pre-IB Mathematics SL & HL

Krzysztof Sikora

May 25, 2018



Copyright © 2013 - 2018 by Krzysztof Sikora.
This work is made available under the terms of the Creative Commons Attribution-ShareAlike 4.0 license, http://creativecommons.org/licenses/by-sa/4.0/.
If you decide to use some of my concepts, I would like you to email me (krzyssikora@gmail.com) so that I will be able to notify you about major changes and new versions of the textbook. I would be happy to share some more of my materials with you. You will not have to pay anything for that. All I expect from you is that you accept the conditions of a free licence that are explained below.

- **Creative Commons** is a nonprofit organization that enables the sharing and use of creativity and knowledge through free legal tools. For this work I decided to use by-nc-sa licence.
- **Attribution** - You must give appropriate credit, provide a link to the license, and indicate if changes were made. You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use.
- **ShareAlike** - If you remix, transform, or build upon the material, you must distribute your contributions under the same license as the original.

Contents

Contents **3**

1 Numbers **5**
1.1 Primes, factors and divisibility . . . 5
1.2 Fractions and decimals . . . 6
1.3 Subsets of real numbers set . . . 7
1.4 Absolute value . . . 7
1.5 Percentages . . . 7
1.6 Approximations. Decimal places and significant figures. . . . 10
1.7 Exponents and roots . . . 10
1.8 Expansions . . . 12

2 Sets **13**
2.1 Sets and subsets . . . 13
2.2 Venn diagrams . . . 14
2.3 Operations on sets . . . 16
2.4 Chapter review . . . 18

3 Lines and vectors **20**
3.1 Basic concepts . . . 20
3.2 *Slope-intercept* equation of a line . . . 20
3.3 *General* equation of a line . . . 22
3.4 Vectors . . . 23
3.5 Simultaneous equations . . . 27
3.6 Applications of linear equations and vectors . . . 27
3.7 Chapter review . . . 31

4 Functions **34**
4.1 Basic properties . . . 34
4.2 Transformations of graphs of functions . . . 38
4.3 Equations and inequalities . . . 44
4.4 Chapter review . . . 45

5 Quadratics **48**
5.1 Solving quadratic equations . . . 48
Factorisation . . . 48
Completing the square . . . 49
Quadratic formula . . . 50
5.2 Parabola . . . 51
5.3 Applications of quadratics . . . 52
Quadratic inequalities . . . 52
Problems involving quadratics . . . 52
Investigating graphs of rational functions . . . 54

6 Trigonometry **55**

6.1 Degrees and radians . . . 55
6.2 Trigonometric ratios . . . 55
6.3 Trigonometric functions . . . 58
6.4 Trigonometric equations . . . 59
6.5 Trigonometry in geometry . . . 60
6.6 Arcs, sectors, segments . . . 63
6.7 Chapter review . . . 64

7 Statistics **67**
7.1 Types of data . . . 67
7.2 Averages, range, quartiles . . . 68
7.3 Grouped data, frequencies . . . 70
7.4 Miscellaneous problems . . . 76

8 Geometry **81**
8.1 Polygons . . . 81
8.2 Circles . . . 81
8.3 Similarity . . . 83
8.4 Solid geometry . . . 84
8.5 Miscellaneous problems . . . 85

9 Numbers II **88**
9.1 Factorials, Pascal's triangle and binomial theorem . . . 88
9.2 Logarithms . . . 91
Algebra of logarithms . . . 92
Logarithmic equations . . . 94
Aplications . . . 94
9.3 Absolute value equations and inequalities . . . 95
9.4 Complex numbers . . . 96
9.5 Mathematical induction . . . 96

10 Quadratics, circles and polynomials **98**
10.1 Vieta's formulae for quadratics . . . 98
10.2 Algebraic fractions . . . 99
10.3 Equation of a circle . . . 100
10.4 Polynomials . . . 101
10.5 Chapter Review . . . 103

Chapter 1

Numbers

1.1 Primes, factors and divisibility

Q1. Write down all prime numbers smaller than 100.

Q2. Which of the following numbers are co-prime? Write down their Greatest Common Divisor if they are not.

(1) 12 and 15
(2) 12 and 25
(3) 21 and 25
(4) 49 and 84
(5) 199 and 201
(6) 198 and 201
(7) 71 and 78
(8) 72 and 77
(9) 52 and 78
(10) 202 and 188
(11) 39 and 42
(12) 39 and 52
(13) 120 and 108
(14) 35 and 38

Q3. Find the Greatest Common Divisor and the Least Common Multiple of the following pairs of numbers.

(1) 18 and 30
(2) 32 and 48
(3) 24 and 36
(4) 54 and 72
(5) 90 and 126
(6) 24 and 56
(7) 56 and 63
(8) 81 and 72
(9) 64 and 56
(10) 99 and 198
(11) 2001 and 2002
(12) 2001 and 2003
(13) 2000 and 2002
(14) 85 and 119
(15) 42 and 70
(16) 49 and 35
(17) 132 and 242
(18) 625 and 1000
(19) 625 and 400
(20) 128 and 80
(21) 216 and 192
(22) 104 and 117

Q4. Express the following integers as products of their prime factors.

(1) 18
(2) 24
(3) 30
(4) 32
(5) 36
(6) 42
(7) 48
(8) 54
(9) 56
(10) 63
(11) 64
(12) 72
(13) 80
(14) 81
(15) 90
(16) 99
(17) 117
(18) 128
(19) 192
(20) 196
(21) 216
(22) 242
(23) 294
(24) 400
(25) 625
(26) 1000
(27) 2013
(28) 92
(29) 96
(30) 100
(31) 144
(32) 162
(33) 224
(34) 240
(35) 289
(36) 484
(37) 529
(38) 576
(39) 675
(40) 676
(41) 1001
(42) 1089
(43) 1125
(44) 2312
(45) 6300

Q5. Find the remainder when number a is divided by number b.

(1) $a = 13, \quad b = 3$
(2) $a = 14, \quad b = 4$
(3) $a = 18, \quad b = 5$
(4) $a = 23, \quad b = 3$
(5) $a = 23, \quad b = 4$
(6) $a = 23, \quad b = 6$
(7) $a = 28, \quad b = 5$
(8) $a = 28, \quad b = 7$
(9) $a = 48, \quad b = 7$
(10) $a = 48, \quad b = 11$
(11) $a = 55, \quad b = 6$
(12) $a = 55, \quad b = 7$
(13) $a = 75, \quad b = 6$
(14) $a = 75, \quad b = 7$
(15) $a = 120, \quad b = 6$
(16) $a = 120, \quad b = 11$

Q6. Without a calculator state, which of the following numbers are divisible by 2, 3, 4, 5, 6, 9, 10.

(1) 5303145
(2) 517380
(3) 4849845
(4) 2316537
(5) 33698267
(6) 3964502
(7) 1167424
(8) 24760990
(9) 26640625
(10) 7094372
(11) 60445656
(12) 6885714

1.2 Fractions and decimals

Q7. Which of the following fractions can be expressed as terminating decimals and which as recurring decimals?

(1) $\frac{4}{9}$
(2) $\frac{15}{6}$
(3) $\frac{15}{16}$
(4) $\frac{3}{40}$
(5) $\frac{9}{4}$
(6) $\frac{3}{32}$
(7) $\frac{3}{33}$
(8) $\frac{49}{14}$
(9) $\frac{14}{49}$
(10) $\frac{12}{15}$
(11) $\frac{15}{12}$
(12) $\frac{14}{18}$
(13) $\frac{18}{14}$
(14) $\frac{39}{15}$
(15) $\frac{35}{15}$
(16) $\frac{15}{35}$
(17) $\frac{14}{35}$
(18) $\frac{81}{1600}$
(19) $\frac{28}{125}$
(20) $\frac{169}{6250}$

Q8. Express the following decimals as fractions in simplified form.

(1) 1.24
(2) 2.85
(3) 0.00625
(4) 3.375
(5) 5.625
(6) 7.44
(7) 12.125
(8) 75.875
(9) $2.\dot{4}$
(10) $3.\dot{1}\dot{2}$
(11) $5.\dot{5}$
(12) $7.\dot{5}\dot{7}$
(13) $14.\dot{1}\dot{4}$
(14) $0.\dot{3}2\dot{1}$
(15) $4.\dot{6}0\dot{6}$
(16) $5.\dot{2}0\dot{4}$
(17) $0.5\dot{1}$
(18) $1.2\dot{1}$
(19) $3.6\dot{3}$
(20) $5.5\dot{2}\dot{4}$
(21) $6.0\dot{1}\dot{5}$
(22) $1.3\dot{1}4\dot{1}$
(23) $7.2\dot{4}5\dot{3}$
(24) $9.0\dot{0}0\dot{6}$

Q9. Express the following fractions as decimals.

(1) $\frac{5}{8}$
(2) $\frac{3}{16}$
(3) $\frac{7}{8}$
(4) $\frac{11}{40}$
(5) $\frac{3}{125}$
(6) $\frac{7}{9}$
(7) $\frac{2}{9}$
(8) $\frac{2}{11}$
(9) $\frac{6}{22}$
(10) $\frac{2}{7}$
(11) $\frac{5}{6}$
(12) $\frac{5}{18}$
(13) $\frac{5}{12}$
(14) $\frac{3}{22}$
(15) $\frac{2}{33}$
(16) $\frac{5}{37}$

1.3 Subsets of real numbers set

Q10. Which of the following numbers are

- rational ($\mathbb{Q}$)
- irrational ($\mathbb{R} \setminus \mathbb{Q}$)
- integer ($\mathbb{Z}$)
- natural ($\mathbb{N}$)?

(1) $6\sqrt{6}$
(2) $\frac{5}{\sqrt{5}}$
(3) $\frac{4}{\sqrt{4}}$
(4) 1.4
(5) $1.\dot{4}$
(6) $3\frac{11}{17}$
(7) $5.0\dot{3}$
(8) $31.2\dot{7}2\dot{3}$
(9) $\sqrt{2} \times \sqrt{32}$
(10) $\sqrt{11} - \sqrt{2}$
(11) $\frac{2}{\sqrt{3}-1}$
(12) $\frac{6-\sqrt{3}}{2\sqrt{3}-1}$

1.4 Absolute value

Q11. Evaluate.

(1) $|7.2|$
(2) $|-3.4|$
(3) $|\pi - 3.4|$
(4) $|\sqrt{5} - 2|$
(5) $|\frac{3}{2\sqrt{2}-3}|$
(6) $|\pi^2 - 10|$
(7) $|3 - 2\sqrt{3}|$
(8) $|(-2)^4|$
(9) $|(-2)^3|$
(10) $|\sqrt{81} - 3^2|$
(11) $|3\sqrt{2} - 4|$
(12) $|5 - 2\sqrt{7}|$
(13) $|7 - 5\sqrt{2}|$
(14) $|\sqrt[3]{9} - 2|$
(15) $\sqrt{(2\sqrt{6} - 5)^2}$
(16) $\sqrt{(2\sqrt{10} - 6)^2}$
(17) $\sqrt{(\pi - 3)^2}$
(18) $\sqrt{((-2)^3 - 5)^2}$
(19) $\sqrt{(4\sqrt{3} - 7)^2}$
(20) $\sqrt{(3\sqrt{11} - 10)^2}$

Q12. Solve the equations.

(1) $|2x - 5| = 4$
(2) $|2 - 3x| = 1$
(3) $|4x + 2| = 3$
(4) $|-x - 5| = 2$
(5) $|3x - 7| = 0$
(6) $|4x - 1| + 2 = 8$
(7) $|3 - 5x| - 4 = 2$
(8) $|3x - 2| + 5 = 1$
(9) $|4 + 3x| - 3 = -2$
(10) $|5 - 2x| - 5 = 1$
(11) $|3 - 11x| + 5 = 7$
(12) $|7x + 6| + 5 = 4$

Q13. Solve the equations.

(1) $2|x + 3| + |x + 3| = 6$
(2) $5|3x + 7| - 2|3x + 7| = 12$
(3) $2|2x - 3| + 4|3 - 2x| = 3$
(4) $2|2x - 6| + |x - 3| = 4$
(5) $5|1 - 2x| - |4x - 2| = 5$
(6) $|3x + 6| + 2|2 + x| = 8$
(7) $4|2x - 1| - 2|3 - 6x| = 3$
(8) $3|7x + 3| + 2| - 6 - 14x| = 21$
(9) $5|6x - 4| - 4|2 - 3x| = 2$
(10) $4|4x - 2| + 5|3 - 6x| = 1$
(11) $3|4x - 8| - 4|6 - 3x| = 2$
(12) $2|6x - 8| - 3|12 - 9x| = 10$

1.5 Percentages

Q14. (1) Evaluate 15% of 320.
(2) Evaluate 20% of 335.
(3) Evaluate 35% of 120.
(4) Evaluate 45% of $151\frac{1}{9}$.
(5) Evaluate 12% of $541\frac{2}{3}$.

Q15. (1) What percentage of the number 250 is the number 15?
(2) What percentage of the number 72 is the number 18?
(3) What percentage of the number 108 is the number 19.44?
(4) What percentage of the number 240 is the number 34?
(5) What percentage of the number 24 is the number 54?
(6) What percentage of the number 26 is the number 62.4?

Q16. (1) Find a number whose 40% is 26.
(2) Find a number whose 26% is 22.1.
(3) Find a number whose 15% is 4.8.
(4) Find a number whose 67% is 321.6.
(5) Find a number whose 52% is 13.

Q17. (1) How many percent is 145 greater than 125?
(2) How many percent is 125 smaller than 150?
(3) How many percent is 87 greater than 75?
(4) How many percent is 75 smaller than 90?
(5) How many percent is 57.5 greater than 46?
(6) How many percent is 46 smaller than 57.5?

Q18. The number of votes for each of the 5 candidates in the school election is given in the table below.

candidate	A	B	C	D	E
votes	49	72	59	54	16

(1) How many percent of votes more got candidate B than candidate D?
(2) How many percentage points more got candidate B than candidate D?
(3) How many percent of votes more got candidate D than candidate E?
(4) How many percentage points more got candidate D than candidate E?
(5) How many percent of votes less got candidate E than candidate B?
(6) How many percentage points less got candidate E than candidate B?
(7) How many percentage points more got candidate B than candidate E?

Q19. (1) Evaluate 34% of 85.
(2) What percentage of the number 95 is the number $61\frac{3}{4}$?
(3) Find the number whose 42% is 27.3.
(4) How many percent is 68.4 greater than 60?
(5) How many percent is 70.4 smaller than 80?

Q20. (1) What percentage of the number 62.5 is the number $11\frac{1}{4}$?
(2) What percentage of the number 25.6 is the number 3.2?
(3) What percentage of the number 65 is the number 53.3?
(4) What percentage of the number 85 is the number 71.4?
(5) What percentage of the number 70 is the number 53.2?
(6) What percentage of the number 750 is the number 255?
(7) What percentage of the number 650 is the number 273?
(8) What percentage of the number 65 is the number $14\frac{14}{25}$?
(9) Find the number whose 17% is 77.52.
(10) Find the number whose 174% is 443.7.
(11) Find the number whose 24% is 16.8.
(12) Find the number whose 16% is 13.6.

(13) Find the number whose 18% is 11.7.

(14) Find the number whose 92% is 151.8.

(15) Find the number whose 72% is 46.8.

(16) Evaluate 42.5% of 28.

(17) Evaluate 98% of 45.

(18) Evaluate 44% of 325.

(19) How many percent is the number 85 greater than 68?

(20) How many percent is the number 68 smaller than 85?

Q21. There are 250 students in a school and 18% of them were born in June or July. How many students were born in June or July?

Q22. There are 350 students in a school. 301 of them are younger than 18. How many percent of students are 18 or older?

Q23. There are 66 blond students in a school and this number makes 12% of all students. How many students are there in the school?

Q24. The initial price of a bike (660 zł) was reduced by 20%. What was the bike's price after the reduction?

Q25. The initial price of a CD-player (250 zł) was reduced by 25%. What was the player's price after the reduction?

Q26. The initial price of a TV-set (1320 zł) was reduced by 10%. What was the price after the reduction?

Q27. A book that had costed 34zł was sold for 27.20 zł. What was the percentage reduction in price?

Q28. A CD that had costed 55 zł was sold for 49.50 zł. What was the percentage reduction in price?

Q29. A book that had costed 24 zł was sold for 21.60 zł. What was the percentage reduction in price?

Q30. A CD was sold for 51 zł after 15% reduction. What was the initial price?

Q31. A laptop was sold for 2340 zł after 25% reduction. What was the initial price?

Q32. A jacket was sold for 748 zł after 12% reduction. What was the initial price?

Q33. Gas price after 10% increase is 6.16 zł per litre. What was the price before the increase?

Q34. A number was increased by 15% and then the result decreased by 12%. By how many percent should the new result be increased so that the final result is larger than the initial number by 26.5%

Q35. A number was increased by $11\frac{1}{9}$%. Should the result be increased or decreased and by how many percent so that we get the result from which the initial number is smaller by 20%?

Q36. Adam went to the cinema with his friends and he spent 49 zł which was 14% of his savings. How much money did he have after the cinema?

Q37. Boris went to the cinema with his friends. He had had 250 zł before the cinema and was left with 205 zł afterwards. What percentage of his savings did he spend at the cinema?

Q38. Calvin went to the cinema with his friends and he spent 72 zł which was 16% of his savings. How much money was he left with after the cinema?

Q39. A fast food restaurant uses on average 85 kg of lamb meat and 120 kg of chicken meat every day. They buy the meat at the butcher's paying 20 zł per kilogram of lamb meat and 15 zł per kilogram of chicken meat. Last Thursday they used 20% less lamb meat but 15% more chicken meat. When did they spend more money for the meat: on average or last Thursday? How many percent more?

Q40. A grandma uses 15 kg of fruits and 9 kg of sugar for producing fruit jam. She used to pay 4 zł for 1 kilogram of fruits and 3.60 zł for 1 kilogram of sugar. Last year the fruit prices rose by 10% while sugar price by 5%. What is the percentage increase in the price of the same amount of jam?

Q41. 15 students want to go on a trip. They have already bought open train tickets, so they can use them any time they want. They have found a hostel where they want to stay and eat meals, too. They will have to pay for 3 nights and for food for 4 full days. One night in low season is 22 zł while in high season it is 5% more expensive. The price for food in low season is 24 zł per day while in high season it is 10% higher. How many percent more do they have to pay in high season than in low season?

Q42. Umberto has a large number of books on his shelf. One day he concluded he got too many of them and got rid of 25% of the books. He kept on buing new books so soon the number increased by 14%. Then he had 684 books. How many books did Umberto have initially? Which number of books - initial or final - was smaller? By how many percent?

Q43. A price of a bike in a tourist shop was raised by 5% while a price of a tent was lowered by 30%. Then a bike and a tent costed 420 zł each. How many percent was a tent more expensive than a bike before the changes?

Q44. A number was increased by 20%. Then the result was decreased by 20% and the final result was 288. What was the initial number? Which of the numbers - initial or final - was smaller? By how many percent?

1.6 Approximations. Decimal places and significant figures.

Q45. Write down approximations of the following numbers correct to 2 decimal places.

(1) 102.438
(2) 2.0109
(3) 3.605
(4) 3.896
(5) 14.1441
(6) 29.998
(7) 0.005
(8) 0.00499

Q46. Write down approximations of the following numbers correct to the nearest ten.

(1) 8.23
(2) 24.5
(3) 6705.02
(4) 344.99
(5) 25.03
(6) 428.07
(7) 654.3
(8) 106.01

Q47. Write down approximations of the following numbers correct to the nearest thousand.

(1) 1502.1
(2) 6520841.2
(3) 12099.01
(4) 4444
(5) 498.50
(6) 129700.1
(7) 42611
(8) 62411

Q48. Write down approximations of the following numbers correct to the number of significant figures shown in braces.

(1) 23407 {1}
(2) 15705 {1}
(3) 0.00234 {1}
(4) 0.0003509 {1}
(5) 4158874 {2}
(6) 25.394 {2}
(7) 0.0020803 {2}
(8) 0.0002549 {2}
(9) 25.394 {3}
(10) 0.0020803 {3}
(11) 0.0002549 {3}
(12) 45555555 {3}

Q49. A number a approximated to 3 significant figures is given. Write down the range of possible values of a.

(1) 2340000
(2) 933
(3) 4050000
(4) 0.0123
(5) 0.0045
(6) 2000
(7) 19
(8) 32500
(9) 0.0999
(10) 0.00246
(11) 0.405
(12) 0.07

1.7 Exponents and roots

Q50. Evaluate without a calculator.

(1) $\sqrt[3]{125}$
(2) $\sqrt[3]{27}$
(3) $\sqrt[3]{216}$
(4) $\sqrt[4]{81}$
(5) $\sqrt[5]{32}$
(6) $\sqrt[5]{243}$
(7) $\sqrt[3]{343}$
(8) $\sqrt[6]{64}$
(9) $\sqrt[3]{\frac{8}{27}}$
(10) $\sqrt[3]{15\frac{5}{8}}$
(11) $\sqrt[3]{2\frac{10}{27}}$
(12) $\sqrt[3]{42\frac{7}{8}}$
(13) $\sqrt[3]{3\frac{3}{8}}$
(14) $\sqrt[3]{4\frac{17}{27}}$
(15) $\sqrt[3]{12\frac{19}{27}}$
(16) $\sqrt[4]{5\frac{1}{16}}$
(17) $\sqrt[4]{39\frac{1}{16}}$
(18) $\sqrt[4]{3\frac{13}{81}}$
(19) $\sqrt[5]{7\frac{19}{32}}$

Q51. Compare the numbers i.e. state whether one number is smaller ("<"), greater (">") or equal ("=") to the other. Do not use a calculator.

(1) $2^5 \quad \ldots \quad 5^2$
(2) $2^5 \quad \ldots \quad (-2)^5$
(3) $(-2)^5 \ldots (-2)^4$
(4) $4^0 \quad \ldots \quad 0^4$
(5) $(-2)^5 \ldots (-2)^7$
(6) $(\frac{1}{2})^5 \quad \ldots \quad (\frac{1}{2})^6$
(7) $(-\frac{1}{2})^5 \ldots -(\frac{1}{2})^6$
(8) $(-2)^5 \ldots 2^4$
(9) $(-2)^5 \ldots (-2)^6$
(10) $(-2)^5 \ldots -2^6$
(11) $(-2)^5 \ldots -2^5$
(12) $(-2)^4 \ldots -2^4$

Q52. Express each of the following as a single power (positive or negative, without parethesis).

