.134

Q10. (i) 0.393
(ii) 0.0144
(iii) 4.54×10^{-5}
(iv) 5.89×10^{-4}
(v) 0.226
(vi) 0.00426

Q11. (i) 0.741
(ii) 2.04×10^{-5}
(iii) 0.655
(iv) 0.474

Q12. (i) 0.0380
(ii) 0.00149
(iii) 0.956
(iv) 0.998
(v) 0.958
(vi) 0.958
(vii) 0.159
(viii) 0.683
(ix) 0.325

Q13. (i) 0.997
(ii) 0.351
(iii) 0.0596
(iv) 0.999
(v) 2.06×10^{-9}
(vi) 4.12×10^{-8}
(vii) 1(0.99992...)

Q14. (i) 0.4
(ii) 0.191
(iii) 7

CHAPTER 43

CONTINUOUS RANDOM VARIABLES

QUESTIONS 43.1.

Q1. (i) —
(ii) $3a$
(iii) $\frac{1}{3}$
(iv) see below
(v) $\frac{2}{3}$
(vi) $\frac{2}{3}$
(vii) no mode
(viii) 1
(ix) 1
(x) $\frac{3}{4}$
(xi) $\frac{1}{6}$ and $\frac{1}{2}$
(xii) $\begin{cases} 0 & \text{for } x < -0.5 \\ \frac{1}{3}(x + \frac{1}{2}) & \text{for } -0.5 \leq x \leq 2.5 \\ 1 & \text{for } x > 2.5 \end{cases}$

Q2. (i) —
(ii) $\frac{4}{b}$
(iii) 4
(iv) see below
(v) 0
(vi) 1
(vii) $4-\sqrt{5}$
(viii) $\frac{11}{6}$
(ix) $\frac{11}{36}$
(x) $\begin{cases} 0 & \text{for } x<1 \\ -\frac{1}{8}(x^2-8x+7) & \text{for } 1\le x\le 3 \\ 1 & \text{for } x>3 \end{cases}$

Q3. (i) —
(ii) $\ln(c+1)$
(iii) $e-1$
(iv) see below
(v) $\ln 2$
(vi) 0
(vii) $\sqrt{e}-1$
(viii) $e-2$
(ix) $\frac{-e^2+4e-3}{2} \approx 0.242$
(x) $\begin{cases} 0 & \text{for } x<0 \\ \ln(x+1) & \text{for } 0\le x\le e-1 \\ 1 & \text{for } x>e-1 \end{cases}$

Q4. (i) —
(ii) $\frac{10k}{3}$
(iii) 0.3
(iv) see below
(v) 0.65
(vi) 1.5
(vii) c.1.76
(viii) 1.8
(ix) 0.24

Q5. (i) —
(ii) $2s$
(iii) $\frac{1}{2}$
(iv) see below
(v) 1
(vi) $\frac{1}{2}$
(vii) $\ln 3$
(viii) $\ln 2$
(ix) $\frac{3}{2}\ln 3-1$
(x) 0.0948

Q6. (i) —
(ii) $2t$
(iii) $\frac{1}{2}$
(iv) see below
(v) 0.518
(vi) 0
(vii) 0
(viii) 0
(ix) 0.352

Q7. (i) —
(ii) $\frac{r}{3}$
(iii) 3
(iv) see below
(v) $\frac{1}{2}$
(vi) $\frac{1}{2}$
(vii) $\frac{1}{2}$
(viii) $\frac{1}{2}$
(ix) 0.0125

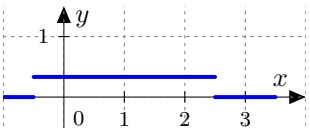
Q8. (i) —
(ii) 1
(iii) —
(iv) see below
(v) 0
(vi) $\ln 2$
(vii) 1
(viii) 1

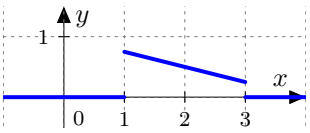
Q9. (i) $\frac{1}{9}$ (ii) 1.77 (iii) $\frac{3}{\sqrt{2}} \approx 2.12$ (iv) 1.82 (v) 0.691

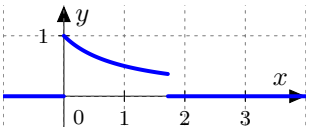
Q10. (i) 4 (ii) 1.21 (iii) 1.18 (iv) 0.917 (v) 1

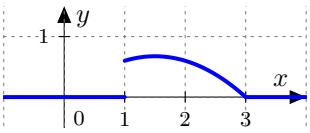
Q11. (i) $a = 0.7, b = 0.2$ (ii) $\frac{3}{14} \approx 0.214$ (iii) $\frac{\sqrt{14}+1}{7} \approx 0.677$

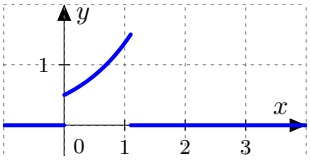
GRAPHS:

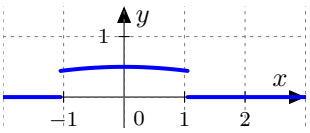
Q1.