(1) $3^{10} \times 3^4$
(2) $3^{17} \div 3^{11}$
(3) $(7^{11})^7$
(4) $(-2)^5 \times (-2)^6$
(5) $((-2)^6)^5$
(6) $((-2)^7)^5$
(7) $\frac{(-3)^{11}}{(-3)^5}$
(8) $\frac{5^6 \times 5^{11}}{5^{10}}$
(9) $\frac{3^{21}}{3^9 \times 3^{10}}$
(10) $3^9 \div 9^3$
(11) $4^8 \times 2^4$
(12) $4^8 \times 16^4$
(13) $4^8 \times 8^4$
(14) $\frac{27^9}{9^8}$
(15) $\frac{32^8}{4^6 \times 8^3}$
(16) $\frac{81^{12} \div 9^5}{27^{10}}$
(17) $81^{-4} \times 9^7$
(18) $81^{-3} \times 27^4$
(19) $16^{-11} \div 8^{-15}$
(20) $8^{12} \times 4^{-15}$
(21) $4^{-5} \div 32^5$
(22) $25^{11} \times 125^{-8}$
(23) $49^{-23} \div 7^{-50}$
(24) $27^{-9} \div 9^{-10}$
(25) $\frac{81^3 \times 27^{-5}}{9^{11} \times 3^{-12}}$
(26) $\frac{4^{-16} \div 16^{-5}}{2^{-11} \times 8^6}$
(27) $\frac{25^3 \times 125^{-8}}{5^{-11} \div 25^{-5}}$
(28) $\frac{8^{-15} \div 16^{-7}}{4^6 \div 32^5}$

Q53. Write down the following numbers in ascending order (i.e. from the smallest to the largest). Do not use a calculator.

a	b	c	d	e	f	g	h	i	j
$(-2)^{13}$	-2^{14}	$(-2)^{14}$	$(-4)^7$	-2^{13}	$(-2)^{15}$	-2^{15}	-4^8	$(-4)^8$	4^8

Q54. Write down the following numbers in ascending order (i.e. from the smallest to the largest). Do not use a calculator.

a	b	c	d	e	f	g	h	i	j
$(-2)^{-15}$	-2^{-15}	-4^{-8}	$(-4)^{-8}$	4^{-8}	$(-2)^{-13}$	-2^{-14}	$(-2)^{-14}$	$(-4)^{-7}$	-2^{-13}

Q55. Express the following as a single power.

(1) $\sqrt{x^7}$
(2) $\sqrt[3]{a^2}$
(3) $\sqrt{\frac{a^{19}}{a^7}}$
(4) $\sqrt[3]{a^{16} \div a^8}$
(5) $\sqrt[3]{a^8 \times a^7}$
(6) $\sqrt[4]{x^{11} \times x^9}$
(7) $\sqrt[5]{b^{34} \div b^9}$
(8) $\sqrt[3]{c^8 \times c^6}$
(9) $\sqrt[6]{y^3 \times y^{15}}$
(10) $\sqrt[4]{d^{43} \div d^{34}}$
(11) $\sqrt[5]{s^3 \times s^5}$
(12) $\sqrt[3]{t^{12} \div t^6}$
(13) $\sqrt[7]{n^{10} \times n^{11}}$
(14) $\sqrt[3]{n^{11}} \div \sqrt[6]{n^{13}}$
(15) $\sqrt[3]{a^{10}} \times \sqrt[6]{a^{10}}$
(16) $\sqrt[3]{a^4 \div a^{-5}}$
(17) $\sqrt[3]{p^{-7} \div p^2}$
(18) $\sqrt[3]{s^{-11} \times s^{-3}}$
(19) $\sqrt[4]{b^{-23} \times b^{11}}$
(20) $\sqrt[5]{x^{-18} \div x^{-8}}$
(21) $\sqrt[4]{y^{-31} \times y^9}$
(22) $\sqrt[3]{t^{-32} \times t^{33}}$
(23) $\sqrt[6]{w^{-8} \div w^{-10}}$
(24) $\sqrt[3]{a^{-14}} \times \sqrt[6]{a^{-14}}$

Q56. Simplify the following.

(1) $x^5 \times x^7$
(2) $\frac{6p^7}{2p^3}$
(3) $x^3 \times 6x^6 \div (2x^8)$
(4) $2x^6 \times 3x^5 \div (9x^9)$
(5) $9a^6 \div (12a^7) \times 2a^3$
(6) $6s^8 \times 2s^{-2} \div (4s^4)$
(7) $18n^{-3} \div (8n^8) \times \frac{2}{3}n^{10}$
(8) $\frac{9w^{11}}{6w^5 \div (4w^{-3})}$
(9) $\frac{(2a^3)^3 \times (3a^{-2})^2}{(6a^2)^2}$
(10) $\frac{98p^5}{7p^6 \times 4p^{-4}}$
(11) $\frac{2a^5 \times 9a^6}{3a^7 \times (2a)^2}$
(12) $\frac{3s^3 \div (2s^{-5})^2}{6s^{-3} \times 5s^{11}}$

Q57. Evaluate without a calculator.

(1) 10^{-5}
(2) 0.75^{-2}
(3) 2^{-5}
(4) $(\frac{1}{2})^{-5}$
(5) $(\frac{3}{4})^{-3}$
(6) $(2\frac{1}{2})^{-1}$
(7) $(1\frac{1}{2})^{-4}$
(8) $(2\frac{2}{3})^{-3}$
(9) 0.1^{-6}
(10) $(\frac{3}{16})^{-1}$
(11) $(5\frac{2}{6})^{-1}$
(12) $(1\frac{3}{4})^{-2}$

Q58. Evaluate without a calculator.

(1) $(\frac{1}{32})^{-0.2}$
(2) $8^{\frac{2}{3}}$
(3) $32^{-\frac{3}{5}}$
(4) $9^{1.5}$
(5) $(\frac{8}{27})^{-1\frac{1}{3}}$
(6) $\sqrt[20]{3^{-10}}$
(7) $(\sqrt[4]{3\frac{3}{8}})^{2\frac{2}{3}}$
(8) $(\sqrt[6]{(-4)^2})^{-3}$
(9) $(\sqrt[9]{5^3})^{-3}$
(10) $(\sqrt{7\frac{19}{32}})^{-0.4}$
(11) $\sqrt[5]{(39\frac{1}{16})^{-1.25}}$
(12) $(-3.375)^{1.\dot{3}}$
(13) $(\sqrt[74]{(\frac{1}{243})^{12}})^{1.2\dot{3}}$
(14) $(\sqrt[5]{0.125^4})^{-4.1\dot{6}}$

Q59. Rationalize the denominator.

(1) $\frac{1}{\sqrt{2}}$
(2) $\frac{1}{\sqrt{3}}$
(3) $\frac{2}{\sqrt{6}}$
(4) $\frac{3}{\sqrt{6}}$
(5) $\frac{6}{\sqrt{3}}$
(6) $\frac{6}{\sqrt{2}}$
(7) $\frac{14}{\sqrt{7}}$
(8) $\frac{14}{\sqrt{2}}$
(9) $\frac{14}{\sqrt{3}}$
(10) $\frac{15}{\sqrt{3}}$
(11) $\frac{15}{\sqrt{6}}$
(12) $\frac{15}{\sqrt{2}}$
(13) $\sqrt{\frac{2}{5}}$
(14) $\sqrt{\frac{5}{2}}$
(15) $\sqrt{\frac{15}{2}}$

Q60. Simplify the following. Do not use a calculator.

(1) $\sqrt{2} \times \sqrt{18}$
(2) $\sqrt{2} + \sqrt{18}$
(3) $\sqrt{3} \times \sqrt{75}$
(4) $\sqrt{3} - \sqrt{75}$
(5) $\sqrt{8} \times \sqrt{98}$
(6) $\sqrt{12} \times \sqrt{24}$
(7) $\sqrt{50} + 2\sqrt{8} - \sqrt{18}$
(8) $\sqrt{363} - 2\sqrt{27} - \sqrt{75}$
(9) $\sqrt{45} - 3\sqrt{20} - 3\sqrt{5}$
(10) $2\sqrt{108} - 2\sqrt{48} - 3\sqrt{75}$
(11) $5\sqrt{\frac{1}{2}} - 2\sqrt{32} + \sqrt{4.5}$
(12) $2\sqrt{8\frac{1}{3}} - \frac{1}{6}\sqrt{75} + \frac{5}{2}\sqrt{27}$

Q61. Rationalize the denominator.

(1) $\frac{1}{\sqrt{2}-1}$
(2) $\frac{2}{\sqrt{3}+1}$
(3) $\frac{6}{2\sqrt{3}+3}$
(4) $\frac{2+\sqrt{3}}{\sqrt{3}-2}$
(5) $\frac{2-\sqrt{3}}{\sqrt{3}-2}$
(6) $\frac{9}{3\sqrt{2}+2\sqrt{3}}$
(7) $\frac{2\sqrt{2}}{\sqrt{7}-3}$
(8) $\frac{\sqrt{5}}{2\sqrt{2}+\sqrt{3}}$

1.8 Expansions

Q62. Expand.

(1) $(x-1)^2$
(2) $(x+2)^2$
(3) $(x-3)^2$
(4) $(x+4)^2$
(5) $(x-4)^2$
(6) $(x-5)^2$
(7) $(x-1.5)^2$
(8) $(x+2.5)^2$
(9) $(x+\frac{1}{2})^2$
(10) $(x+\frac{1}{3})^2$
(11) $(x-\frac{2}{3})^2$
(12) $(2x-1)^2$
(13) $(3x+1)^2$
(14) $(2x-3)^2$
(15) $(3x+2)^2$
(16) $(2x+5)^2$
(17) $(\frac{1}{2}x-1)^2$
(18) $(\frac{3}{2}x+2)^2$
(19) $(2-5x)^2$
(20) $(6x+1)^2$

Q63. Expand and simplify.

(1) $(a^2-ab)^2$
(2) $(x^3+2xy^2)^2$
(3) $(2xs-3s^3)^2$
(4) $(3a^2b^3+2ab^4)^2$
(5) $(2pq^2-3p^3q)^2$
(6) $(\frac{st^2}{2}-2s^2t)^2$
(7) $(\frac{3}{2}ac+\frac{2}{3}ac^5)^2$
(8) $(\frac{2}{3}a-0.75a^3c^3)^2$

Q64. Expand and simplify.

(1) $(2+\sqrt{2})^2$
(2) $(3-\sqrt{2})^2$
(3) $(4-3\sqrt{3})^2$
(4) $(2\sqrt{3}-3\sqrt{2})^2$
(5) $(2\sqrt{6}-7\sqrt{2})^2$
(6) $(8\sqrt{2}+3\sqrt{6})^2$
(7) $(5\sqrt{3}+2\sqrt{6})^2$
(8) $(3\sqrt{6}-7\sqrt{3})^2$
(9) $(3\sqrt{6}-5\sqrt{10})^2$
(10) $(2\sqrt{15}+3\sqrt{6})^2$
(11) $(\sqrt{22}-\sqrt{33})^2$
(12) $(\frac{1}{\sqrt{2}}+3\sqrt{2})^2$

Chapter 2

Sets

2.1 Sets and subsets

Q1. Translate the following sentences written in symbolic form (regardless of their logical value) into words.

(1) $\forall t \in \mathbb{R}\ \exists s \in \mathbb{Z}\ s \leq t$
(2) $\forall n \in \mathbb{N}\ (-1)^{2n+1} = -1$
(3) $\forall n \in \mathbb{N}\ (-1)^{2n} = 1$
(4) $\exists n \in \mathbb{N}\ \forall s \in \mathbb{R}\ s \geq s + n$
(5) $\forall x \in \mathbb{R}^+\ \exists! y \in \mathbb{R}\ y^2 = x$
(6) $\forall n \in \mathbb{N}\ (2 \mid n) \Rightarrow (4 \mid n)$
(7) $\exists n \in \mathbb{N}\ (2 \mid n) \Rightarrow (4 \mid n)$
(8) $\exists n \in \mathbb{N}\ ((2 \nmid n) \wedge (4 \mid n))$
(9) $(n \in \mathbb{Z}) \Rightarrow (2^n \in \mathbb{Z})$
(10) $(n \in \mathbb{Q}) \Leftrightarrow (2^n \in \mathbb{Q})$

Q2. Write the following statements in symbolic form. Are they true?

(1) If a number is natural, it is an integer, too.
(2) All integers are rational.
(3) There exists a real number that is neither positive nor negative.
(4) Not all natural numbers are positive.
(5) If an integer is divisible by 4 then it is divisible by 2, too.
(6) If an integer is divisible by 2 and by 3, it is divisible by 6.
(7) Every positive real number is a square of a certain real number.
(8) Every positive integer is a square of a certain integer.
(9) For any two real numbers there exist a positive integer smaller than the difference between them.
(10) If a real number is smaller than its square then it is negative.
(11) An even power of an integer is always positive.
(12) An odd power of an integer is not always positive.
(13) If one real number is smaller than the other then the square of the first number is smaller than the square of the other, too.
(14) There exists a positive real number that is smaller than its square.
(15) All positive real numbers are larger than their squares.
(16) If an integer is divisible by 2, its square is divisible by 4.
(17) If an integer is divisible by 4, its square is divisible by 16.

Q3. Write down all elements of the following sets.

(1) $\{x \in \mathbb{Z} \mid (x^2 = 16) \vee (x^2 = 25)\}$
(2) $\{x \in \mathbb{Z} \mid x^2 \leq 2\}$
(3) $\{x \in \mathbb{Z} \mid 2 < x^2 \leq 20\}$
(4) $\{x \in \mathbb{Z} \mid 4 < x^2 \leq 16\}$
(5) $\{x \in \mathbb{N} \mid x^3 < 100\}$
(6) $\{x \in \mathbb{Z} \mid (x < 6) \wedge (x \geq 2)\}$
(7) $\{x \in \mathbb{Z} \mid (x^2 \leq 1) \vee (|x - 1| = 2)\}$
(8) $\{x \in \mathbb{Z} \mid (2x - 5 < 4) \wedge (2 - 3x \leq 6)\}$
(9) $\{x \in \mathbb{R} \mid (|x + 1| = 2) \vee (|x + 2| = 1)\}$
(10) $\{x \in \mathbb{R} \mid (|2x + 1| = 3) \vee (|3x - 1| = 2)\}$
(11) $\{x \in \mathbb{R} \mid (x^2 < 10) \wedge (|3x - 2| + 1 = 0)\}$
(12) $\{x \in \mathbb{Z} \mid (2x + 19 \geq 2) \wedge (14 - 3x > 5)\}$

Q4. Write the following sets in simplest form (e.g. listing all elements).

(1) $\mathbb{N} \cap \{x \in \mathbb{R} \mid x < 5\}$
(2) $(\mathbb{Z} \setminus \mathbb{N}) \cap \{x \in \mathbb{R} \mid x + 8 \geq 0\}$
(3) $\{x \in \mathbb{R} \mid x^2 < 8\} \cap \mathbb{Z}$
(4) $\mathbb{Z} \setminus \mathbb{Q}$
(5) $\mathbb{Z} \cup \mathbb{N}$
(6) $\mathbb{Z} \cap \mathbb{N}$
(7) $(\mathbb{R} \setminus \mathbb{Z}) \cap \mathbb{N}$
(8) $(\mathbb{Z} \setminus \mathbb{N}) \cap \{x \in \mathbb{R} \mid x + 6 > 0\}$
(9) $\mathbb{N} \cap \{x \in \mathbb{R} \mid x \leq 3\}$
(10) $\{x \in \mathbb{Z} \mid x^2 < 30\} \cap \{x \in \mathbb{Z} \mid x^2 > 8\}$
(11) $\{x \in \mathbb{Z} \mid (2 \mid x)\} \cap \{x \in \mathbb{Z} \mid (3 \mid x)\} \cap \{x \in \mathbb{R} \mid x^2 < 100\}$
(12) $\{x \in \mathbb{Z} \mid (2 \mid x)\} \cap \{x \in \mathbb{Z} \mid (4 \mid x)\} \cap \{x \in \mathbb{R} \mid x^2 < 100\}$

Q5. List all subsets of a set $\{1, 3\}$.

Q6. List all subsets of a set $\{2, 5, 8\}$.

Q7. List all 3-element subsets of a set $\{1, 2, 3, 4\}$.

Q8. List all 2-element subsets of a set $\{1, 2, 3, 4\}$.

Q9. List all 2-element subsets of a set $\{1, 2, 3, 4, 5\}$.

Q10. List all 4-element subsets of a set $\{1, 2, 3, 4, 5\}$.

Q11. List all proper subsets of a set $\{0, 2, 4, 8\}$ that contain 0.

Q12. How many proper subsets of a set $\{1, 2, 4, 8, 16\}$ contain 1?

Q13. How many proper subsets of a set $\{1, 2, 4, 8, 16\}$ do not contain 1?

Q14. How many proper subsets of a set $\{1, 2, 4, 8, 16\}$ contain 1 but do not contain 16?

2.2 Venn diagrams

Q15. Shade the following sets in Venn diagrams.

(1) $A \cap B'$
(2) $A \cup B'$
(3) $(A \cap B')'$
(4) $(A \setminus B)'$
(5) $(A \cup B) \setminus (A \cap B)$
(6) $(A \cup B) \setminus A$
(7) $(A \cap B) \setminus (A \cup B)$
(8) $B \setminus (A \cap B)$
(9) $(A \cup B) \cap C'$
(10) $(A \cap B) \cup C$
(11) $A' \cup (B \cap C)$
(12) $(A \cup B) \setminus C$
(13) $A \cup (B \setminus C)$
(14) $(A \cap B) \setminus C$
(15) $A \cap (B \setminus C)$
(16) $A \setminus (B \cap C)$
(17) $A \setminus (B \cup C)$

Q16. Let A, B and C be subsets of a universal set U.
Use Ven diagrams to verify whether the following statements are true.

(1) $A \setminus B = A \cap B'$
(2) $A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$
(3) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$
(4) $A \setminus (B \cup C) = (A \setminus B) \cup C$
(5) $A \cap (B \cup C) = (A \cap B) \cup C$
(6) $A \cap (B \cup C)' = (A \setminus B) \setminus C$
(7) $(A \cup B)' = A' \cap B'$
(8) $(A \cap B)' = A' \cup B'$
(9) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(10) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Q17. Among 48 students in a winter sport camp 31 ski, 23 snowboard and 6 do both.

(1) Show the situation in the Venn diagram.

How many students:

(2) ski but do not snowboard?

(3) snowboard but do no ski?

Q18. Among 50 students 38 study biology, 14 study physics and 9 study none of the above.

(1) Show the situation in the Venn diagram.

How many students:

(2) study both biology and physics?

(3) study biology but not physics?

Q19. In a group of 20 adults 8 have driving licence for cars, 12 for motorcycles and 2 for both cars and motorcycles.

(1) How many people can drive cars but not motorcycles?

(2) How many people in the group do not have a driving licence?

Q20. In a group of 12 tourists everyone wears sandals or socks. 7 people wear sandals and 6 wear socks. The number of people who wear both sandals and socks is smallest possible. What is the number?

Q21. 15 students were sitting in a cantine during a break. 2 of them ate nothing. 11 ate sandwiches and 5 ate pizza. How many students:

(1) ate both sandwiches and pizza?

(2) ate sandwiches only?

Q22. Out of 50 quadrilaterals drawn 19 are rectangles, 32 are rhombuses and 4 are squares.

(1) How many rhombuses are not squares?

(2) How many quadrilaterals are not parallelograms?

Q23. In a group of 40 students none of 9 students who do Physics does Chemistry or Biology. If 17 students do Chemistry, 19 students do Biology and every student must do at least one of the sciences, find how many students:

(1) do Chemistry but not Biology,

(2) do Biology but not Chemistry.

Q24. In a school 112 students do not learn German while 96 students do not learn French. 60 students learn none of the two languages and that number makes 37.5% of the whole school. How many students:

(1) are there in the school?

(2) learn both languages?

Q25. In an IB school each student must do at least one science subject: Biology, Chemistry or Physics.
31 students do Biology, 16 students do Chemistry and 12 students do Physics.
No one does both Biology and Physics, 1 Chemistry student does Physics and 11 Chemistry students do Biology.
How many students do only:

(1) Biology, (2) Chemistry, (3) Physics?

Q26. In IB1 year 17 students do the Maths HL course, 23 students do Economics and 14 students do Physics. 11 Maths HL students do Physics, 7 Physics students do Economics and 12 Economics students do Maths HL. There are 6 students who do all three subjects. How many students:

(1) do at least one of the three subjects,

(2) do Maths HL only (i.e. they do not do Economics nor Physics),

(3) do Economics only,

(4) do Physics only?

Q27. A group of 50 tourists had a lunch in a restaurant. After lunch they all decided to have some dessert. 17 ate banana cake, 23 had chocolate cake while 15 - an apple pie.
1 person had both banana cake and apple pie, 2 tourist had both banana and choclate cakes and 3 people had both apple pie and chocolate cake.
How many tourists:

(1) had apple pie only,

(2) had all three desserts?

Q28. 15 IB students consider Biology, Chemistry and English as their Extended Essays subjects.
11 students think of Biology, 5 about Chemistry and 8 about English.
5 students who think of Biology consider English, too. 3 students thinking of Chemistry consider English and the same number of potential EE authors in Chemistry considers Biology.
There are 2 students who hesitate among all 3 subjects.
How many students have decided on each of the three subjects?

Q29. The Society School of Sussex offers three new courses for its students: Advaced Algebra, Basic Botany and Creative Computer Science.
Thirty students signed up for Advanced Algebra. Out of these ten chose Basic Botany. Out of twenty students who decided to do Creative Computer Science four chose Basic Botany, too. Out of fifteen students who do both Advanced Algebra and Creative Computer Science one student signed up for all three courses.
Knowing that eighteen students undertook Basic Botany and each student signed up for at least one course find the total number of students.

Q30. Each student of the Natural Sciences Summer School has to sign up for at least two courses out of Biology, Chemistry and Physics. The three courses are attended by 57, 48 and 29 students respectively. Knowing that eight students do not do Biology, seventeen students do not do Chemistry and thirty six students do not do Physics find out how many out of sixty five students in the school signed up for all three courses.

Q31. Each student of the Social Sciences Summer School has to sign up for at least two courses out of History, Economics and Philosophy. The History course is attended by 62 students. 41 students decided to do both History and Economics and 24 students decided to do both History and Philosophy. 15 students do both Economics and Philosophy.
How many students attend the Social Sciences Summer School?

2.3 Operations on sets

Q32. Consider the sets:

$$A = \{1, 3, 5, 7, 9, 11\},$$
$$B = \{0, 2, 4, 6, 8, 10\}$$
$$C = \{0, 3, 6, 9\}$$
$$\text{and } U = \{0, 1, 2, 3, \ldots, 10, 11, 12\} \text{ - the universal set.}$$

Find the following sets by listing their all elements.

(1) $A \setminus B$
(2) $B \setminus A$
(3) $A \setminus C$
(4) $C \setminus B$
(5) $A \cap B$
(6) $A \cap C$
(7) $B \cap C$
(8) A'
(9) B'
(10) $A \cup B$
(11) $B' \cup C$
(12) $A' \cap B'$
(13) $(A \cup B)'$
(14) $(A \cup B \cup C)'$
(15) $A' \cap B' \cap C'$

Write down the following.

(16) $n(A)$
(17) $n(B)$
(18) $n(C)$
(19) $n(A \cap B)$
(20) $n(A \cup C)$
(21) $n(C')$

Q33. Given the sets $A = \{2, 3, 4, 7, 9, 12\}$ and $B = \{x \in \mathbb{N} |\ (x < 15) \wedge (2|x)\}$. Find:

(1) $n(A \cup B)$, (2) $A \cap B$, (3) $A \setminus B$, (4) $B \setminus A$.

Q34. Given the sets $A = \{3, 4, 6, 10, 13, 18\}$ and $B = \{x \in \mathbb{N} |\ (x < 22) \wedge (3|x)\}$. Find:

(1) $n(A \cup B)$, (2) $A \cap B$, (3) $A \setminus B$, (4) $B \setminus A$.

Q35. Consider the subsets of $U = \{x \in \mathbb{Z} |\ -15 \leq x \leq 15\}$:

$$\begin{aligned} A =& \{x \in U |\ x \text{ is a multiple of } 3\} \\ B =& \{x \in U |\ x \text{ is a multiple of } 4\} \\ C =& \{x \in U |\ x \text{ is a multiple of } 12\} \end{aligned}$$

Are the following statements true or false?

(1) $A \subset B$ (3) $C \subset A$ (5) $C \subset B$
(2) $B \subset C$ (4) $B \subset A$ (6) $A \subset C$

Write down the numbers:

(7) $n(A)$ (8) $n(B)$ (9) $n(C)$

Find the sets:

(10) $A \setminus C$ (13) $C \setminus B$ (16) $B \cap C$
(11) $B \setminus C$ (14) $(A \cup B \cup C)'$ (17) $A \cup C$
(12) $C \setminus A$ (15) $B \cup C$ (18) $A \cap C$

Q36. Given the sets:

$$\begin{aligned} A =& \{2, 5, 6, 8, 10, 11\} \\ B =& \{3, 4, 5, 7, 8, 9\} \\ \text{and } C =& \{1, 2, 4, 8, 9\} \end{aligned}$$

subsets of the universal set $U = \{1, 2, 3, \ldots, 12\}$.

(1) Write down $n(A \cap B)$
(2) Write down $n(A \cup B)$
(3) Write down $n(B \cup C)$
(4) Give an example of a set D such that $n(D) = 5$ and $A \setminus D = \{2, 5, 10, 11\}$
(5) Give an example of a set E such that $n(E) = 6$ and $E \setminus B = \{1, 6, 10\}$
(6) Give an example of a set F such that $n(F) = 7$ and $C \cap F = \{2, 4, 8\}$
(7) Find $(A \cup B)'$.
(8) Find $(A \cup B \cup C)'$.
(9) Find $A' \cap B'$.

Q37. Given the sets $A = [-7, 5]$, $B = [-5, 11[$ and $C = \{x \in \mathbb{R} |\ x < 6\}$.
Find the following writing your answer as an interval or a union of mutually exclusive intervals. Mark the solutions on the real number line.

(1) $A \cap B \cap C$ (2) $C \setminus (A \cap B)$ (3) $A \cup B \setminus C$

Q38. Given the sets $A = [-3, 4]$, $B = \{x \in \mathbb{R} |\ x < 2\}$ and $C = [-8, -3]$.
Find the following writing your answer as an interval or a union of mutually exclusive intervals. Mark the solutions on the real number line.

(1) $A \cap B \cap C$ (2) $C \setminus (A \cap B)$ (3) $A \cap B \setminus C$

Q39. For the sets:

$$\begin{aligned} A =& [-2, 0[\cup]3, 5] \cup [7, 9] \\ \text{and } B =&]-\infty, -2] \cup]0, 3[\cup [5, 7[\end{aligned}$$

find the following intervals. Write your answer as an interval or a union of mutually exclusive intervals. Mark the solutions on the real number line.

(1) $A \cup B$ (2) $A \cap B$ (3) $A \setminus B$ (4) $B \setminus A$

Q40. Consider the sets:

$$\begin{aligned} A =&]-3, 2[\\ B =&]-1, 3] \cup]5, 7] \\ C =& \{x \in \mathbb{R} |\ x \leq -1\} \\ \text{and } D =& \{x \in \mathbb{R} |\ x > -3\} \text{ - all subsets of universal set } U = \mathbb{R}. \end{aligned}$$

Find the following intervals. Write your answer as an interval or a union of mutually exclusive intervals. Mark the solutions on the real number line.

(1) $A \cap B \cap C$
(2) $C \setminus (A \cup B)$
(3) $A \cup B \setminus C$
(4) $D \setminus B$
(5) $A \cap B'$
(6) $B \cup C'$
(7) $(B \cup C)'$
(8) $(A \cap D)'$
(9) $A \setminus B$
(10) $C \setminus D$

Q41. Use de Morgan's laws to complete the following.

(1) $(A \cup B)' =$
(2) $(A \cap B)' =$
(3) $(A' \cap B)' =$
(4) $(A \cup B')' =$
(5) $A' \cup B' =$
(6) $A' \cap B' =$
(7) $A' \cup B =$
(8) $A \cap B' =$

2.4 Chapter review

Q1. Consider the following sets:

$$\begin{aligned} A &= \{x \in \mathbb{R}^- |\ \ |x| < 5\} \\ B &= \{x \in \mathbb{R}^- |\ \ |x| \geq 3\} \\ C &=]-\infty, -4[\cup [-3, +\infty[\\ D &=]-\infty, -5[\cup [3, +\infty[\\ E &= \{x \in \mathbb{Z} |\ \ (2x + 1 < 9) \wedge (1 - 3x < 6)\} \\ F &= D' \cap \mathbb{Z} \end{aligned}$$

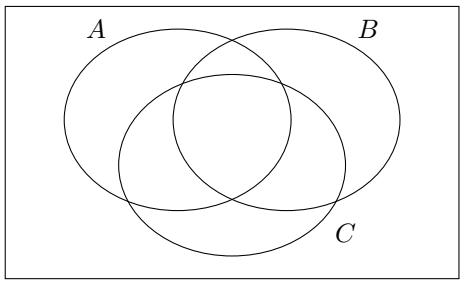
Write down in the easiest possible way (i.e. listing the elements or as an interval or as a union of mutually exclusive intervals) the following sets.

(1) $B \cap C$
(2) $E \cap F$
(3) $E \setminus F$
(4) $E \cap A'$
(5) $(D \cup E) \cap F'$
(6) $(A \cup C) \setminus D$
(7) $(C \setminus E) \cup F$
(8) $(C \setminus D) \cup A$

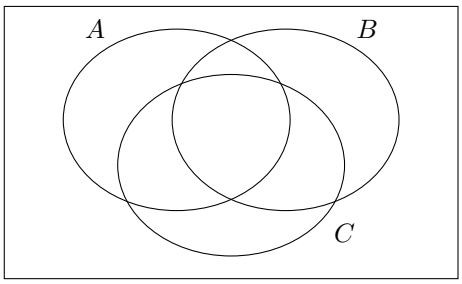
Q2. Write down all the proper subsets of a set $E = \{a, \star, \circ, \triangle, 5\}$ that are mutually exclusive with $\{b, 5, \star, 7\}$. How many subsets does set E have?

Q3. Shade:

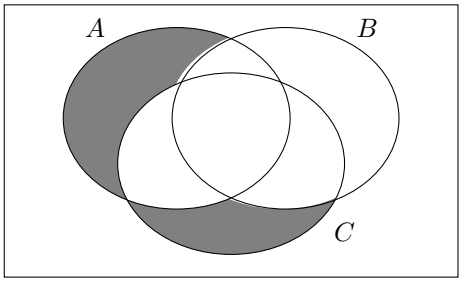
$A' \cap (B \cup C)'$



$(A \setminus B)' \cap C$



Q4. Name the shaded region for the Venn diagram:



Q5. The group of students consists of 25 women and 30 men. Sixtenn people in the group are red-haired. Sixteen women in the group do not have red hair. Draw the Venn diagram to show the situation and hence write down the number red-haired men.

Q6. A group of 38 students were asked to read three books: "Moby-Dick", "Slaughterhouse-five" and "Ulysses".

It turned out that only 18 of them read "Moby-dick" and 22 read "Slaughterhouse-five".
All students who read "Ulysses" read at least one of the other two books.

15 students read neither "Moby-dick" nor "Ulysses".
4 read "Moby-dick" and "Slaughterhouse-five".
6 read "Moby-dick" and "Ulysses".
8 read "Slaughterhouse-five" and "Ulysses".

Draw a Venn diagram to show the situation and answer the questions below.

(a) How many students read ""Slaughterhouse-five only?

(b) How many students read all three books?

(c) How many students did not read any of the books?

Chapter 3

Lines and vectors

3.1 Basic concepts

Q1. Find the midpoint of the line segment AB.

(1)	$A(9,2)$,	$B(3,12)$	(5)	$A(8,-7)$,	$B(6,4)$
(2)	$A(4,-5)$,	$B(12,3)$	(6)	$A(-8,0)$,	$B(-8,-6)$
(3)	$A(-3,-4)$,	$B(9,-6)$	(7)	$A(-3,5)$,	$B(4,8)$
(4)	$A(-5,8)$,	$B(3,-4)$	(8)	$A(23,-11)$,	$B(45,-3)$

Q2. Find the distance between points A and B. Give exact, simplified answer.

(1)	$A(2,5)$,	$B(5,9)$	(6)	$A(-5,-5)$,	$B(-4,2)$
(2)	$A(-3,5)$,	$B(2,-7)$	(7)	$A(6,-3)$,	$B(4,8)$
(3)	$A(7,2)$,	$B(-1,8)$	(8)	$A(2,5)$,	$B(6,-3)$
(4)	$A(2,-1)$,	$B(6,3)$	(9)	$A(9,3)$,	$B(-5,5)$
(5)	$A(-4,5)$,	$B(-1,-4)$	(10)	$A(1,-2)$,	$B(9,13)$

3.2 *Slope-intercept* equation of a line

Q3. Find the gradient (slope) of the line segment AB. Give the answer as an integer or an exact, simplified fraction.

(1)	$A(2,5)$,	$B(5,9)$	(5)	$A(6,3)$,	$B(12,1)$
(2)	$A(2,-1)$,	$B(-6,5)$	(6)	$A(-5,-1)$,	$B(-2,5)$
(3)	$A(4,-4)$,	$B(9,1)$	(7)	$A(1,4)$,	$B(5,-4)$
(4)	$A(3,-4)$,	$B(6,-3)$	(8)	$A(-2,-2)$,	$B(-6,4)$

Q4. Write down the coordinates of three points that lie on the line passing through A with a gradient m.

(1) $m = \frac{2}{3}$,	$A(0, 0)$	(6) $m = -\frac{1}{2}$,	$A(-3, -2)$
(2) $m = -\frac{3}{2}$,	$A(0, 0)$	(7) $m = \frac{3}{4}$,	$A(-2, -3)$
(3) $m = 2$,	$A(1, -2)$	(8) $m = -\frac{5}{3}$,	$A(-1, 5)$
(4) $m = -3$,	$A(-4, 1)$	(9) $m = \frac{7}{3}$,	$A(-8, -12)$
(5) $m = \frac{1}{4}$,	$A(2, -1)$	(10) $m = -\frac{2}{5}$,	$A(-8, 5)$

Q5. Sketch the line with gradient m and passing through point A.

(1) $m = \frac{2}{3}$,	$A(0, 0)$	(6) $m = -\frac{1}{2}$,	$A(-3, -2)$
(2) $m = -\frac{3}{2}$,	$A(0, 0)$	(7) $m = \frac{3}{4}$,	$A(-2, -3)$
(3) $m = 2$,	$A(1, -2)$	(8) $m = -\frac{5}{3}$,	$A(-1, 5)$
(4) $m = -3$,	$A(-4, 1)$	(9) $m = \frac{7}{3}$,	$A(-8, -12)$
(5) $m = \frac{1}{4}$,	$A(2, -1)$	(10) $m = -\frac{2}{5}$,	$A(-8, 5)$

Q6. Check algebraically if point A lies on line k.

(1) $A(3, 5)$,	$k : y = \frac{2}{3}x + 3$	(6) $A(6, -8)$,	$k : y = -\frac{1}{2}x - 5$
(2) $A(-4, 8)$,	$k : y = -\frac{3}{2}x + 2$	(7) $A(6, -1.5)$,	$k : y = \frac{3}{4}x - 6$
(3) $A(-3, -4)$,	$k : y = 2x - 1$	(8) $A(-6, -8)$,	$k : y = -\frac{5}{3}x + 2$
(4) $A(2, -9)$,	$k : y = -3x - 3$	(9) $A(-3, -6)$,	$k : y = \frac{7}{3}x + 1$
(5) $A(8, 6)$,	$k : y = \frac{1}{4}x + 4$	(10) $A(5.5, 0.8)$,	$k : y = -\frac{2}{5}x + 3$

Q7. Check the missing coordinate of point A given it lies on line k.

(1) $A(6, y)$,	$k : y = \frac{2}{3}x + 3$	(6) $A(x, -3.5)$,	$k : y = -\frac{1}{2}x - 5$
(2) $A(-3, y)$,	$k : y = -\frac{3}{2}x + 2$	(7) $A(x, 4.5)$,	$k : y = \frac{3}{4}x - 6$
(3) $A(2, y)$,	$k : y = 2x - 1$	(8) $A(x, 12)$,	$k : y = -\frac{5}{3}x + 2$
(4) $A(-1, y)$,	$k : y = -3x - 3$	(9) $A(x, 15)$,	$k : y = \frac{7}{3}x + 1$
(5) $A(6, y)$,	$k : y = \frac{1}{4}x + 4$	(10) $A(x, 4.8)$,	$k : y = -\frac{2}{5}x + 3$

Q8. Find the equation of the line passing through A and B.

(1) $A(2, 1)$,	$B(-1, -5)$	(6) $A(3, 0)$,	$B(-6, 15)$
(2) $A(-3, 8)$,	$B(1, -4)$	(7) $A(-3, -4)$,	$B(6, 17)$
(3) $A(4, 3)$,	$B(-6, 0.5)$	(8) $A(4.5, 0.2)$,	$B(6.5, -0.6)$
(4) $A(6, -7)$,	$B(-2, -3)$	(9) $A(6, 5)$,	$B(-3, -1)$
(5) $A(6, -1.5)$,	$B(12, 3)$	(10) $A(-3, 7.5)$,	$B(4, -3)$

Q9. Find the line parallel to k and passing through A.

(1) $k : y = 2x - 3$,	$A(2, 2)$	(6) $k : y = -\frac{5}{3}x + 5$,	$A(7, -11\frac{1}{3})$
(2) $k : y = -3x - 1$,	$A(3, -6)$	(7) $k : y = \frac{7}{3}x + 3$,	$A(8, 19\frac{2}{3})$
(3) $k : y = \frac{1}{4}x + 2$,	$A(4, \frac{1}{2})$	(8) $k : y = -\frac{2}{5}x + 2$,	$A(9, -5\frac{1}{5})$
(4) $k : y = -\frac{1}{2}x - 4$,	$A(5, 0)$	(9) $k : y = \frac{2}{3}x + 1$,	$A(10, 9\frac{2}{3})$
(5) $k : y = \frac{3}{4}x - 6$,	$A(6, 4)$	(10) $k : y = -\frac{3}{2}x + 3$,	$A(11, -20\frac{1}{2})$

Q10. Find the line perpendicular to k and passing through A.

(1) $k : y = 2x - 2$, $A(2, 2)$
(2) $k : y = -3x + 3$, $A(3, -6)$
(3) $k : y = \frac{1}{4}x - 1\frac{1}{4}$, $A(-3, -2)$
(4) $k : y = -\frac{1}{2}x - 1\frac{1}{2}$, $A(-3, 0)$
(5) $k : y = \frac{3}{4}x - 0\frac{1}{2}$, $A(6, 4)$
(6) $k : y = -\frac{5}{3}x - 3\frac{2}{3}$, $A(-4, 3)$
(7) $k : y = \frac{7}{3}x + 3\frac{2}{3}$, $A(-2, -1)$
(8) $k : y = -\frac{2}{5}x + 5\frac{3}{5}$, $A(9, 2)$
(9) $k : y = \frac{2}{3}x - 9\frac{2}{3}$, $A(10, -3)$
(10) $k : y = -\frac{3}{2}x - 5$, $A(-4, 1)$

Q11. Find the value of parameter m for which the linear function is increasing.

(1) $y = mx + 5$
(2) $y = (m + 1)x + 1$
(3) $y = (m - 2)x - 3$
(4) $y = (2m - 5)x + m$

Q12. Find the value of parameter m for which the linear function is decreasing.

(1) $y = (4 - m)x + 6$
(2) $y = (4m + 3)x - 2$
(3) $y = m^2x + 5$
(4) $y = (m^2 - 4)x - 3$

Q13. Mr Headway drove from Worshow to Kraken by car. He left at 12 o'clock and then every half an hour he registered the distance covered so far. The table shows his records.

time $[h]$	0.5	1	1.5	2	2.5	3	3.5
distance $[km]$	45	92	136	184	231	274	320

Basing on the data in the table answer the following questions:

(i) What was the average mr Headway's speed?
(ii) In which half-hour period his average speed was greatest? What was this average speed?
(iii) In which one-hour period his average speed was greatest? What was this average speed?

3.3 *General* equation of a line

Q14. Express the given general equation of a line in slope-intecept form.

(1) $4x + 6y + 15 = 0$
(2) $6x + 3y - 4 = 0$
(3) $x - 2y + 3 = 0$
(4) $9x + 12y + 4 = 0$
(5) $3x - y - 2 = 0$
(6) $5x - 3y - 4 = 0$
(7) $4x - 10y - 35 = 0$
(8) $14x + 6y + 3 = 0$
(9) $x + 4y - 2 = 0$
(10) $9x - 6y + 8 = 0$

Q15. Express the given slope-intecept equation of a line in the form $Ax + By + C = 0$, where $A, B, C \in \mathbb{Z}, A > 0$ and $gcd(A, B, C) = 1$.

(1) $y = 2x - 1\frac{1}{3}$
(2) $y = -3x + 2$
(3) $y = \frac{1}{4}x + \frac{1}{2}$
(4) $y = -\frac{1}{2}x - 1\frac{1}{2}$
(5) $y = \frac{3}{4}x + \frac{1}{3}$
(6) $y = -\frac{5}{3}x + 1\frac{1}{3}$
(7) $y = \frac{7}{3}x - \frac{1}{2}$
(8) $y = -\frac{2}{5}x + 3\frac{1}{2}$
(9) $y = \frac{2}{3}x - 2\frac{1}{2}$
(10) $y = -\frac{3}{2}x - 1\frac{1}{3}$

Q16. Find the equation of the line passing through points P and Q. Give your answer in the form $Ax + By + C = 0$, where $A, B, C \in \mathbb{Z}, A > 0$ and $gcd(A, B, C) = 1$.

(1) $P(3, -4.5)$, $Q(-7.5, 2.5)$
(2) $P(-0.5, 3)$, $Q(3, -7.5)$
(3) $P(-1, 1)$, $Q(4, 3.5)$
(4) $P(7, -6.5)$, $Q(-2, 1)$
(5) $P(6, 16)$, $Q(2, 4)$
(6) $P(-4, -8)$, $Q(2, 2)$

(7) $P(-5,-11)$, $Q(10,1)$
(8) $P(6,-14.5)$, $Q(-3,6.5)$
(9) $P(10,-2)$, $Q(6,-1)$
(10) $P(-2,-1.5)$, $Q(2,3)$

Q17. Find the general equation of the line **parallel** to k and passing through point P.

(1) $k: 4x+6y+15=0$, $P(2,2)$
(2) $k: 6x+3y-4=0$, $P(3,-6)$
(3) $k: x-2y+3=0$, $P(-3,-2)$
(4) $k: 9x+12y+4=0$, $P(-3,0)$
(5) $k: 3x-y-2=0$, $P(6,4)$
(6) $k: 5x-3y-4=0$, $P(-4,3)$
(7) $k: 4x-10y-35=0$, $P(-2,-1)$
(8) $k: 14x+6y+3=0$, $P(9,2)$
(9) $k: x+4y-2=0$, $P(10,-3)$
(10) $k: 9x-6y+8=0$, $P(-4,1)$

Q18. Find the general equation of the line **perpendicular** to k and passing through point P.

(1) $k: 4x+6y+15=0$, $P(2,2)$
(2) $k: 6x+3y-4=0$, $P(3,-6)$
(3) $k: x-2y+3=0$, $P(-3,-2)$
(4) $k: 9x+12y+4=0$, $P(-3,0)$
(5) $k: 3x-y-2=0$, $P(6,4)$
(6) $k: 5x-3y-4=0$, $P(-4,3)$
(7) $k: 4x-10y-35=0$, $P(-2,-1)$
(8) $k: 14x+6y+3=0$, $P(9,2)$
(9) $k: x+4y-2=0$, $P(10,-3)$
(10) $k: 9x-6y+8=0$, $P(-4,1)$

Q19. (*) Find the distance between point P and line k.

(1) $k: 4x+3y-17=0$, $P(10,9)$
(2) $k: 3x+2y+9=0$, $P(-8,-12)$
(3) $k: x-2y-1=0$, $P(-1,-6)$
(4) $k: 5x+12y+15=0$, $P(\frac{9}{2},18)$
(5) $k: 3x-y-14=0$, $P(9,3)$
(6) $k: 6x-3y-12=0$, $P(\frac{61}{2},-\frac{21}{2})$
(7) $k: 4x-10y-5=0$, $P(-\frac{37}{4},19)$
(8) $k: 6x+4y+1=0$, $P(\frac{9}{2},6)$
(9) $k: x+4y+2=0$, $P(12,5)$
(10) $k: 2x-5y+13=0$, $P(2,-14)$
(11) $k: 2x-0.5y+3.5=0$, $P(11,0)$
(12) $k: \frac{1}{4}x-5y-\frac{11}{2}=0$, $P(\frac{5}{2},-11)$
(13) $k: 2x-\frac{1}{5}y+\frac{41}{5}=0$, $P(16,-1)$
(14) $k: \frac{1}{3}x-2y+\frac{10}{3}=0$, $P(-2,-11)$
(15) $k: x-\frac{2}{3}y+\frac{14}{3}=0$, $P(-1,-1)$

3.4 Vectors

Q20. Find vector $\vec{AB}$.

(1) $A(3,-1)$, $B(-1,3)$
(2) $A(0,-8)$, $B(2,-4)$
(3) $A(3,-2)$, $B(6,4)$
(4) $A(-2,-10)$, $B(-1,-13)$
(5) $A(4,9)$, $B(6,15)$
(6) $A(8,7)$, $B(5,14)$
(7) $A(-4,-5)$, $B(-9,-15)$
(8) $A(-5,8)$, $B(3,-2)$
(9) $A(-3,-9)$, $B(5,-6)$
(10) $A(10,7)$, $B(6,15)$

Q21. Find point B.

(1) $A(-10,-5)$, $\vec{AB}=\begin{pmatrix}-8\\4\end{pmatrix}$
(2) $A(5,-4)$, $\vec{AB}=\begin{pmatrix}-2\\-3\end{pmatrix}$
(3) $A(4,-7)$, $\vec{AB}=\begin{pmatrix}-3\\7\end{pmatrix}$
(4) $A(2,0)$, $\vec{AB}=\begin{pmatrix}4\\10\end{pmatrix}$
(5) $A(4,9)$, $\vec{AB}=\begin{pmatrix}-3\\-6\end{pmatrix}$
(6) $A(5,0)$, $\vec{AB}=\begin{pmatrix}-6\\-9\end{pmatrix}$
(7) $A(-3,5)$, $\vec{AB}=\begin{pmatrix}-3\\-3\end{pmatrix}$
(8) $A(-3,-9)$, $\vec{AB}=\begin{pmatrix}3\\8\end{pmatrix}$
(9) $A(-8,2)$, $\vec{AB}=\begin{pmatrix}3\\8\end{pmatrix}$
(10) $A(5,6)$, $\vec{AB}=\begin{pmatrix}10\\-2\end{pmatrix}$

Q22. Find vector $\vec{w}$ given $\vec{u}$ and $\vec{v}$.

(1) $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\vec{w} = 2\vec{u} + 3\vec{v}$

(2) $\vec{u} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\vec{w} = 4\vec{u} - 5\vec{v}$

(3) $\vec{u} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\vec{w} = -2\vec{u} - 3\vec{v}$

(4) $\vec{u} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$, $\vec{w} = 5\vec{u} + 6\vec{v}$

(5) $\vec{u} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, $\vec{w} = 3\vec{u} + 4\vec{v}$

(6) $\vec{u} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, $\vec{w} = -4\vec{u} + 5\vec{v}$

Q23. Find vector $\vec{u}$.