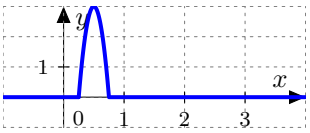
Q2.


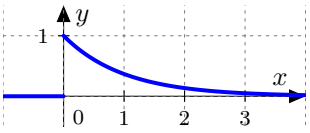
Q3.


Q4.


Q5.


Q6.


Q7.


Q8.


CHAPTER 44

NORMAL DISTRIBUTION

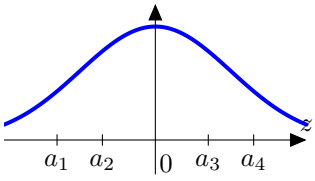
QUESTIONS 44.1.

Q1. (i) (1) 0.692
(2) 0.998
(3) 1
(4) 0.5
(5) 0.268
(6) 0.032
(7) 0.841
(8) 0.979
(9) 0.813

(10) 0.0108
(11) 0.0793
(12) 0.00135
(13) 0.153
(14) 0.138
(15) 0.206
(16) 0.683
(17) 0.695
(18) 0.856

(ii) (1) 0.345
(2) 0.931
(3) 3.09
(4) -0.192
(5) -0.504
(6) -1.41
(7) -0.643
(8) -0.482
(9) -1.28
(10) 0.675
(11) 0.410
(12) 1.96

Q2. (i)



(ii) (1) $1 - p_1$
(2) $1 - p_4$
(3) $p_4 - p_2$
(4) $p_3 - p_2$
(5) p_1
(6) $1 - p_4$
(7) $2p_3 - 1$
(8) $1 - 2p_1$
(9) $p_4 - p_3$
(10) $p_2 - p_1$
(11) $p_3 + p_4 - 1$
(12) $p_3 + p_4 - 1$

(iii) (1) $-\infty, a_1$
(2) $-\infty, a_4$
(3) a_1, a_2
(4) $a_1, +\infty$
(5) $a_3, +\infty$
(6) a_1, a_4
(7) a_2, a_3
(8) a_3, a_4

Q3. (1) 0
(2) 1
(3) $P(Z < 1) \approx 0.841$
(4) $P(Z < \frac{a-5}{2}), a \approx 4.65$

Q4. (1) 0.841
(2) 0.933
(3) 0.309
(4) 0.629
(5) 0.212
(6) 0.341
(7) 0.323
(8) 0.657
(9) 0.497

Q5. (1) 0.988
(2) 0.749
(3) 0.027
(4) 0.922
(5) 0.254
(6) 0.377
(7) 0.251
(8) 0.716
(9) 0.652

Q6. (1) 0.742
(2) 0.298
(3) 0.742
(4) 0.115
(5) 0.324
(6) 0.505

Q7. (1) 8.36
(2) 9.19
(3) 9.31
(4) 5.81
(5) 4.94
(6) 2.6

Q8. (1) 65.8
(2) 65.6
(3) 64.6
(4) 61.4
(5) 62.0
(6) 60.4

Q9. (1) 40.1%
(2) 1.22%
(3) 93.1%
(4) $m = 1005g$
(5) $d = 999g$

Q10. (1) 59.3%
(2) 0.0228
(3) 0.53
(4) 4 min.15 sec.
(5) 3 min.29 sec.

Q11. (1) $h_1 \approx 26.8cm$
(2) $h_2 \approx 29.8cm$
(3) 3.08

Q12. (1) 0.276
(2) 6.61×10^{-6}
(3) 0.516
(4) 503

Q13. (1) 43.7
(2) 1.22%
(3) 62.3 and higher
(4) 4.78×10^{-6}

Q14. $172cm$

Q15. $57.3kg$

Q16. 1.17

Q17. 1.69

Q18. $\mu = 60.5, \sigma = 17.4$

Q19. $\mu = 70.6, \sigma = 27.4$

Q20. $\mu = 1.67, \sigma = 0.658$

Q21. $\mu = 858, \sigma = 111$

Q22. $\mu = 17.7, \sigma = 3.37$

Q23. $\mu = 20, \sigma = 10.4$

Q24. $\mu = 16.3, \sigma = 5.69$

Q25. $\mu = 131, \sigma = 11.3$

Q26. $\mu = 25, \sigma = 7.72$