(1) $A(3,-4)$, $B(-2,3)$, $C(1,-1)$, $\vec{u} = \vec{AB} - \vec{BC}$

(2) $A(3,-4)$, $B(-2,3)$, $C(1,-1)$, $\vec{u} = \vec{BC} - \vec{AC}$

(3) $A(4,-3)$, $B(3,-2)$, $C(2,-1)$, $\vec{u} = 2\vec{AB} + \vec{BC}$

(4) $A(4,-3)$, $B(3,-2)$, $C(2,-1)$, $\vec{u} = -\vec{AB} + \vec{AC}$

(5) $A(-7,1)$, $B(4,4)$, $C(-7,8)$, $\vec{u} = -2\vec{AB} + \vec{BC}$

(6) $A(-2,0)$, $B(-3,1)$, $C(-7,-5)$, $\vec{u} = -3\vec{AB} + 2\vec{AC}$

(7) $A(-1,3)$, $B(4,8)$, $C(7,3)$, $\vec{u} = -\vec{BC} + 2\vec{AC}$

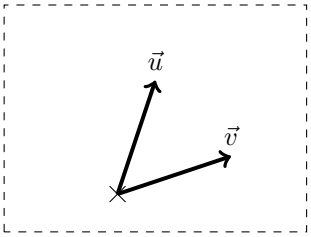
(8) $A(-6,2)$, $B(-4,-5)$, $C(5,-4)$, $\vec{u} = 2\vec{AB} + \vec{AC}$

(9) $A(-4,0)$, $B(-7,-7)$, $C(-3,5)$, $\vec{u} = \vec{BC} - 2\vec{AC}$

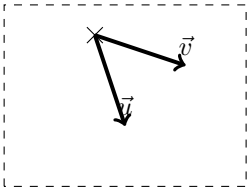
(10) $A(-2,-1)$, $B(6,3)$, $C(1,-8)$, $\vec{u} = -\vec{AB} + 3\vec{BC}$

Q24. In each of the diagrams below draw vectors $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$.

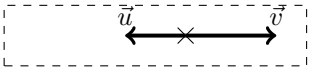
(1)



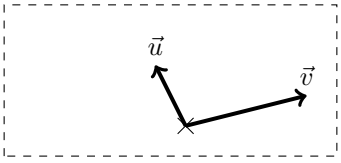
(2)



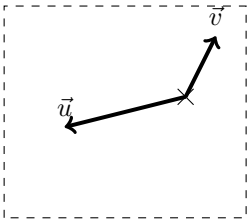
(3)



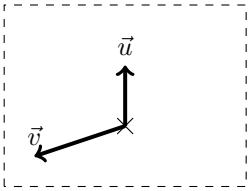
(4)



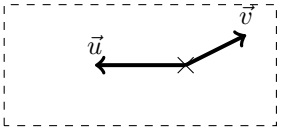
(5)



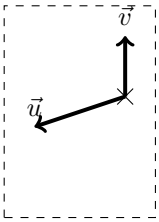
(6)



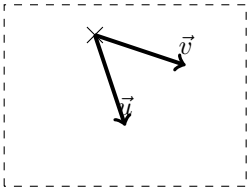
(7)



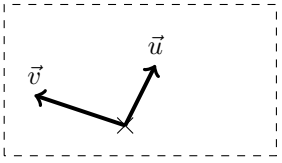
(8)



(9)

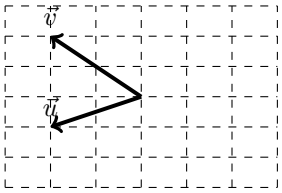


(10)

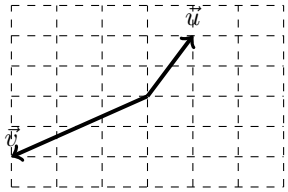


Q25. Draw vector $\vec{w}$ in the grid below questions.

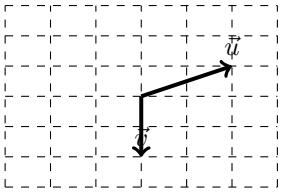
(1) $\vec{w} = 2\vec{u} + 3\vec{v}$



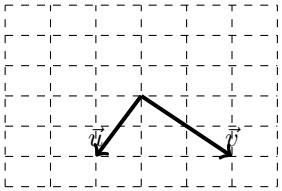
(2) $\vec{w} = 4\vec{u} - 5\vec{v}$



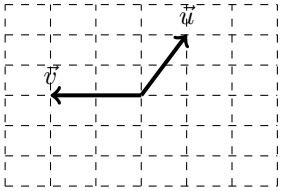
(3) $\vec{w} = -2\vec{u} - 3\vec{v}$



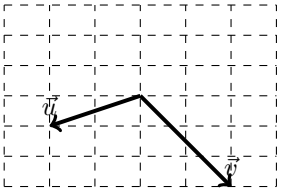
(4) $\vec{w} = 5\vec{u} + 6\vec{v}$



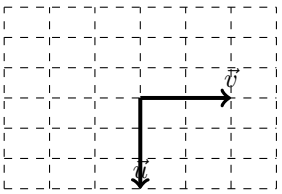
(5) $\vec{w} = 3\vec{u} + 4\vec{v}$



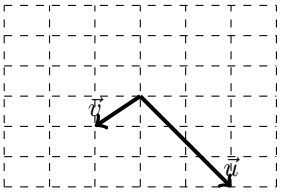
(6) $\vec{w} = -4\vec{u} + 5\vec{v}$



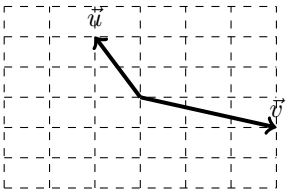
(7) $\vec{w} = 2\vec{u} - 3\vec{v}$



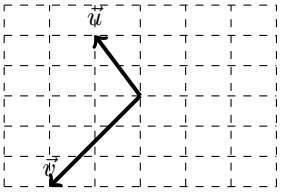
(8) $\vec{w} = -3\vec{u} + 2\vec{v}$

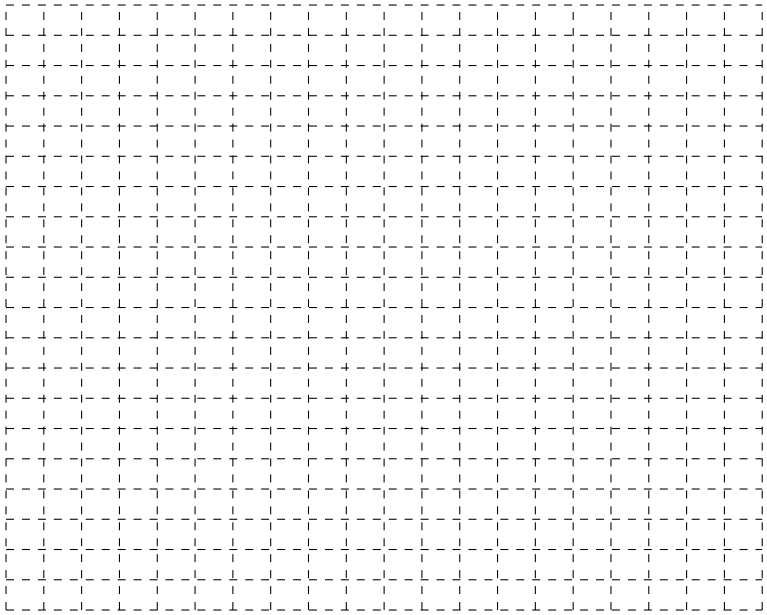


(9) $\vec{w} = 4\vec{u} - 1\vec{v}$



(10) $\vec{w} = -3\vec{u} + \vec{v}$





Q26. Find vectors **parallel** to $\vec{v}$ with magnitude d.

(1) $\vec{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $d = 10$

(2) $\vec{v} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$, $d = 15$

(3) $\vec{v} = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$, $d = 52$

(4) $\vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $d = 30$

(5) $\vec{v} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$, $d = 6\sqrt{5}$

(6) $\vec{v} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$, $d = 20$

(7) $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $d = 26$

(8) $\vec{v} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$, $d = 10\sqrt{2}$

(9) $\vec{v} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$, $d = 6$

(10) $\vec{v} = \begin{pmatrix} 2 \\ -2\sqrt{3} \end{pmatrix}$, $d = 6$

Q27. Find the values of c for which vecors $\vec{u}$ and $\vec{v}$ are parallel.

(1) $\vec{u} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 6 \\ 12 + c \end{pmatrix}$

(2) $\vec{u} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -6 \\ 2c + 4 \end{pmatrix}$

(3) $\vec{u} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 6 \\ 1 - 2c \end{pmatrix}$

(4) $\vec{u} = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3c - 6 \\ -6 \end{pmatrix}$

(5) $\vec{u} = \begin{pmatrix} 6 \\ -15 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -8 \\ 3c + 11 \end{pmatrix}$

(6) $\vec{u} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 12 \\ 1 + 3c \end{pmatrix}$

Q28. Find vectors **perpendicular** to $\vec{v}$ with magnitude d.

(1) $\vec{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $d = 15$

(2) $\vec{v} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$, $d = 10$

(3) $\vec{v} = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$, $d = 39$

(4) $\vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $d = 20$

(5) $\vec{v} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$, $d = 3\sqrt{5}$

(6) $\vec{v} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$, $d = 30$

(7) $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $d = 39$

(8) $\vec{v} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$, $d = 20\sqrt{2}$

(9) $\vec{v} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$, $d = 9$

(10) $\vec{v} = \begin{pmatrix} 2 \\ -2\sqrt{3} \end{pmatrix}$, $d = 8$

Q29. Write down two vectors that move line k onto line m.

(1) $k : y = \frac{2}{3}x + 3$, $m : y = \frac{2}{3}x + 2$

(2) $k : y = -\frac{3}{2}x + 2$, $m : y = -\frac{3}{2}x - 1$

(3) $k : y = 2x - 1$, $m : y = 2x - 4$

(4) $k : y = -3x - 3$, $m : y = -3x + 3$

(5) $k : y = \frac{1}{4}x + 4$, $m : y = \frac{1}{4}x - 5$

(6) $k : y = -\frac{1}{2}x - 5$, $m : y = -\frac{1}{2}x - 1$

(7) $k : y = \frac{3}{4}x - 6$, $m : y = \frac{3}{4}x + 2$

(8) $k : y = -\frac{5}{3}x + 2$, $m : y = -\frac{5}{3}x + 3$

(9) $k : y = \frac{7}{3}x + 1$, $m : y = \frac{7}{3}x + 4$

(10) $k : y = -\frac{2}{5}x + 3$, $m : y = -\frac{2}{5}x - 1$

(11) $k : 4x + 3y - 14 = 0$, $m : 4x + 3y - 23 = 0$

(12) $k : 5x + 2y - 3 = 0$, $m : 5x + 2y - 7 = 0$

(13) $k : x - 2y - 1 = 0$, $m : x - 2y - 11 = 0$

(14) $k : 7x + 12y + 21 = 0$, $m : 7x + 12y + 9 = 0$

(15) $k : 3x - y - 14 = 0$, $m : 3x - y - 16 = 0$

(16) $k : 5x - 3y + 29 = 0$, $m : 5x - 3y + 17 = 0$

(17) $k : 4x - 10y - 2 = 0$, $m : 4x - 10y + 18 = 0$

(18) $k : 3x + 5y - 37 = 0$, $m : 3x + 5y - 67 = 0$

(19) $k : x + 4y + 2 = 0$, $m : x + 4y + 14 = 0$

(20) $k : 9x - 6y + 42 = 0$, $m : 9x - 6y + 36 = 0$

3.5 Simultaneous equations

Q30. Solve the simultaneous equations.

(1) $\begin{cases} y = \frac{1}{2}x + 1 \\ y = -3x + 6 \end{cases}$

(2) $\begin{cases} y = \frac{1}{2}x + \frac{3}{2} \\ y = -\frac{1}{2}x + \frac{5}{2} \end{cases}$

(3) $\begin{cases} x - y = 5 \\ y = -\frac{1}{2}x - 2 \end{cases}$

(4) $\begin{cases} x - 3y + 5 = 0 \\ 2x + y + 3 = 0 \end{cases}$

(5) $\begin{cases} 2x + 2y - 1 = 0 \\ y = 2x + 5 \end{cases}$

(6) $\begin{cases} y = -2x - 3 \\ 33 - x(3y - 1) = 3y(1 - x) \end{cases}$

(7) $\begin{cases} x + 2y = 7 \\ 2x + 6 = 6y \end{cases}$

(8) $\begin{cases} y + x + 1 = 0 \\ 2y - x = 7 \end{cases}$

(9) $\begin{cases} 2x - y + 5 = 0 \\ x + y = 2 \end{cases}$

(10) $\begin{cases} 2y + x = 5 \\ 3x = y - 6 \end{cases}$

(11) $\begin{cases} y + 2x = 2 \\ y - x = 2 \end{cases}$

(12) $\begin{cases} 3y = x + 1 \\ y + 2x + 2 = 0 \end{cases}$

(13) $\begin{cases} 2y - 3x = 6 \\ 2y + x + 10 = 0 \end{cases}$

(14) $\begin{cases} 3x - 2y + 8 = 0 \\ 3y + 2x = 6 \end{cases}$

(15) $\begin{cases} y = 2x + 10 \\ 5y + 2x + 4 = 0 \end{cases}$

(16) $\begin{cases} 2x + 3y + 5 = 0 \\ 3x - 4y = 52 \end{cases}$

Q31. Solve the simultaneous equations.

(1) $\begin{cases} 2(x - 2y) - (x - 2y) + 3 = 0 \\ 3(5x - 2y) + 2(6y - 3x) + 3 = 0 \end{cases}$

(2) $\begin{cases} 15y = (x + \frac{5}{2})^2 - (x + \frac{5}{2})(x - \frac{5}{2}) + \frac{25}{2} \\ x + y = 1 \end{cases}$

(3) $\begin{cases} 2x - (y - 2)^2 = 3 - (y - 1)(y + 1) \\ y - \frac{2x-y}{6} = \frac{1}{2} - \frac{2y-x}{4} \end{cases}$

(4) $\begin{cases} y - (2x - 3)^2 = 3 - (2x - 1)(2x + 1) \\ 2x - \frac{y-2}{3} = 2 - \frac{y-3x}{6} \end{cases}$

(5) $\begin{cases} 3(x + 1) + (y + \frac{1}{4})^2 = (y - \frac{1}{4})^2 \\ y + 2x = 4(x + 2) \end{cases}$

(6) $\begin{cases} (2x + \frac{1}{2})^2 - (y - \frac{1}{2})^2 = (2x - y)(2x + y) \\ (x - \frac{1}{2})^2 - (2y + \frac{1}{2})^2 = (x - 2y)(2y + x) - 6 \end{cases}$

(7) $\begin{cases} (x + 1)^2 - (y - 1)^2 = (x - y)(x + y) - 5 \\ (y + 2)^2 - (x - 2)^2 = (y - x)(x + y) \end{cases}$

(8) $\begin{cases} (2 - 4x)^2 + 48y = (4x - 2)(2 + 4x) \\ (2y - 3)^2 + 4x = (2y - 3)(2y + 3) \end{cases}$

3.6 Applications of linear equations and vectors

Q32. Let C be point $(-5, 4)$. Let A and B be points on line $y = \frac{1}{2}x - 1$ with x-coordinates -4 and 4 respectively.

(i) Find the coordinates of A and B

(ii) Find the length of line segment AB.

(iii) Find the area of triangle ABC

Q33. Let C be point $(2, 4)$. Let A and B be points on line $y = -\frac{1}{3}x + \frac{2}{3}$ with x-coordinates -4 and 8 respectively.

(i) Find the coordinates of A and B

(ii) Find the length of line segment AB.

(iii) Find the area of triangle ABC

Q34. Let A and B be points with y-coordinate 6 on lines $y = -\frac{1}{3}x + 4$ and $y = 2x - 3$ resectively.

(i) Find the x-coordinates of A and B.

(ii) Find the coordinates of point C - the intersection of the lines.

(iii) Find the area of triangle ABC.

Q35. (i) (Use a calculator for this question.) Find the perimeter of the triangle bounded by the lines

$$\begin{cases} 2x + y + 6 = 0 \\ x - 2y - 2 = 0 \\ x + 2y - 6 = 0 \end{cases}$$

(ii) Find the area of the triangle.

Q36. (i) (Use a calculator for this question.) Find the perimeter of the triangle bounded by the lines

$$\begin{cases} x - 3y + 9 = 0 \\ 3x + y + 17 = 0 \\ 3x - y - 5 = 0 \end{cases}$$

(ii) Find the area of the triangle.

Q37. Find the area of triangle ABC, where $A = (-3, -2), B = (3, 1)$ and $C = (-4, 5)$.

Q38. Find the area of triangle ABC, where $A = (-7, 3), B = (2, -3)$ and $C = (1, 2)$.

Q39. There are 20 kids in a class. Each girl has four pens and each boy has two pens. Altogether the kids have 64 pens. How many girls and how many boys are there in the class?

Q40. There are 52 vehicles in a parking lot. Each car has four wheels, each motorcycle has 2 wheels. Altogether there are 180 wheels in the parking lot. How many cars and how many motorcycles are there in the parking lot?

Q41. A manager bought a number of laptops for his office. He bought latops of two types: the cheaper ones for 2500 each and the more expensive ones for 3500 each. The number of cheaper laptops bought exceeds the number of the more expensive ones by 4. The manager spent for the laptops 52000 altogether. How many laptops (in total) did he buy?

Q42. In a ski resort hotel there are rooms of two types: for two and for four persons.
There are 12 more 4-bed rooms than 2-bed rooms.
The hotel offers places for 150 skiers altogether.
How many of them can take places in 2-bed rooms?

Q43. There are buses and cars in a parking lot. Each bus' capacity is 20 people while each car's capacity is 5 people. Altogether there can be 240 people placed in all the vehicles.
If each bus' capacity was 20% larger and each car's capacity was 20% smaller the total number of people that can be placed in all vehicles would stay unchnged.
How many cars and how many buses are there in the parking lot?

Q44. The lines

$$\begin{cases} 3x + 10y - 25 &= 0 \\ -4x + 3y + 23 &= 0 \end{cases}$$

intersect at point A.
The lines

$$\begin{cases} x + 3y + 12 &= 0 \\ 9x + 2y + 37 &= 0 \end{cases}$$

intersect at point C.
Find the equation of line passing through points A and C. Use a calculator.

Q45. Consider points $A = (-4, -6)$, $B = (8, -4)$ and $C = (1, 4)$.
Find the centroid[1] of triangle ABC. Use a calculator.

Q46. A taxi fare consists of a constant amount of 8pln and of 2.80pln paaid for each kilmetre driven. Let x denote the number of kilmetres driven and y denote the fare for the whole journey.

(i) Find an expression for y in terms of x.

(ii) What is the fare for a 12 km journey?

(iii) How far can one go for 30pln?

Q47. (Use a calculator for this question.) In the year 1940 the population of Molvania was 25.6 mln people. From that time the population increased at a constant rate and reached the level of 46 mln in 2012.

(i) What was the population of Molvania in 2000?

[1] A centroid of a triangle is the intersection of the medians. A median is a line connecting a vertex with the midpoint of the opposite side.

(ii) What was the population of Molvania in 1950?

(iii) What would be the population in 2020 if the rate of increase remained constant?

(iv) When would the number of people of Molvania exceed 100 mln if the rate of increase remained constant?

Q48. Let O denote the position of the Operating Centre. Three ports are situated as follows:

- port A is 2 miles North of O,
- port B is 4 miles East and 6 miles South of O,
- port C is 30 miles East and 2 miles North of O.

At 10am three boats, a, b and c left ports A, B and C respectively. They all sailed with constant speeds. At 11am their positions were:

- a: 3 miles East and 4 miles North of O,
- b: 6 miles East and 2 miles South of O,
- c: 24 miles East and 6 miles North of O.

(i) Which of the boats is the fastest? What is its velocity?

(ii) What is the position of intersection of routes of boats a and b? Will they meet there? If YES - at what time? If NO - which of the two boats will arrive at the point of intersection first?

(iii) What is the position of intersection of routes of boats a and C? Will they meet there? If YES - at what time? If NO - which of the two boats will arrive at the point of intersection first?

Q49. Peteborough is a village with one asphalt road only. The road is a straight line that connects two most distant points in the village: A which is located 3 kilometres West and 3 kilometres North of the Old Church and B placed 7 kilometres East and 2 kilometres South of the Old Church. Old Pete lives in a house 2 kilometres West of the church. He wants to get to the local market that is 5 kilometres East and 4 kilometres North of the church. To get there as quickly as possible he decided to take the shortest route to the asphalt road and then follow the road to the point closest to the market. From the point he would go straight to the destination.

(i) What will be the length of his juorney? Give answer correct to 100 metres.

Pete can drive $30kmh^{-1}$ on an asphalt road but only $5kmh^{-1}$ kmph off the road.

(ii) How long will his juorney be? Give answer correct to the nearest minute.

Q50. (Use a calculator for this question.) A new highway connects cities A and B in a straight line. A is placed 180 kilometres West and 60 kilometres South of C, the capital. B is placed 40 kilometres East and 40 kilometres North of C. Towns T and U are placed 126 kilometres West and 12 kilometres South of C and 56 kilometres West and 9 kilometres North of C respectively. The capital C and towns T and U are connected with the highway with the shortest possible routes. A car leaves A and is heading for B but on its way it is to visit T, U and finally C.

(i) What is the total distance that the car would travel?

(ii) Assuming that the car's average speed on a highway is $105\frac{km}{h}$ and off the highway it is $70\frac{km}{h}$, what would be the total time of the car's journey? Give the answer correct to the nearest minute.

(iii) What is the average speed of the car for the whole journey?

Q51. Triangle ABC is inscribed in a circle with radius 5. Two sides of the triangle are contained in the lines $y = \frac{1}{2}x + \frac{5}{2}$ and $y = -2x - 5$. First of the two sides is twice longer than the other. Find the area of the triangle.

Q52. Points $A(-5, 3)$ and $B(3, -1)$ are vertices of a rectangle $ABCD$. The diagonals of the rectangle intersect at point $M(\frac{1}{2}, 4)$. Find the cooridnates of the other two vertices of the rectangle.

Q53. Points $A(3, -3)$ and $B(7, -1)$ are endpoints of a base of an isosceles triangle ABC. The area of the triangle is 15. Find two possible positions of vertex C.

Q54. Points $A = (-5, -4)$, $B = (3, -2)$ and $C = (4, 2)$ are the consecutive vertices of parallelogram $ABCD$. Find the coordinates of D.

Q55. Points $A = (-7, 4)$ and $B = (3, -2)$ are two consecutive vertices of parallelogram $ABCD$. Point $M = (1, 2.5)$ is the point of intersection of the diagonals of the parallelogram. Find the coordinates of vertices C and D.

Q56. In triangle ABC the points A and B are $(-10, -4)$ and $(-7, 7)$ respectively. Point $S = (-5, 2)$ is the centroid of the triangle (i.e. the point of intersection of the medians, i.e. lines connecting vertices with midpoints of opposite sides). Find the coordinates of vertex C.

Q57. The area of the square $OABC$, where O is the origin, is 10. Find the coordinates of vertices A, B and C if they are above the x-axis and A lies on the line given by $y = \frac{1}{3}x$.

Q58. The x-coordinates of vertices of a rectangle $ABCD$ are negative. Knowing that $A = (-4, -5)$, $B = (-1, -3)$ and that the area of $ABCD$ is 26 find the coordinates of C and D.

Q59. Consider points $A = (-5, -10)$, $B = (1, -7)$ and $P = (9, 0)$.
Line k is parallel to line segment AB and passes through P.
Line segment AB is a side of a parallelogram $ABCD$, whose vertices C and D lie on line k.

(i) Find the equation of k.

(ii) Knowing that the perimeter of the parallelogram $ABCD$ is $10\sqrt{5}$ find the length of side BC.

(iii) Hence find the coordinates of C and D.

Q60. Consider the points $A = (-7, 4)$ and $B = (2, -2)$.
Points C and D are such that quadrilateral $ABCD$ is a **rectangle** with area 78.

(1) Find vector $\vec{AB}$.

(2) Find magnitude of $\vec{AB}$.

(3) Evaluate the length BC.

(4) Hence find vector $\vec{BC}$.

(5) Hence find the possible positions of points C and D.

Q61. $ABCD$ ia an **isosceles trapezium**, where AB is parallel to CD.
Points A and B are $(-5, -3)$ and $(7, 1)$ respectively.
The area of the trapezium is 87.5.
Point $M = (-1.5, 6.5)$ is the midpoint of side CD.

(1) Find vector $\vec{AB}$.

(2) Find magnitude of $\vec{AB}$.

(3) Find the distance between sides AB and CD.

(4) Find the length CD.

(5) Hence find the possible positions of points C and D.

Q62. Given points $A = (1, 4)$ and $C = (10, 1)$ find the coordinates of points B and D such that the **rhombus** $ABCD$ has the area of 75.

Q63. Consider the line k defined by equation $3x + 2y - 9 = 0$ and the point $A(-2, 1)$.
Line m is parallel to k and passes through A.

(i) Find the equation of m in general form.

Line p is perpendicular to k and passes through A.

(ii) Find the equation of p in general form.

B is the point of intersections of lines k and p.

(iii) Find the coordinates of B.

C is the x-intercept of line k.

(iv) Find the coordinates of C.

D is a point on line m such that the area of trapezium $ABCD$ is 19.5.

(v) Find the length AD. Give your answer in the form $q\sqrt{13}$, where $q \in \mathbb{Q}$.

(vi) Find the coordinates of D

Q64. Point $M = (-2, \frac{1}{2})$ is the midpoint of line segment AB where $A = (1, 2)$.

(i) Find the coordinates of B.

Triangle ABC is an isosceles triangle with area 15.

(ii) Find the coordinates of C.

Q65. Consider points $A(-3, 1)$, $B(5, 3)$ and $C(4, 7)$.

(i) Show that $A\hat{B}C$ is a right angle.

(ii) Find the equation of the circle circumscribed on triangle ABC.

(iii) Find the coordinates of point D such that $ABCD$ is a rectangle.

Q66. Consider the points $A(-9, 3)$, $B(3, -3)$ and $C(1, 3)$.

(i) Find an equation of the line passing through A and B. Give your answer in general form.

(ii) Write down a vector perpendicular to line AB and hence find the distance from C to the line.

D is a point such that CD is parallel to AB and the area of the trapezium $ABCD$ is 45.

(iii) Find the length CD.

(iv) Hence find the coordinates of D.

Q67. Consider the points $A(-12, -1)$, $B(-3, 5)$ and $C(-9, 7.5)$.

(i) Find the coordinates of point D, the reflection of C in line AB.

(ii) Find the area of the quadrilateral $ADBC$. Give your answer correct to 4 d.p.

Q68. Consider the points $A(-1, 5)$ and $B(5, 1)$.

(i) Find the coordinates of points C and D such that $ABCD$ is a square.

(ii) Find the coordinates of points C and D such that $ABCD$ is a rectangle with area 78.

Q69. (*) In an isosceles trapezium $ABCD$ ($AB||CD$) vertices A, B and C have coordinates $(-8, -2)$, $(4, 2)$ and $(-1, 5)$ respectively. Find the coordinates of M, the point of intersection of diagonals of the trapezium.

3.7 Chapter review

non-calculator questions

Q1. Consider points $A(-2, -3)$, $B(7, -6)$ and $C(5, 3)$.

(i) Find point S that lies on the line passing through A and B and is closest to C.

(ii) Find point D, the reflection of point C in line AB.

(iii) Find the area of quadrilateral $ADBC$.

(iv) Find the area of triangle DCA.

Q2. Consider line k defined by equation $x - 2y + 4 = 0$.
Line m is perpendicular to k and passess through point $(1, -5)$.

(i) Find the equation of m. Give answer in general form.

(ii) Find the coordinates of P, the point of intersection of k and m.

Points A and B are points with y-coordinate equal -4 on lines k and m respectively.

(iii) Find the x-coordinates of A and B.

Points C and D are points with y-coordinate equal 4 on lines k and m respectively.

(iv) Find the x-coordinates of C and D.

(v) Which of the triangles: ABD or BCD has larger area? Why?

(vi) Find the coordinates of point E such that quadrilateral $ABCE$ is a parallelogram.

Q3. Consider triangle ABC where $A = (-1, -3)$, $B = (8, -6)$ and $C = (13, 4)$.

(i) Find the perimeter of the triangle.

Point M is the midpoint of BC. Point S is the centroid of triangle ABC.

(ii) Find the coordinates of M.

(iii) Hence find the coordinates of S.

Line k is parallel to BC and passes through S.

(iv) Find the equation of k.

Line k intersects sides AB and AC at points D and E respectively.

(v) **Write down** the perimeter of triangle ADE. (You do not have to find coordinates of D or E.)

Q4. In a rhombus $ABCD$ points A and C have coordinates $(-11, 5)$ and $(1, -3)$ respectively. The area of a rhombus is 78.

(i) Find the length of diagonal BD.

(ii) Find the coordinates of M, the midpoint of AC.

(iii) Hence, or otherwise, find the position of two other vertices of the rhombus.

calculator questions

answers must be given exactly or correct to 3 s.f.

Q5. Consider the following lines:

(1) $x - 2y + 1 = 0$

(2) $5x + 6y - 35 = 0$

(3) $17x - 8y + 82 = 0$

Lines (1) and (2) intersect at point B. Lines (1) and (3) intersect at point A.

(i) Find the coordinates of A and B.

C is a point on line 2 with y-coordinate 5.

(ii) Find the coordinates of C.

Line (4) is parallel to (1) and passes through point C. Point D is the intersection of lines (3) and (4).

(iii) Find the equation of line (4).

(iv) Find the coordinates of D.

(v) Find the perimeter of quadrilateral $ABCD$.

(vi) Find the distance from point C to line (1).

(vii) Find the area of quadrilateral $ABCD$.

Q6. Consider the points: $A = (-3, -3)$, $B = (12, -6)$ and $C = (1, 5)$.

(i) Find vector parallel to $\vec{AC}$ with magnitude of $\vec{AB}$.

(ii) Find vector perpendicular to $\vec{AB}$ with magnitude of $\vec{AC}$.

(iii) Evaluate $\frac{\sqrt{17}}{2\pi}(\vec{AB} + \vec{BC} + \vec{CA})$.

(iv) Evaluate $2\vec{AB} - 3\vec{AC} + 2\vec{BC}$.

Q7. Consider line $13x + 17y - 159 = 0$. Let points A and B be the x-intercept and y-intercept of the line respectively.

(i) Find the coordinates of A and B.

C is the midpoint of AB.

(ii) Write down the coordinates of C.

Let point M be such that quadrilateral $OAMB$ is a rectangle, where O is the origin.

(iii) Find the area of $OAMB$.

Useful formulae:

midpoint of a line segment with endpoints $A(x_A, y_A)$ and $B(x_B, y_B)$	$\left(\frac{x_A+x_B}{2}, \frac{y_A+y_B}{2}\right)$
distance between the points $A(x_A, y_A)$ and $B(x_B, y_B)$	$\sqrt{(x_A-x_B)^2+(y_A-y_B)^2}$
gradient of a line segment with endpoints $A(x_A, y_A)$ and $B(x_B, y_B)$	$\frac{y_B-y_A}{x_B-x_A}$ or $\frac{y_A-y_B}{x_A-x_B}$
distance from point (p, q) to line $Ax+By+C=0$	$\frac{\lvert Ap+Bq+C\rvert}{\sqrt{A^2+B^2}}$

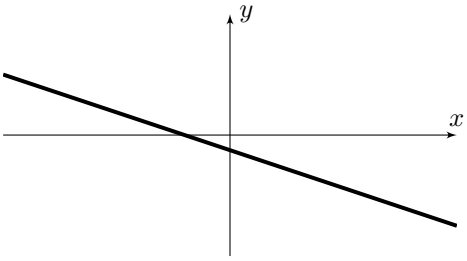
Chapter 4

Functions

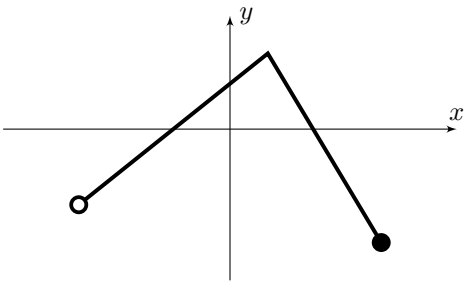
4.1 Basic properties

Q1. Does a diagram show a graph of a function? (where the domain is shown on the x-axis and the range on the y-axis)

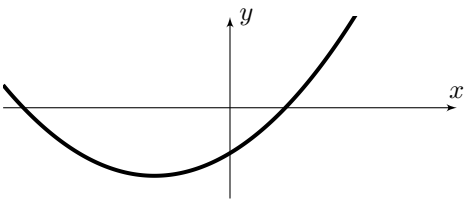
(1)



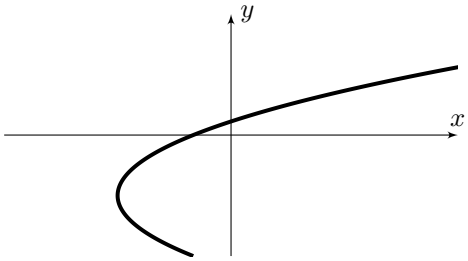
(2)



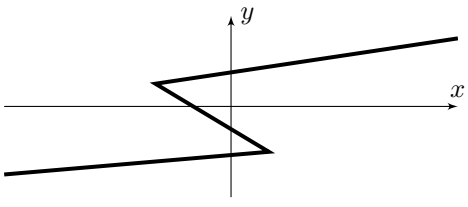
(3)



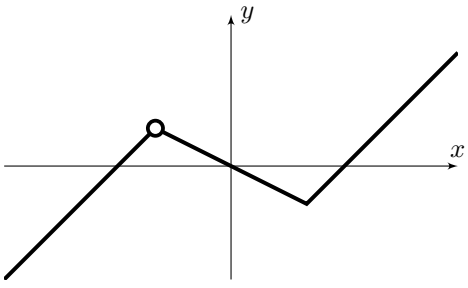
(4)



(5)



(6)



Q2. Each diagram below shows a graph of function f. From the graph read:

(i) the domain of f,

(ii) the range of f,

(iii) zeroes of f,

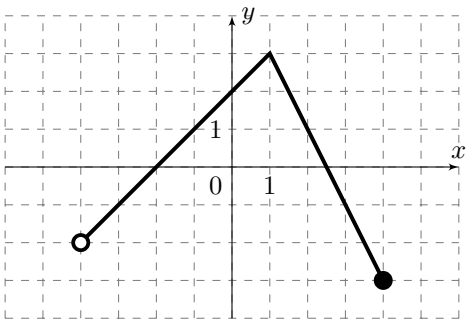
(iv) set(-s) of arguments for which f increases,

(v) set(-s) of arguments for which f decreases,

(vi) set(-s) of arguments for which f is constant,

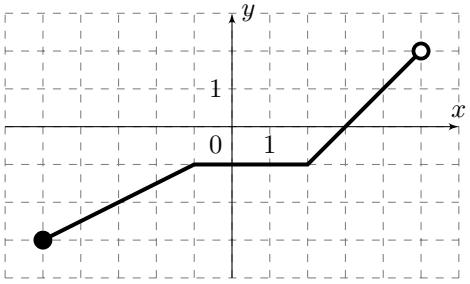
(vii) the set of arguments for which the values are...
(*note that the question finishes differently for each of the diagrams*)

(1)



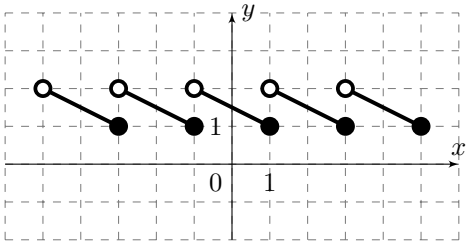
(vii) ... equal 2.

(2)



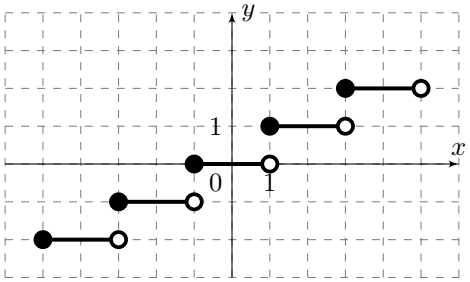
(vii) ... equal -1.

(3)



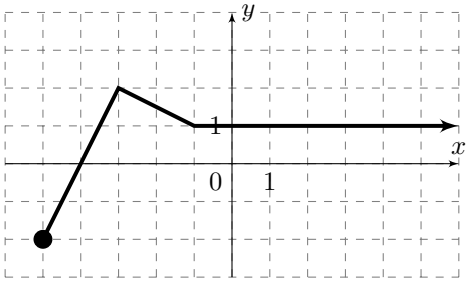
(vii) ... equal 3.

(4)



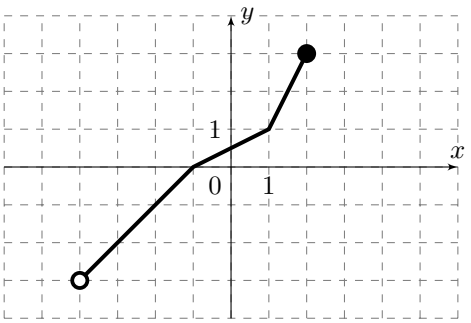
(vii) ... equal 1.5.

(5)



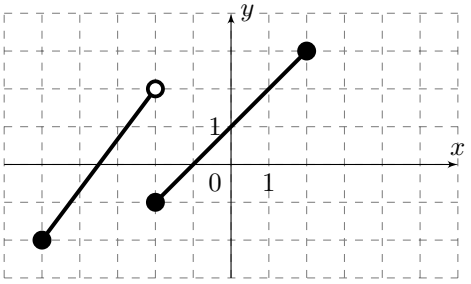
(vii) ... greater than 1.

(6)



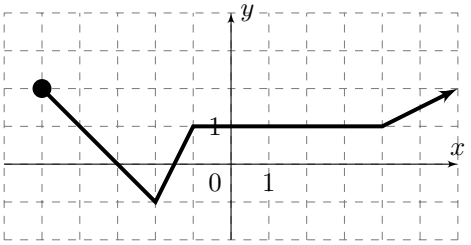
(vii) ... smaller or equal -1.

(7)



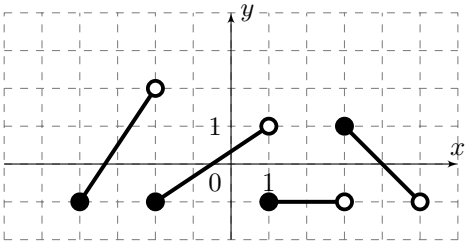
(vii) ... equal 2.

(8)



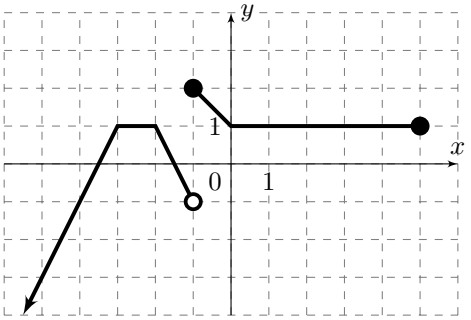
(vii) ... non-positive.

(9)

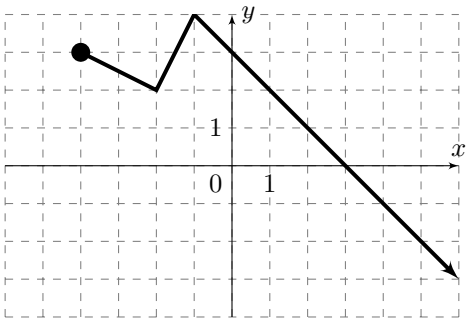


(vii) ... equal 1.

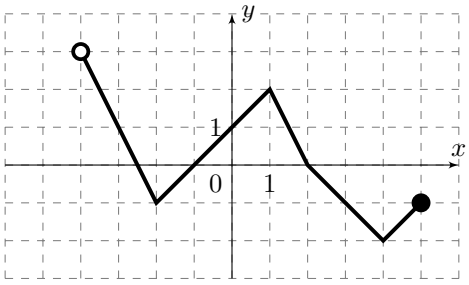
(10)



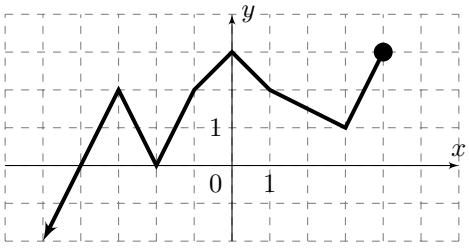
(vii) ... smaller or equal -1.

(11)


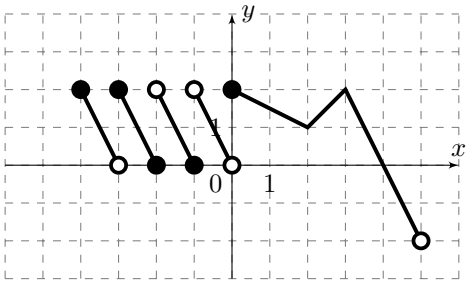
(vii) ... equal 3.

(12)


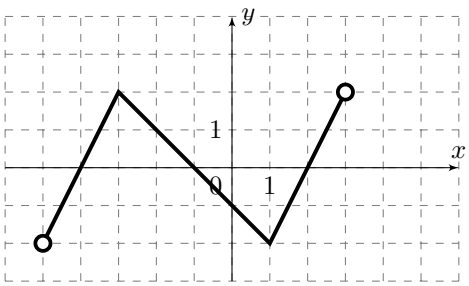
(vii) ... equal -1.

(13)


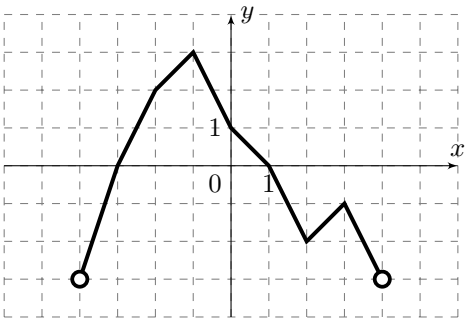
(vii) ... equal 2.

(14)


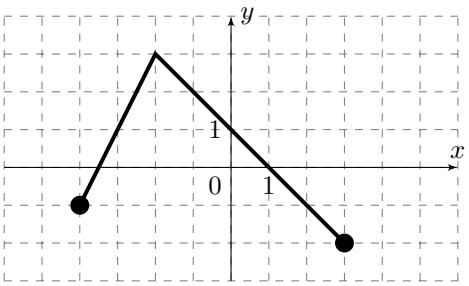
(vii) ... equal 2.

(15)


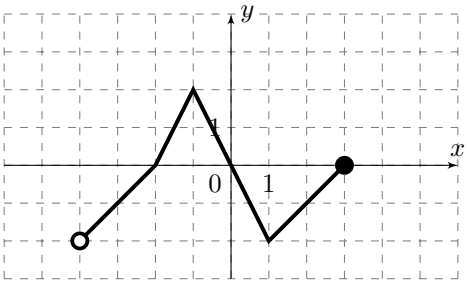
(vii) ... equal 2.

(16)


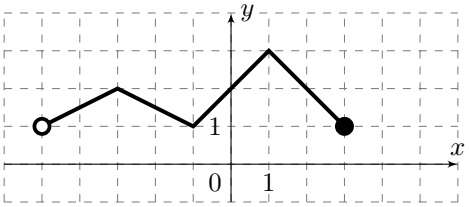
(vii) ... equal -3.

(17)


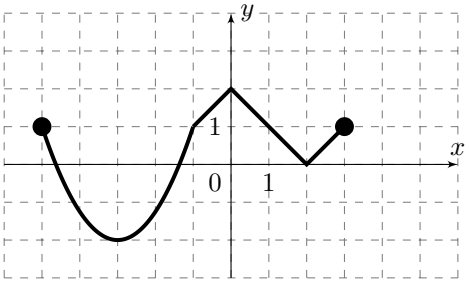
(vii) ... smaller than 1.

(18)


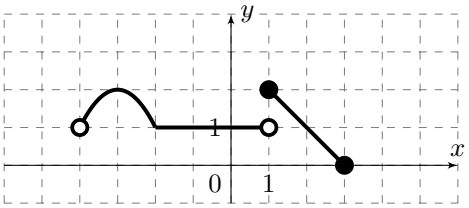
(vii) ... smaller than -1.

(19)


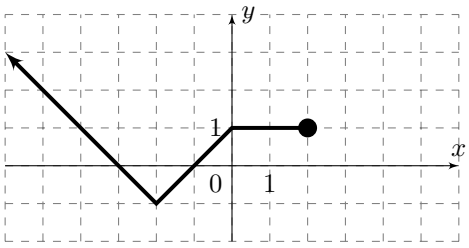
(vii) ... smaller than 2.

(20)


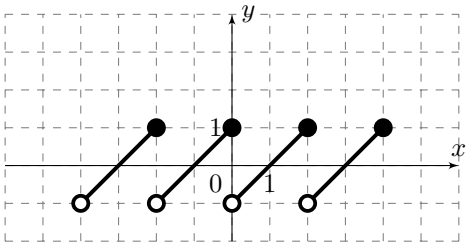
(vii) ... smaller than 1.

(21)


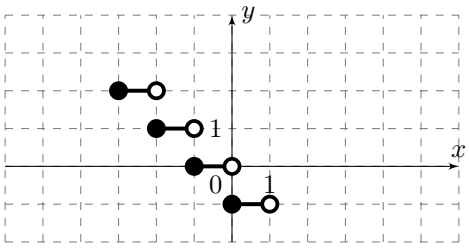
(vii) ... equal 1.

(22) 

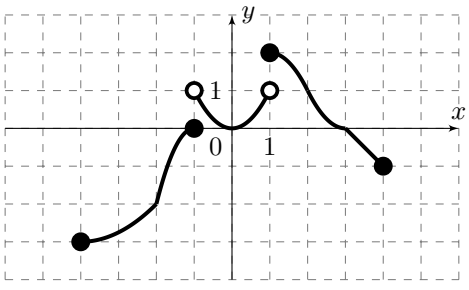
(vii) ... negative.

(23) 

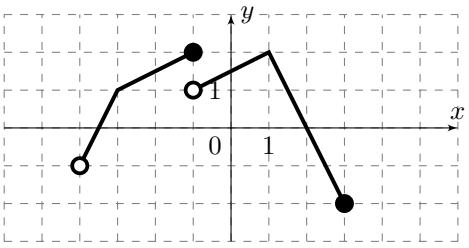
(vii) ... larger or equal 1.

(24) 

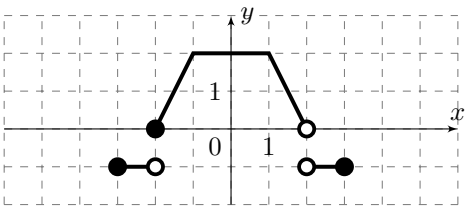
(vii) ... smaller or equal 0.

(25) 

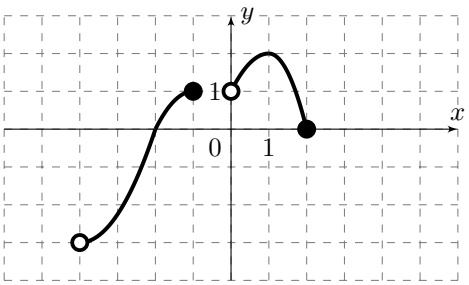
(vii) ... smaller or equal 1.

(26) 

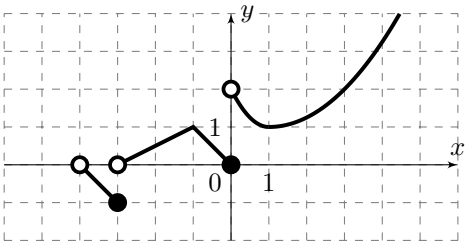
(vii) ... greater or equal -2.

(27) 

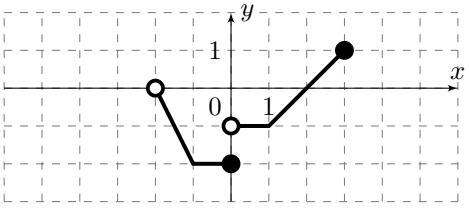
(vii) ... smaller or equal 0.

(28) 

(vii) ... positive.

(29) 

(vii) ... larger or equal 1.

(30) 

(vii) ... smaller or equal -2.

Q3. Write down the **domain** and the **range** of $y = f(x)$.

(1) $f(x) = x^2$
(2) $f(x) = x^3$
(3) $f(x) = \frac{1}{x}$
(4) $f(x) = \sqrt{x}$
(5) $f(x) = \frac{1}{2}x - 3$
(6) $f(x) = |x|$

Q4. Find the zeroes of the following functions. Do not use a calculator.

(1) $y = 2x + 5$
(2) $y = \sqrt{3 - x}$
(3) $y = 3 + \frac{2}{x-1}$
(4) $y = 9 - x^2$
(5) $y = (2x - 1)^2$
(6) $y = |x + 1| - 2$
(7) $y = \frac{2}{3}x + \pi$
(8) $y = \sqrt{3x + 1}$
(9) $y = \frac{-3}{x+4} - 2$
(10) $y = x^2 - 4$
(11) $y = (x + 1)^2 - 4$
(12) $y = 3 + |4 - 2x|$
(13) $y = \sqrt{3}x + 6$
(14) $y = \sqrt{2x} - 3$
(15) $y = x^3 - 8$
(16) $y = (x - 1)^3$
(17) $y = 3 - |2x - 4|$
(18) $y = \frac{\sqrt{x}-3}{2\sqrt{x}-1}$

Q5. Find the largest possible domain and the range of the following functions. You may use a calculator.

(1) $y = 2x + 5$
(2) $y = \sqrt{3 - x}$
(3) $y = 3 + \frac{2}{x-1}$
(4) $y = (x - 3)^2 - 4$
(5) $y = |x + 1| - 2$
(6) $y = \frac{2}{3}x + \pi$
(7) $y = \sqrt{3x + 1}$
(8) $y = \frac{-3}{x+4} - 2$
(9) $y = x^2 - 9$
(10) $y = 3 + |4 - 2x|$
(11) $y = \sqrt{3}x + 6$
(12) $y = \sqrt{2x} - 3$
(13) $y = (x - 3)^2$
(14) $y = 3 - |2x - 4|$
(15) $y = 4 - \frac{3}{2\sqrt{x}-1}$

Q6. Find $f(a)$ for given function f and the value of a.

(1) $f(x) = 2x + 5$, $a = -2$
(2) $f(x) = \sqrt{3 - x}$, $a = -6$
(3) $f(x) = 3 + \frac{2}{x-1}$, $a = 5$
(4) $f(x) = 9 - x^2$, $a = -7$
(5) $f(x) = (2x - 1)^2$, $a = 2.5$
(6) $f(x) = |x + 1| - 2$, $a = -3$
(7) $f(x) = \frac{2}{3}x + \pi$, $a = 6\pi$
(8) $f(x) = \sqrt{3x + 1}$, $a = 8$
(9) $f(x) = \frac{-3}{x+4} - 2$, $a = -1$
(10) $f(x) = x^2 - 4$, $a = \frac{5}{3}$
(11) $f(x) = (x + 1)^2 - 4$, $a = -\frac{10}{3}$
(12) $f(x) = 3 + |4 - 2x|$, $a = 1.4$
(13) $f(x) = \sqrt{3}x + 6$, $a = 4\sqrt{3}$
(14) $f(x) = \sqrt{2x} - 3$, $a = 12$
(15) $f(x) = x^3 - 8$, $a = -2$
(16) $f(x) = (x - 1)^3$, $a = -2$
(17) $f(x) = 3 - |2x - 4|$, $a = \frac{5}{6}$
(18) $f(x) = \frac{\sqrt{x}-3}{2\sqrt{x}-1}$, $a = 4$

Q7. Find the expression for $f(-x)$ for each of the functions given below. Write the answers in simplest form.

(1) $y = 2x + 5$
(2) $y = \sqrt{3 - x}$
(3) $y = 3 + \frac{2}{x-1}$
(4) $y = 9 - x^2$
(5) $y = (2x - 1)^2$
(6) $y = |x + 1| - 2$
(7) $y = \frac{2}{3}x + \pi$
(8) $y = \sqrt{3x + 1}$
(9) $y = \frac{-3}{x+4} - 2$
(10) $y = x^2 - 4$
(11) $y = (x + 1)^2 - 4$
(12) $y = 3 + |4 - 2x|$
(13) $y = \sqrt{3}x + 6$
(14) $y = \sqrt{2x} - 3$
(15) $y = x^3 - 8$
(16) $y = (x - 1)^3$
(17) $y = 3 - |2x - 4|$
(18) $y = \frac{\sqrt{x}-3}{2\sqrt{x}-1}$

Q8. Verify whether the given function is even, odd or neither.

(1) $f(x) = 6x$
(2) $f(x) = \sqrt{3 - x}$
(3) $f(x) = \frac{2}{3x}$
(4) $f(x) = 9 - x^2$
(5) $f(x) = 4x^2 + 16$
(6) $f(x) = |x + 1| - 2$
(7) $f(x) = \frac{2}{x^5 - 5x}$
(8) $f(x) = (4x)^3$
(9) $f(x) = \frac{x-3}{x+3}$
(10) $f(x) = x^2 - 4$
(11) $f(x) = (x + 1)^2$
(12) $f(x) = 3 - |2x|$
(13) $f(x) = \frac{x^2-13}{4-x^4}$
(14) $f(x) = \frac{x-2x^3}{4}$
(15) $f(x) = 2x^4 - 3x^2 + 5$
(16) $f(x) = (x - 1)^3$
(17) $f(x) = |5x^3| + 4$
(18) $f(x) = \sqrt{x} + 5$

4.2 Transformations of graphs of functions

Q9. Sketch in the same set of axes the graphs of $y = f(x)$ and $y = -f(x)$. Label the graphs.

(1) $f(x) = x^2$
(2) $f(x) = x^3$
(3) $f(x) = \frac{1}{x}$
(4) $f(x) = \sqrt{x}$
(5) $f(x) = \frac{1}{2}x - 3$
(6) $f(x) = |x|$

Q10. Sketch in the same set of axes the graphs of $y = f(x)$ and $y = f(-x)$. Label the graphs.

(1) $f(x) = x^2$
(2) $f(x) = x^3$
(3) $f(x) = \frac{1}{x}$
(4) $f(x) = \sqrt{x}$
(5) $f(x) = \frac{1}{2}x - 3$
(6) $f(x) = |x|$

Q11. Sketch in the same set of axes the graphs of $y = f(x)$, $y = 2f(x)$ and $y = \frac{1}{2}f(x)$. Label the graphs clearly.

(1) $f(x) = x^2$
(2) $f(x) = x^3$
(3) $f(x) = \frac{1}{x}$
(4) $f(x) = \sqrt{x}$
(5) $f(x) = \frac{1}{2}x - 3$
(6) $f(x) = |x|$

Q12. Sketch in the same set of axes the graphs of $y = f(x)$, $y = f(2x)$ and $y = f(\frac{1}{2}x)$. Label the graphs clearly.

(1) $f(x) = x^2$
(2) $f(x) = x^3$
(3) $f(x) = \frac{1}{x}$
(4) $f(x) = \sqrt{x}$
(5) $f(x) = \frac{1}{2}x - 3$
(6) $f(x) = |x|$

Q13. (i) Sketch the graph of $y = f(x)$.

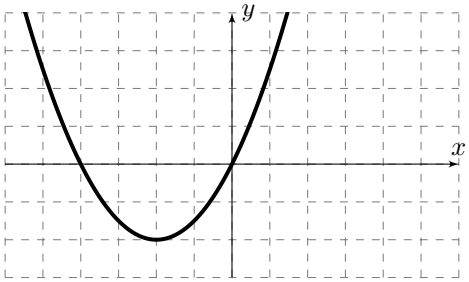
(ii) Sketch in the same set of axes the graph of $y = g(x)$ which is obtained from the graph of $y = f(x)$ by translating with vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

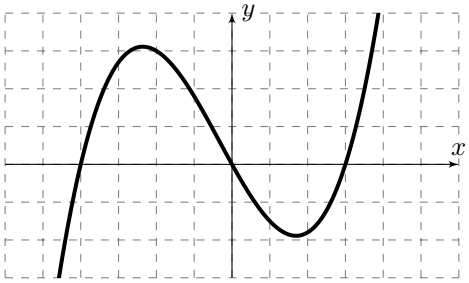
(iii) Write down the equation of $g(x)$.

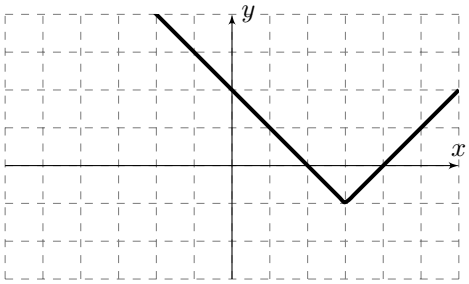
(iv) Write down the domain and the range of g.

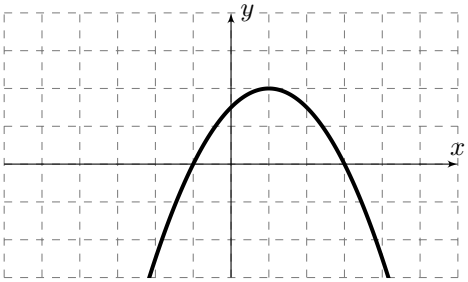
(1) $f(x) = x^2$,	$a = 3$,	$b = -1$	(7) $f(x) = x^2$,	$a = -4$,	$b = 2$
(2) $f(x) = x^3$,	$a = -2$,	$b = -3$	(8) $f(x) = x^3$,	$a = -2$,	$b = 1$
(3) $f(x) = \frac{1}{x}$,	$a = 4$,	$b = 1$	(9) $f(x) = \frac{1}{x}$,	$a = -1$,	$b = -2$
(4) $f(x) = \sqrt{x}$,	$a = -1$,	$b = 2$	(10) $f(x) = \sqrt{x}$,	$a = -4$,	$b = -2$
(5) $f(x) = -\frac{2}{3}x - 2$,	$a = -3$,	$b = 1$	(11) $f(x) = \frac{1}{2}x + 1$,	$a = 2$,	$b = 3$
(6) $f(x) = \|x\|$,	$a = -5$,	$b = 1$	(12) $f(x) = \|x\|$,	$a = -3$,	$b = -4$

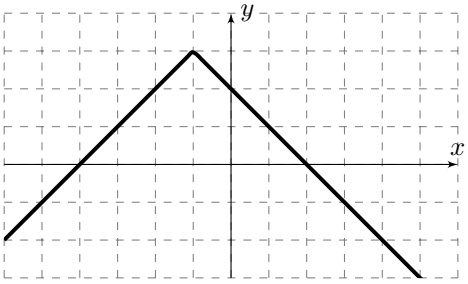
Q14. The diagram shows the graph of $y = f(x)$. Copy the graph and sketch in the same set of axes the graph of $y = |f(x)|$. Label the graphs clearly.

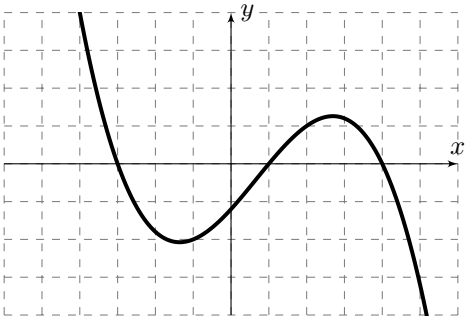
(1)


(2)


(3)


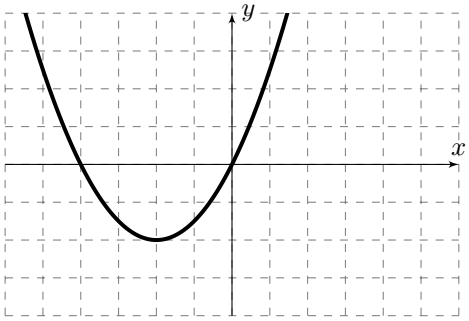
(4)


(5)


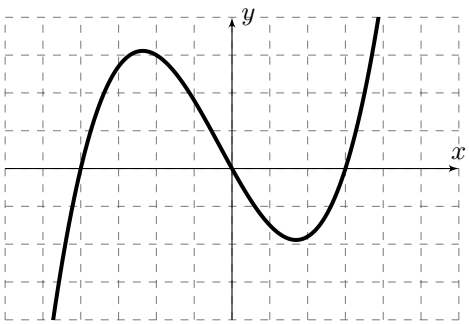
(6)


Q15. The diagram shows the graph of $y = f(x)$. Copy the graph and sketch in the same set of axes the graph of $y = f(|x|)$. Label the graphs clearly.

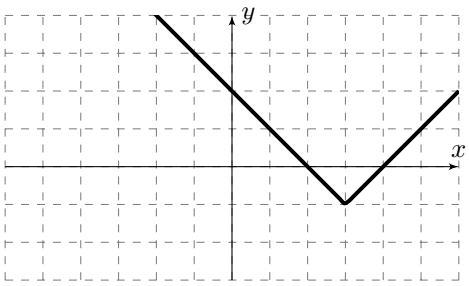
(1)



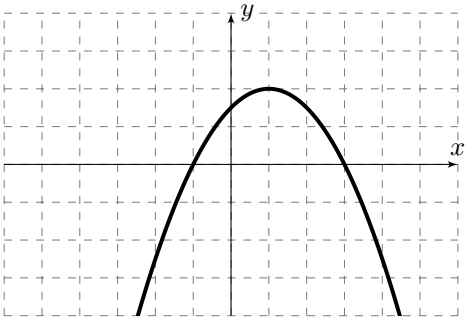
(2)



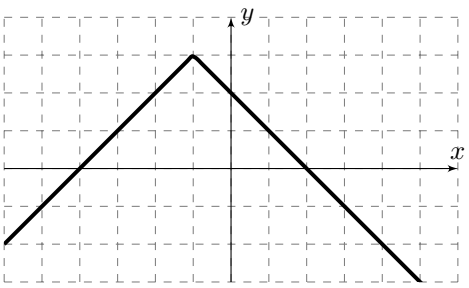
(3)



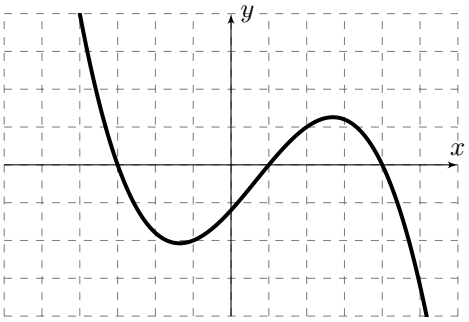
(4)



(5)



(6)

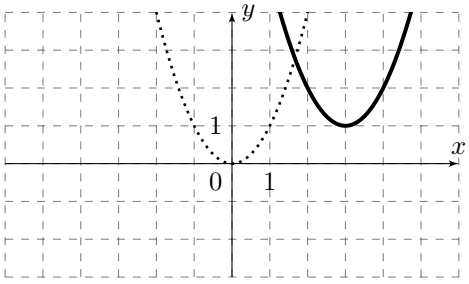


Q16. (i) Recognize the equations of the parent function (dotted line / curve).

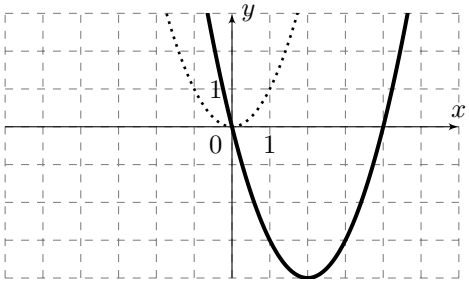
(ii) State the single transformation or the sequence of transformations that have to be applied so that the parent function is transformed to the other one (solid line / curve).

(iii) Give the equation of the new function.

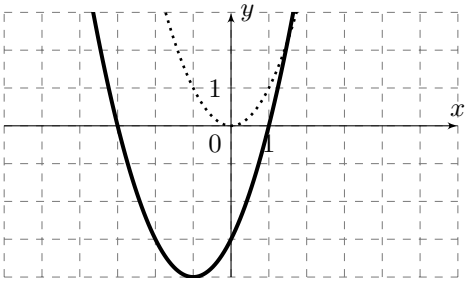
(1)



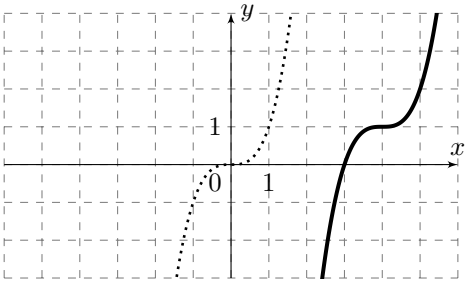
(2)

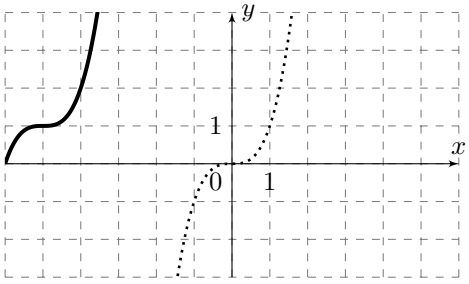


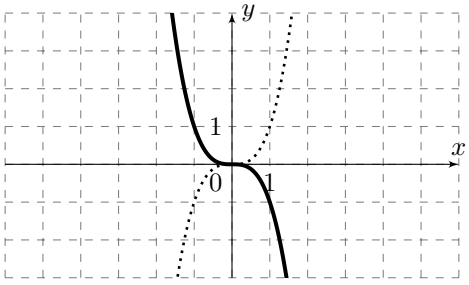
(3)

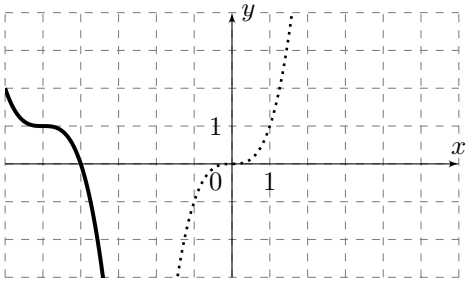


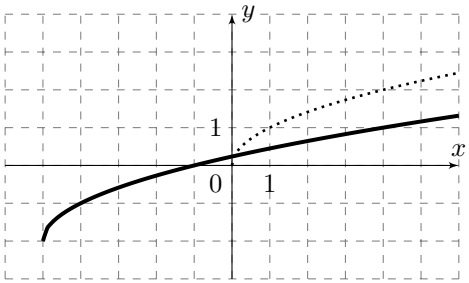
(4)

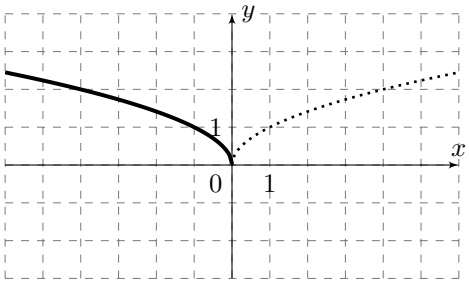


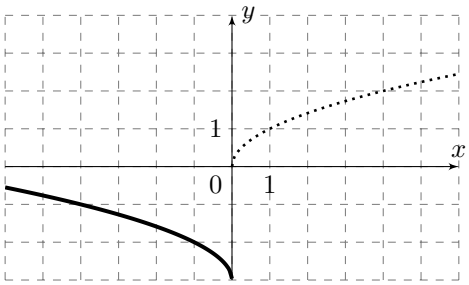
(5) 

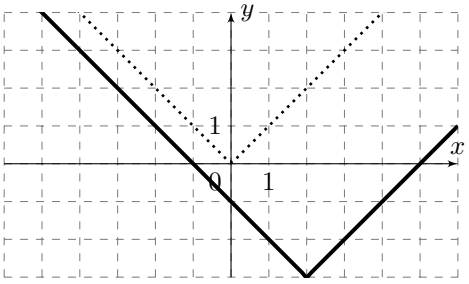
(6) 

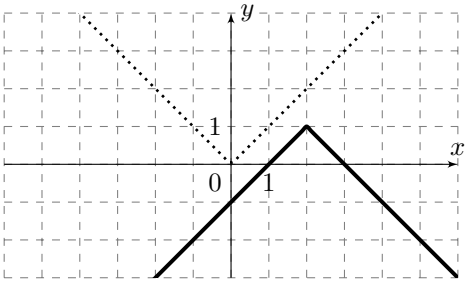
(7) 

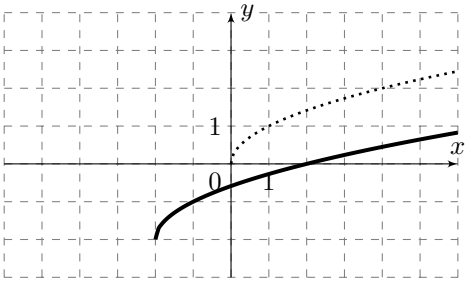
(8) 

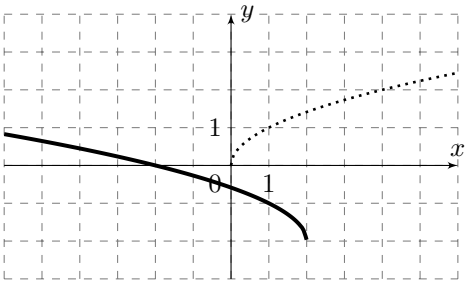
(9) 

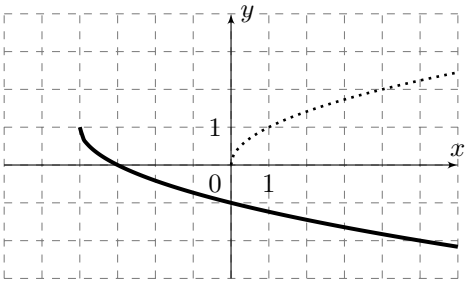
(10) 

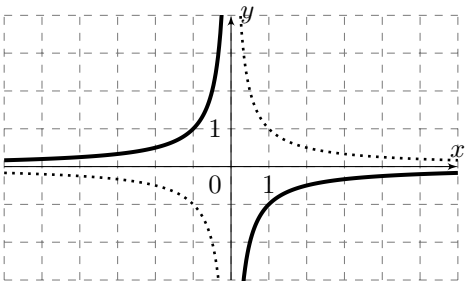
(11) 

(12) 

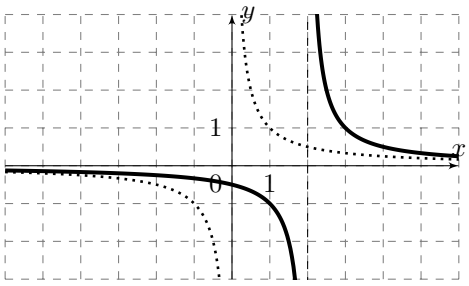
(13) 

(14) 

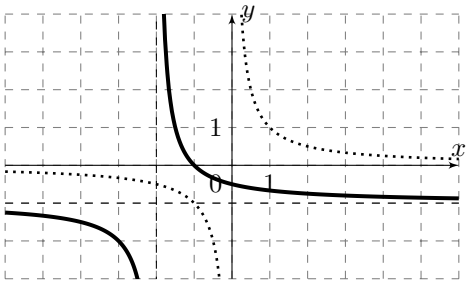
(15) 

(16) 

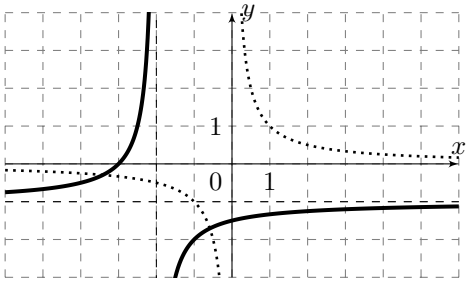
(17)



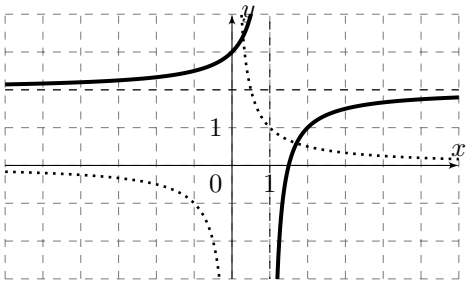
(18)



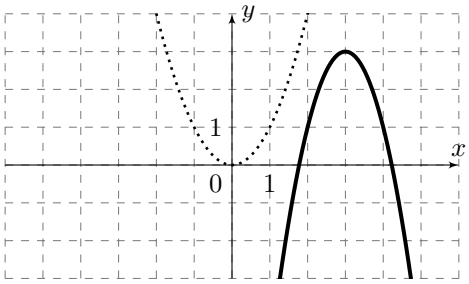
(19)



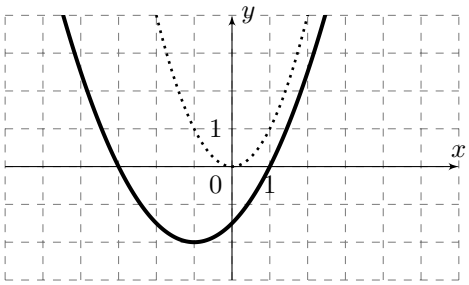
(20)



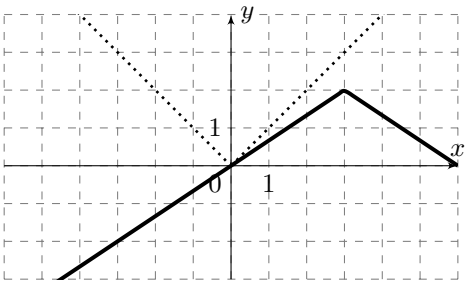
(21)



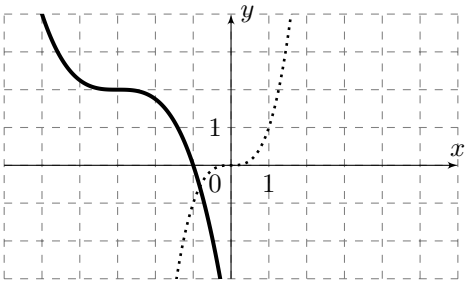
(22)



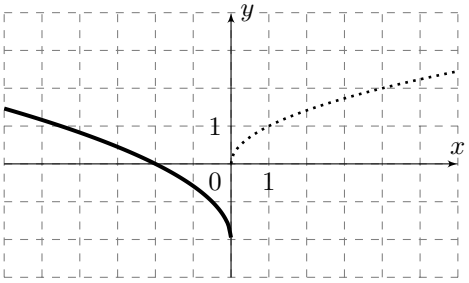
(23)



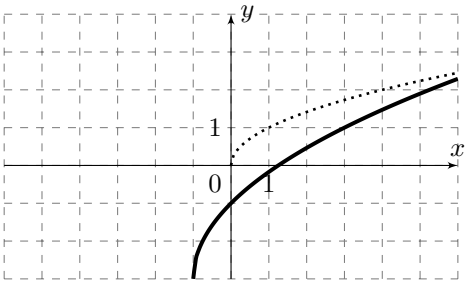
(24)



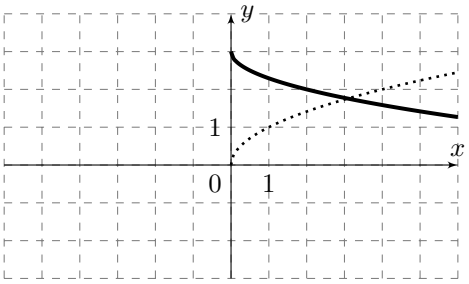
(25)



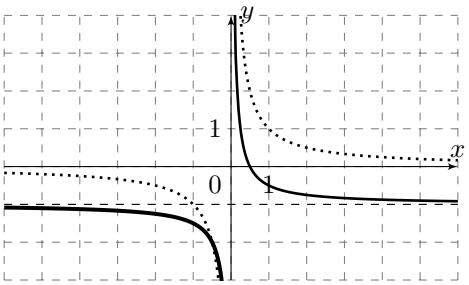
(26)



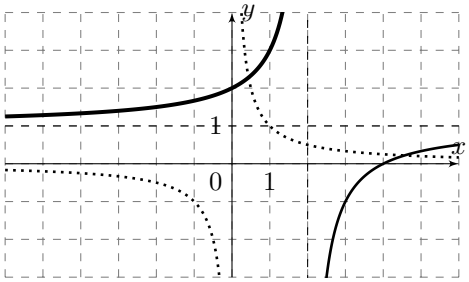
(27)



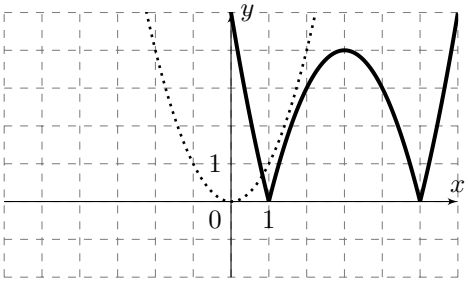
(28)



(29)



(30)



Q17. (i) Describe clearly the sequence of geometric transformations that transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

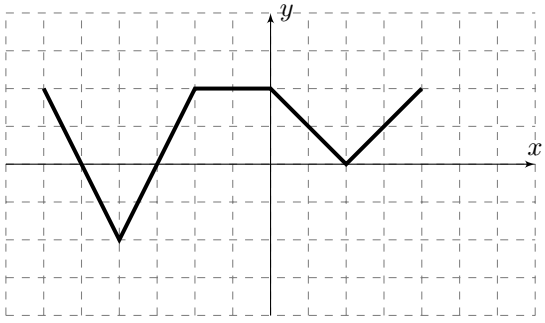
(ii) Graph both functions in the same set of axes.

(1) $f(x) = x^2$, $g(x) = 2(x-3)^2$,
(2) $f(x) = \frac{1}{x}$, $g(x) = \frac{3}{x+1}$,
(3) $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x+2}$,
(4) $f(x) = x^3$, $g(x) = (x-2)^3 - 2$,
(5) $f(x) = 3x + 2$, $g(x) = 3|x| + 2$,
(6) $f(x) = |x|$, $g(x) = -2|x-3|$,
(7) $f(x) = x^2$, $g(x) = (2x)^2 + 1$,
(8) $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{3x} + 2$,
(9) $f(x) = \sqrt{x}$, $g(x) = \sqrt{-x} - 1$,
(10) $f(x) = x^3$, $g(x) = -x^3 - 3$,
(11) $f(x) = 2 - x$, $g(x) = |2 - |x||$,
(12) $f(x) = |x|$, $g(x) = 2|x+1| - 3$,
(13) $f(x) = x^2$, $g(x) = -2(x+3)^2 + 2$,
(14) $f(x) = \frac{1}{x}$, $g(x) = -\frac{2}{x+1} + 1$,
(15) $f(x) = \sqrt{x}$, $g(x) = -2\sqrt{x-3} + 1$,
(16) $f(x) = x^3$, $g(x) = -(x-2)^3 - 2$,
(17) $f(x) = x + 2$, $g(x) = 3|x+2|$,
(18) $f(x) = |x|$, $g(x) = |2x| - 1$,
(19) $f(x) = x^2$, $g(x) = 3 - (x-3)^2$,
(20) $f(x) = \frac{1}{x}$, $g(x) = \frac{2}{x-1} + 2$,
(21) $f(x) = \sqrt{x}$, $g(x) = 2 - \sqrt{x-2}$,
(22) $f(x) = x^3$, $g(x) = \frac{1}{2}(x+1)^3 - 2$,
(23) $f(x) = 2x - 1$, $g(x) = -|2x-1|$,
(24) $f(x) = |x|$, $g(x) = 3 - |\frac{x}{2}|$,
(25) $f(x) = x^2$, $g(x) = \frac{1}{2}(x-3)^2 - 1$,
(26) $f(x) = x^2$, $g(x) = 2 - \frac{1}{2}(x+3)^2$,
(27) $f(x) = x^2$, $g(x) = 3(x+1)^2 - 3$,
(28) $f(x) = x^2$, $g(x) = 4 - 2(x+2)^2$,
(29) $f(x) = \frac{1}{2}x + 1$, $g(x) = \frac{1}{2}|x| + 1$,
(30) $f(x) = \frac{1}{2}x + 1$, $g(x) = |\frac{1}{2}x + 1|$,
(31) $f(x) = x^2$, $g(x) = 2(|x| - 2)^2 - 1$,
(32) $f(x) = x^2$, $g(x) = |2(x+2)^2 - 4|$.

Q18. Point A lies on the graph of function $y = f(x)$. When the graph of f was transformed the the graph of g, point A changed into point A'. Find the coordinates of A'.

(1) $A = (2, 1)$, $g(x) = f(x-2) + 3$,
(2) $A = (-2, 1)$, $g(x) = 2f(x+1)$,
(3) $A = (-3, -1)$, $g(x) = -f(2x) + 2$,
(4) $A = (3, -2)$, $g(x) = \frac{1}{2}f(-x) - 3$,
(5) $A = (-2, -4)$, $g(x) = |2f(-x) + 1|$,
(6) $A = (4, -2)$, $g(x) = 2f\left(-\frac{1}{2}x\right) - 1$.

Q19. The diagram below shows the graph of $y = f(x)$.

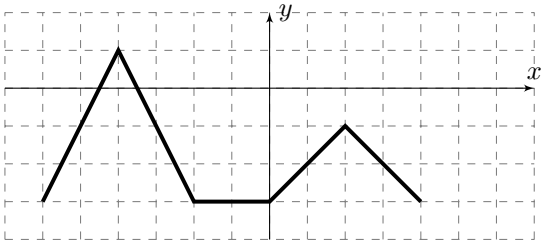


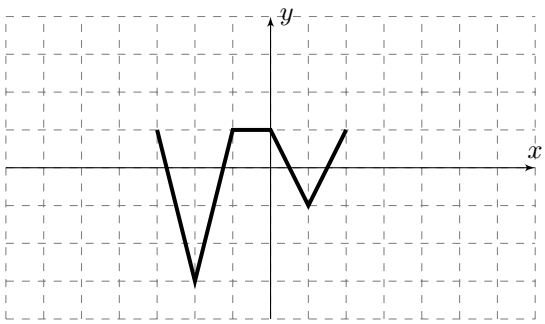
(i) Graph the functions with equations shown.

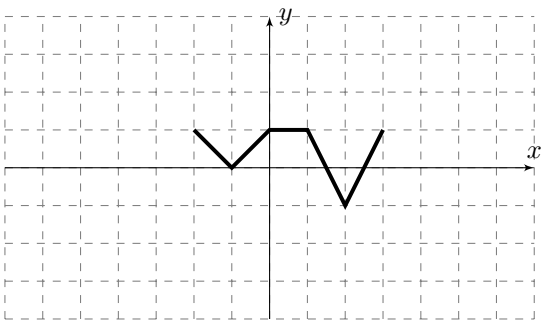
(1) $y = f(-x) - 2$
(2) $y = -f(x+1) - 1$
(3) $y = f(\frac{x}{2})$
(4) $y = \frac{1}{2}f(x-1) + 2$

(ii) Write down the equations of the graphs shown below.

(1)



(2)


(3)


4.3 Equations and inequalities

Q20. Solve the equations.

(1) $|3x+1| = 2$
(2) $|1-2x| = 3$
(3) $|4x+2| = 5$
(4) $|-3x-2| = 6$
(5) $\frac{2}{x-1} + 1 = 3$
(6) $2 - \frac{2}{x+2} = 0$
(7) $\sqrt{x-2} = 3$
(8) $3 - \sqrt{3x} = 1$
(9) $\frac{1}{x+2} = \frac{4}{3-x}$
(10) $\frac{2}{x-2} = -\frac{4}{4x-2}$
(11) $\frac{1}{x-2} - 1 = 2 - \frac{2}{x-2}$
(12) $1 - \frac{4}{2x+1} = \frac{2}{2x+1} - 2$

Q21. (i) Solve the equation $x^2 = 4$.
(ii) Hence solve $(x-2)^2 + 3 = 7$.

Q22. (i) Solve the equation $x^3 + 27 = 0$.
(ii) Hence solve $\frac{1}{3}(x+2)^3 + 9 = 0$.

Q23. (i) Solve the equation $\frac{1}{x} = 2$.
(ii) Hence solve $\frac{-2}{x-3} + 4 = 0$.

Q24. (i) Solve the equation $\sqrt{x} = 3$.
(ii) Hence solve $\sqrt{2x} + 1 = 4$.

Q25. (i) Sketching the appropriate graphs manually state the **number** of solutions of the following equations.
(ii) Use a GDC to solve each equation. Give all answers correct to 3 significant figures.

(1) $\frac{-2}{x-1} = \frac{x}{2}$
(2) $\frac{-2}{x-1} = \frac{x^2}{2}$
(3) $\frac{-2}{x-1} = -x^3$
(4) $\frac{1}{x} = (x-4)^2 + 1$
(5) $\frac{1}{x} = -(x-4)^2 + 1$
(6) $\frac{1}{x} = (x-1)^2 - 1$
(7) $-\frac{2}{x} = x^2 - 1$
(8) $-\frac{2}{x} = (x-2)^2 - 2$
(9) $x^2 = 4 - 2x^2$
(10) $(x-1)^2 - 2 = 3 - 2(x+1)^2$
(11) $\frac{1}{5}(x-4)^2 - 3 = 4 - (x-3)^2$
(12) $2(x+2)^2 - 4 = (x+1)^2 - 1$

Q26. Use a GDC to solve the following inequalities.

(1) $x^3 - 2x^2 - x + 2 \leq x^2$
(2) $\frac{1}{2}x^3 + x^2 - x - 2 > -x^2 + x + 1$
(3) $x^3 - 4x^2 + x + 1 \geq \frac{2x}{x^2+1}$
(4) $\frac{2x-3}{x-2} > x^2 - 1$
(5) $\frac{x^3}{x^2+x+1} > \frac{x}{2}$ for $-1 \leq x \leq 2$
(6) $\frac{x^3}{x^2+x+1} \leq \frac{x}{2}$ for $-0.5 \leq x \leq 1$
(7) $x - x^2 + 1 \geq \frac{2-x}{x+1}$
(8) $4 - 2x - x^2 \geq \frac{-x^2}{x+1}$
(9) $4 - 2x - x^2 \leq \frac{-x^2}{x+1}$ for $-3 < x < 3$
(10) $\frac{x^3+x+3}{x^2+1} > \frac{1}{2}x + 2$ for $-2 \leq x \leq 4$
(11) $\frac{x^3+x+3}{x^2+1} \leq \frac{1}{2}x + 2$ for $0 < x < 5$
(12) $2^{2+x-x^2} < 4 - 2(x+1)^2$
(13) $\left(\frac{3}{2}\right)^{3x-x^2} \geq x^2 - 4x + 5$

4.4 Chapter review

non-calculator questions

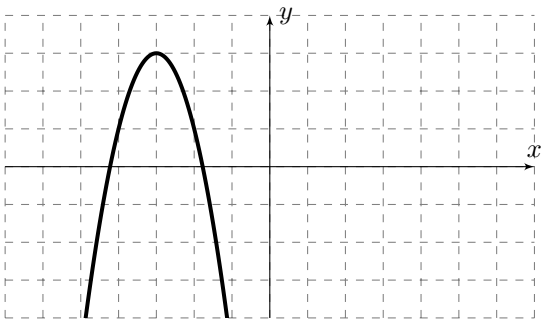
Q1. (i) Describe clearly the sequence of geometric transformations that transform the graph of $y = f(x)$ into the graph of $y = g(x)$. [1]

(ii) Sketch the graph of $y = g(x)$ showing clearly the position of at least three points.

(1)	$f(x) = x^2$	$g(x) = \frac{1}{3}(x+3)^2 - 1$	(5)	$f(x) = \frac{1}{x}$	$g(x) = \frac{-3}{x+2} - 1$
(2)	$f(x) = \sqrt{x}$	$g(x) = \sqrt{-\frac{1}{2}x} + 1$	(6)	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{2x} + 2$
(3)	$f(x) = \sqrt{x}$	$g(x) = -\frac{1}{2}\sqrt{x-3} - 1$	(7)	$f(x) = -\frac{2}{3}x + 2$	$g(x) = -\frac{2}{3}\lvert x\rvert + 2$
(4)	$f(x) = \lvert x\rvert$	$g(x) = -3\lvert x+1\rvert + 2$	(8)	$f(x) = x^2$	$g(x) = \lvert(x+3)^2 - 3\rvert$

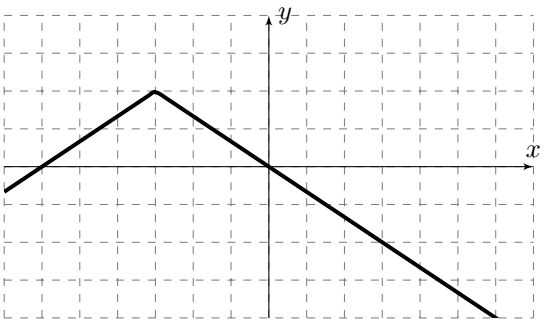
Q2. Write down an equation for each of the graphs shown.

(1)



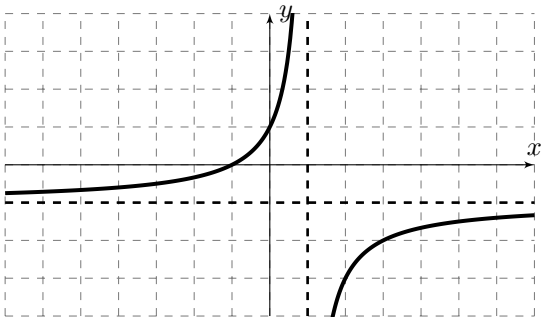
$y =$ .....

(4)



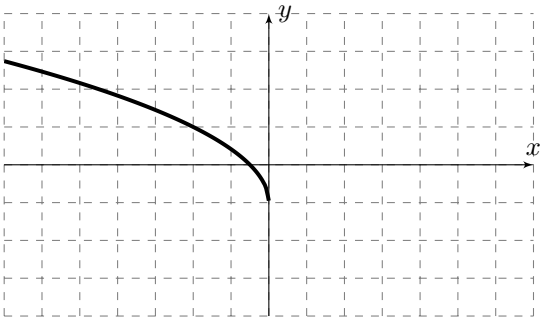
$y =$ .....

(2)



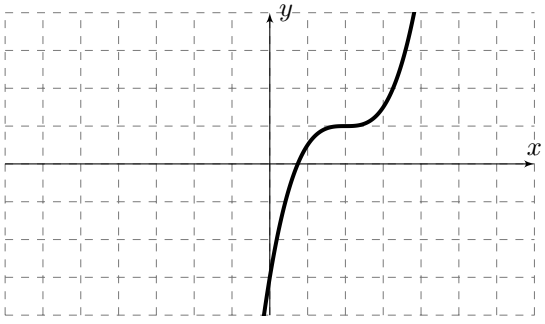
$y =$ .....

(5)



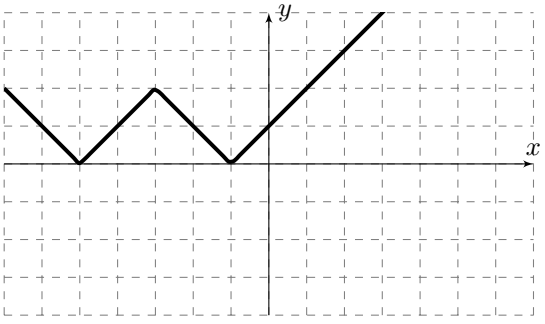
$y =$ .....

(3)



$y =$ .....

(6)

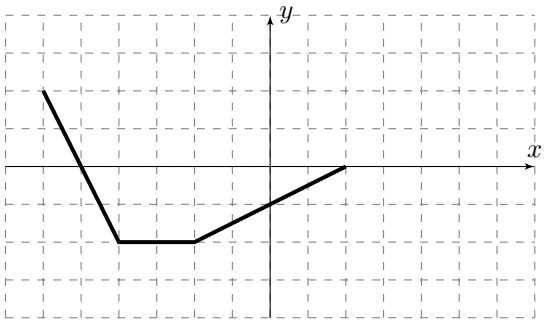


$y =$ .....

Q3. Verify if the given function is even, odd or neither. Justify your answer clearly.
(1) $f(x) = \frac{2x^2 - 3|x| + 1}{x^4 - 3x^2 + 2}$ (2) $g(x) = 2x^3 - \frac{3}{x - x^5}$ (3) $h(x) = \frac{3 - 2|x|}{|x^2 - 3x + 1|}$

[1] use simple terms like "shift by...", "reflection in...", "horizontal / vertical dilation by..."

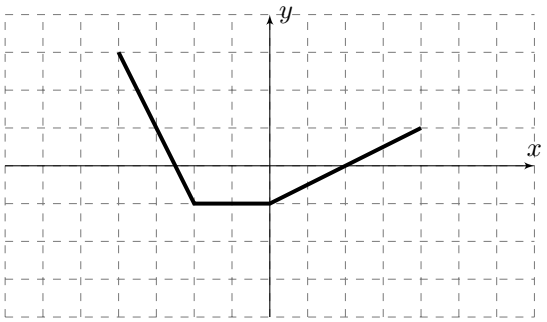
Q4. The diagram below shows the graph of $y = f(x)$.

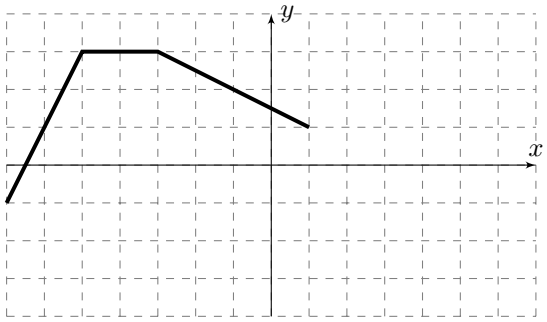


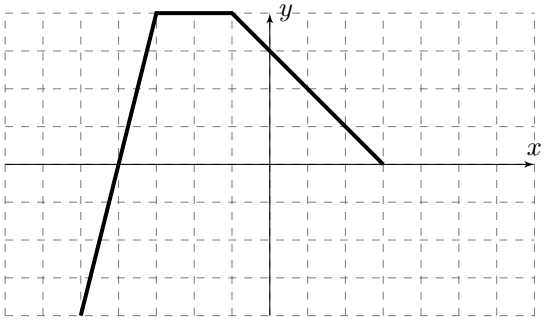
(i) Graph the functions with equations shown.

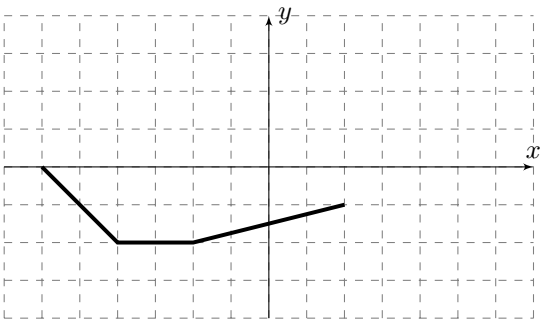
(1) $y = f(x-4)+1$
(2) $y = -f(x)+1$
(3) $y = f(2x)$
(4) $y = -2f(x-1)$
(5) $y = \frac{1}{2}f(-x)$
(6) $y = |f(x-1)|$
(7) $y = f(|x|)-1$

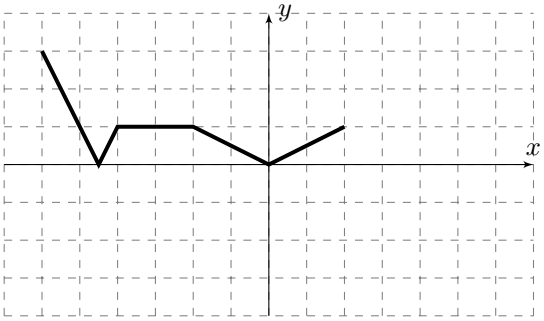
(ii) Write down the equations of the graphs shown below.

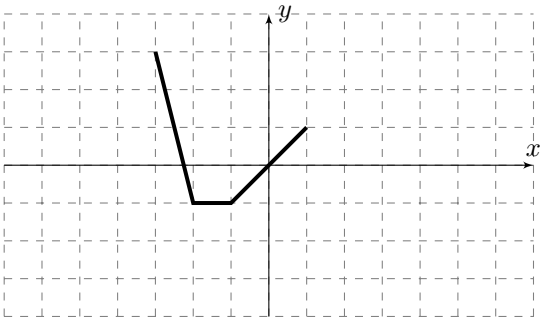
(1)


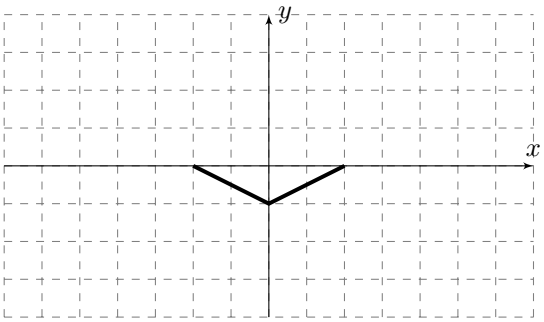
(2)


(3)


(4)


(5)


(6)


(7)


Q5. Consider the functions:

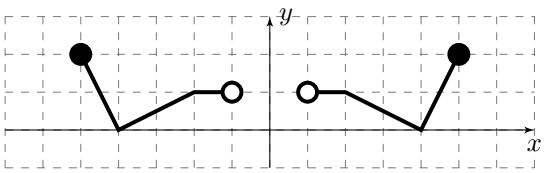
$$f_1(x) = -\sqrt{-3-2x} \qquad f_5(x) = \frac{-2}{x+1} + 3$$
$$f_2(x) = -2|x+3| + 4 \qquad f_6(x) = 2 - (x+1)^2$$
$$f_3(x) = 3|x| - 1 \qquad f_7(x) = \frac{3-x}{\sqrt{3+x}}$$
$$f_4(x) = 2(x+3)^2 - 5 \qquad f_8(x) = \frac{\sqrt{3-x}}{3+x}$$

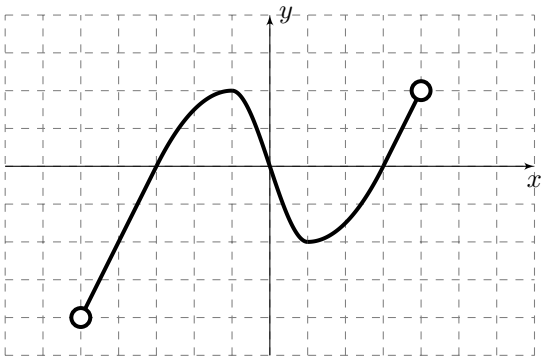
(i) State the domain of all functions defined above.

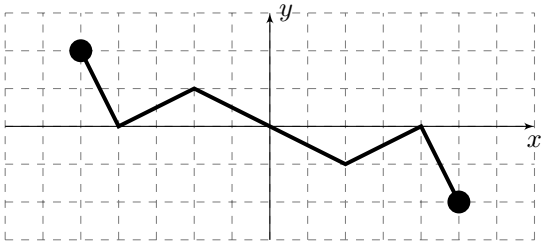
(ii) State the range of functions f_1 to f_6.

Q6. For each of the functions with graphs shown below read from the graph:

(i) the domain,
(ii) the range,
(iii) zeroes,
and sets of arguments
for which the function is:
(iv) decreasing,
(v) increasing,
(vi) constant.
State whether the function is:
(vii) even
(viii) odd
(ix) one-to-one

(1)


(2)


(3)


calculator questions

Q7. Find all points of intersection of the graphs of[2]:

(1) $y = \frac{1}{20}x^2 - \frac{3}{10}x - 2$ and $y = \frac{1}{x}$
(2) $y = 2x + 1$ and $y = |(x-6)^3 - 4|$
(3) $y = \frac{3-x}{2x-1}$ and $y = -0.1x^2 + 0.2x + 20$
(4) $y = 0.01x^3 - 2.2x + 1$ and $y = \frac{5x}{12}$

Q8. Solve the inequalities for $-20 \le x \le 20$.

(1) $\frac{1}{20}x^2 - \frac{3}{10}x - 2 > \frac{1}{x}$
(2) $2x + 1 < |(x-6)^3 - 4|$
(3) $\frac{3-x}{2x-1} \ge y = -0.1x^2 + 0.2x + 20$
(4) $0.01x^3 - 2.2x + 1 \le y = \frac{5x}{12}$

[2] all x-coordinates of the points of intersection are between -20 and 20

Chapter 5

Quadratics

5.1 Solving quadratic equations

Factorisation

Q1. Expand and simplify.

(1) $(x+2)(x+1)$
(2) $(x+3)(x+1)$
(3) $(x+5)(x+2)$
(4) $(x-1)(x-3)$
(5) $(x-2)(x-4)$
(6) $(x-1)(x-12)$
(7) $(x+3)(x-2)$
(8) $(x+2)(x-3)$
(9) $(x+4)(x-2)$
(10) $(x+3)(x-3)$
(11) $(x+3)(x-4)$
(12) $(x+6)(x-2)$
(13) $(x+4)(x-6)$
(14) $(x+8)(x-3)$
(15) $(x+7)(x-3)$

Q2. Expand and simplify.

(1) $(x+1)(2x+1)$
(2) $(2x+5)(x+2)$
(3) $(2x-1)(x-3)$
(4) $(2x-3)(x-1)$
(5) $(2x+1)(x-1)$
(6) $(x+1)(3x-2)$
(7) $(3x+1)(x-2)$
(8) $(3x+2)(x-1)$
(9) $(2x+5)(3x-1)$
(10) $(3x-5)(2x-1)$
(11) $(3x+5)(2x-1)$
(12) $(5x-2)(3x-2)$
(13) $(5x+1)(3x-4)$
(14) $(3x+2)(5x-2)$
(15) $(4x-3)(3x-4)$

Q3. Factorise.

(1) x^2-2x
(2) x^2+3x+2
(3) x^2+5x+6
(4) x^2+4x+3
(5) x^2+4x
(6) x^2+5x+4
(7) x^2+6x+8
(8) $x^2+7x+12$
(9) x^2+6x+5
(10) $x^2+7x+10$
(11) $x^2+8x+12$
(12) $x^2+13x+12$
(13) x^2-4x+3
(14) x^2-3x+2
(15) x^2-5x+6
(16) x^2-7x+6
(17) x^2-6x+8
(18) x^2-9x+8
(19) $x^2-7x+12$
(20) $x^2-8x+12$
(21) $x^2-13x+12$

Q4. Factorise.

(1) $x^2 + x - 6$
(2) $x^2 - x - 6$
(3) $x^2 + 2x - 8$
(4) $x^2 - 2x - 8$
(5) $x^2 - 7x - 8$
(6) $x^2 - 9$
(7) $x^2 + 5x - 6$
(8) $x^2 - x - 12$
(9) $x^2 - 4x - 12$
(10) $x^2 + 4x - 12$
(11) $x^2 + x - 12$
(12) $x^2 - 2x - 24$
(13) $x^2 - 10x - 24$
(14) $x^2 + 5x - 24$
(15) $x^2 + 23x - 24$

Q5. Factorise.

(1) $x^2 + 3x + 2$
(2) $x^2 + 5x + 6$
(3) $x^2 + 4x + 3$
(4) $x^2 + 6x + 8$
(5) $x^2 + 5x + 4$
(6) $x^2 + 5x + 6$
(7) $x^2 + 8x + 15$
(8) $x^2 + 7x + 12$
(9) $x^2 + 8x + 12$
(10) $x^2 - 3x + 2$
(11) $x^2 - 5x + 6$
(12) $x^2 - 9x + 8$
(13) $x^2 - 6x + 8$
(14) $x^2 - 5x + 4$
(15) $x^2 - 4x + 4$
(16) $x^2 - 8x + 15$
(17) $x^2 - 7x + 12$
(18) $x^2 - 8x + 12$
(19) $x^2 - 2x - 8$
(20) $x^2 - 7x - 8$
(21) $x^2 + 4x - 12$
(22) $x^2 - x - 12$
(23) $x^2 + 2x - 15$
(24) $x^2 - 10x - 24$
(25) $x^2 + 5x - 24$
(26) $x^2 - 2x - 24$
(27) $x^2 + 23x - 24$

Q6. Factorise.

(1) $2x^2 + 3x + 1$
(2) $2x^2 + 5x + 2$
(3) $2x^2 + 5x + 3$
(4) $2x^2 + 7x + 6$
(5) $2x^2 + 7x + 3$
(6) $2x^2 + 9x + 10$
(7) $2x^2 - 3x + 1$
(8) $2x^2 - 5x + 2$
(9) $2x^2 - 7x + 3$
(10) $2x^2 - 5x + 3$
(11) $2x^2 + x - 1$
(12) $2x^2 - x - 1$
(13) $3x^2 + 5x - 2$
(14) $3x^2 + x - 2$
(15) $3x^2 - 5x - 2$
(16) $3x^2 - x - 2$
(17) $3x^2 - 10x + 8$
(18) $3x^2 - 14x + 8$
(19) $3x^2 - 25x + 8$
(20) $3x^2 - 10x - 8$
(21) $12x^2 - 25x + 12$

Q7. Factorise.

(1) $6x^2 - 5x + 1$
(2) $6x^2 - 7x + 2$
(3) $6x^2 + x - 2$
(4) $6x^2 - 7x - 3$
(5) $6x^2 - 5x - 6$
(6) $9x^2 + 3x - 2$
(7) $6x^2 + 13x - 5$
(8) $6x^2 - 13x + 5$
(9) $6x^2 + 7x - 5$
(10) $15x^2 - 23x + 4$
(11) $15x^2 - 16x + 4$
(12) $15x^2 - 17x - 4$
(13) $15x^2 + 4x - 4$
(14) $6x^2 + 7x - 5$
(15) $6x^2 + 13x - 5$
(16) $6x^2 - 13x - 5$
(17) $6x^2 - 7x - 5$
(18) $10x^2 + 9x - 9$
(19) $10x^2 + 13x - 9$
(20) $10x^2 - 43x - 9$
(21) $15x^2 + x - 2$

Completing the square

Q8. Expand and simplify.

(1) $(x-1)^2 + 2$
(2) $(x+2)^2 + 4$
(3) $2(x+1)^2 - 4$
(4) $3(x+4)^2 - 20$
(5) $-4(x+0.5)^2 + 4$
(6) $5(x-1)^2 - 10$
(7) $-\frac{1}{2}(x-2)^2 - \frac{3}{2}$
(8) $\frac{2}{3}(x-2)^2 - \frac{3}{2}$
(9) $\frac{1}{4}(x+2)^2 + 2$

Q9. Complete the square.

(1) $x^2 + 2x + 2$
(2) $x^2 + 2x$
(3) $x^2 - 4x + 1$
(4) $x^2 + 4x + 5$
(5) $x^2 - 6x + 10$
(6) $x^2 + 6x + 7$
(7) $x^2 + 8x + 8$
(8) $x^2 - 3x - 3$
(9) $x^2 + 5x + 6$
(10) $2x^2 - 4x + 2$
(11) $2x^2 + 4x$
(12) $2x^2 + 8x + 2$
(13) $2x^2 - 8x + 12$
(14) $2x^2 - 6x + 1$
(15) $\frac{1}{2}x^2 - x + 2$
(16) $-2x^2 - 10x + 2$
(17) $-x^2 - 4x + 2$
(18) $-\frac{1}{2}x^2 + 2x - 1$
(19) $2x^2 + 4x + 6$
(20) $-2x^2 - 2x$
(21) $-\frac{1}{3}x^2 + x + \frac{1}{2}$

Q10. For each equation:

(i) complete the square, (ii) **hence** solve the equation.

(1) $x^2 + 2x + 3 = 0$
(2) $x^2 + 4x - 1 = 0$
(3) $2x^2 + 8x + 3 = 0$
(4) $3x^2 + 6x + 0 = 0$
(5) $-4x^2 + 8x - 1 = 0$
(6) $-5x^2 - 10x + 4 = 0$
(7) $\frac{1}{3}x^2 + 2x + \frac{7}{2} = 0$
(8) $-\frac{2}{3}x^2 - 4x - 1 = 0$
(9) $-\frac{3}{4}x^2 + 1x - 1 = 0$
(10) $\frac{4}{3}x^2 + 16x + 38 = 0$
(11) $-\frac{5}{3}x^2 + 4x - \frac{7}{5} = 0$
(12) $\frac{2}{5}x^2 + \frac{8}{5}x - \frac{2}{5} = 0$

Quadratic formula

Q11. Solve the equations using a quadratic formula. Give your answers correct to 3 s.f. Check them graphing an appropriate function on a GDC.

(1) $x^2 + 3x + 1 = 0$
(2) $x^2 - 2x + 2 = 0$
(3) $x^2 + 8x + 6 = 0$
(4) $x^2 - 6x + 4 = 0$
(5) $x^2 + 4x + 2 = 0$
(6) $x^2 + 5x + 5 = 0$
(7) $x^2 - 2x - 4 = 0$
(8) $x^2 + 5x - 1 = 0$
(9) $x^2 + 4x - 6 = 0$
(10) $x^2 + 6x - 4 = 0$
(11) $x^2 - 6x - 11 = 0$
(12) $x^2 + 4x - 8 = 0$
(13) $2x^2 - 3x - 4 = 0$
(14) $2x^2 - 11x + 6 = 0$
(15) $-3x^2 - 7x + 6 = 0$
(16) $3x^2 - 6x + 8 = 0$
(17) $-2x^2 - 9x + 8 = 0$
(18) $-4x^2 - 7x + 12 = 0$
(19) $1.5x^2 - 6.6x + 7.26 = 0$
(20) $2.3x^2 - 13x + 12 = 0$
(21) $1.5x^2 + 2x - 6 = 0$
(22) $-3x^2 - 5x + 4 = 0$
(23) $3.9x^2 + 15.6x + 15.6 = 0$
(24) $4x^2 - 7x + 12 = 0$

Q12. Solve the equations using a quadratic formula. Give exact answers.

(1) $x^2 + 3x + 1 = 0$
(2) $x^2 - 2x + 2 = 0$
(3) $x^2 + 8x + 6 = 0$
(4) $x^2 - 6x + 4 = 0$
(5) $x^2 + 4x + 2 = 0$
(6) $x^2 + 5x + 5 = 0$
(7) $2x^2 - 7x - 4 = 0$
(8) $2x^2 + x - 6 = 0$
(9) $2x^2 + 6x - 7 = 0$
(10) $2x^2 - 6x - 11 = 0$
(11) $3x^2 + 4x - 8 = 0$
(12) $-2x^2 - 3x - 6 = 0$
(13) $-2x^2 - 3x - 4 = 0$
(14) $-2x^2 - 11x + 6 = 0$
(15) $-3x^2 - 7x + 6 = 0$
(16) $3x^2 - 6x + 8 = 0$
(17) $-2x^2 - 9x + 5 = 0$
(18) $-4x^2 - 7x + 1 = 0$
(19) $5x^2 + 2x - 1 = 0$
(20) $-3x^2 + 4x + 1 = 0$
(21) $2x^2 + 9x - 1 = 0$
(22) $-3x^2 - 5x + 1 = 0$
(23) $4x^2 + 10x + 3 = 0$
(24) $4x^2 - 8x - 3 = 0$

Q13. For what values of k the equation $x^2 - 2(k+1)x - k + 1 = 0$ has two equal roots?

Q14. For what values of m the equation $x^2 + mx + 9 = 0$ has two different real roots?

Q15. For what values of m the equation $x^2 + (m+1)x + m = 0$ has two different real roots?

Q16. For what values of m the equation $(m-3)x^2 + (3-m)x + 1 = 0$ has no real roots?

5.2 Parabola

Q17. For each of the quadratics below find:

- x-intercepts,
- y-intercept,
- vertex.

Hence sketch the graphs. Check your answers on a GDC.

(1) $y = x^2 + 3x + 2$
(2) $y = x^2 - 2x - 3$
(3) $y = x^2 + 8x + 6$
(4) $y = x^2 - 3x + 6$
(5) $y = x^2 + 4x + 2$
(6) $y = 0.5x^2 + 5x + 5$
(7) $y = 2x^2 - 2x - 4$
(8) $y = -x^2 + 5x - 1$
(9) $y = -2x^2 + 4x + 1$
(10) $y = -x^2 + 3x + 4$
(11) $y = -3x^2 + 2x + 1$
(12) $y = -6x^2 - x + 2$

Q18. Find the equation of a parabola that has x-intercepts $(-4, 0)$ and $(2, 0)$ and passes through point $(-3, 5)$. Give answer in the form $y = ax^2 + bx + c$.

Q19. Find the equation of a parabola that has x-intercepts $(1, 0)$ and $(5, 0)$ and passes through point $(2, 6)$. Give answer in the form $y = ax^2 + bx + c$.

Q20. Find the equation of a parabola that has x-intercepts $(-2, 0)$ and $(3, 0)$ and passes through point $(2, -2)$. Give answer in the form $y = ax^2 + bx + c$.

Q21. Find the equation of a parabola that has x-intercepts $(0.5, 0)$ and $(3.5, 0)$ and passes through point $(3, 2.5)$. Give answer in the form $y = ax^2 + bx + c$.

Q22. Find the equation of a parabola that has x-intercepts $(2, 0)$ and $(5, 0)$ and passes through point $(4, 3)$. Give answer in the form $y = ax^2 + bx + c$.

Q23. Find the equation of a parabola that has a vertex at $(-5, -1)$ and passes through point $(-3, 3)$. Give answer in the form $y = ax^2 + bx + c$.

Q24. Find the equation of a parabola that has a vertex at $(2, 5)$ and passes through point $(1, 3.5)$. Give answer in the form $y = ax^2 + bx + c$.

Q25. Find the equation of a parabola that has a vertex at $(4, -2)$ and passes through point $(2, -6)$. Give answer in the form $y = ax^2 + bx + c$.

Q26. Find the equation of a parabola that has a vertex at $(3, 1)$ and passes through point $(1, -5)$. Give answer in the form $y = ax^2 + bx + c$.

Q27. Find the equation of a parabola that has a vertex at $(-1, 8)$ and an x-intercept at $x = 1$. Give answer in the form $y = ax^2 + bx + c$.

Q28. Find the equation of a parabola that passes through points $(-1, 2)$ and $(2, -1)$ and is symmetrical in line $x = 1$. Give answer in the form $y = ax^2 + bx + c$.

Q29. Find the equation of a parabola that passes through points $(-4, 1)$ and $(-1, 7)$ and is symmetrical in line $x = -3$. Give answer in the form $y = ax^2 + bx + c$.

Q30. Find the equation of a parabola that passes through points $(-4, 3)$ and $(1, 0.5)$ and is symmetrical in line $x = -2$. Give answer in the form $y = ax^2 + bx + c$.

Q31. Express the equation $y = x^2 + 2x - 3$ in a vertex and a product form.

Q32. Express the equation $y = 2x^2 - 3x - 2$ in a vertex and a product form.

Q33. Express the equation $y = -3(x + 1)^2 + \frac{3}{4}$ in a general and product form.

Q34. Express the equation $y = \frac{1}{2}(x - 2)^2 - 2$ in a general and product form.

Q35. Express the equation $y = -\frac{1}{2}(x + 4)(x + 2)$ in a general and vertex form.

Q36. Express the equation $y = -2(x + \frac{3}{2})(x - 1)$ in a general and vertex form.

Q37. Complete the table.

	GENERAL form $y = ax^2 + bx + c$	VERTEX form $y = a(x-p)^2 + q$	FACTOR form $y = a(x-x_1)(x-x_2)$	zeroes $x_1,\ x_2$	vertex (p, q)	discriminant $\Delta = b^2 - 4ac$
(1)	$y = x^2 - 2x - 15$					
(2)			$y = 2(x-1)(x-5)$			
(3)		$y = 2(x + \frac{5}{4})^2 - \frac{49}{8}$				
(4)	$y = x^2 ...$				$(-2, -3)$	
(5)	$y = -2x^2 ...$			$-7,\ -2$		
(6)	$y = -4x^2 + 28x - 49$					
(7)	$y = -\frac{1}{2}x^2 - 2x - 3$					
(8)		$y = -4(x-1)^2 + 2$				
(9)	$y = -2x^2 + 2x\sqrt{2} + 7$					

5.3 Applications of quadratics

Quadratic inequalities

Q38. Solve the inequalities. Give exact answers.

(1) $x^2 + 3x + 2 \geq 0$
(2) $3x^2 + 10x + 3 \leq 0$
(3) $2x^2 - 5x + 2 < 0$
(4) $2x^2 + 3x - 2 > 0$
(5) $3x^2 + 4x + 1 \geq 0$
(6) $-3x^2 - 4x + 7 \leq 0$
(7) $3x^2 + 10x + 3 < 0$
(8) $-4x^2 + 4x + 3 > 0$
(9) $-2x^2 + x + 1 \geq 0$
(10) $3x^2 + 4x + 3 \leq 0$
(11) $-x^2 + 2x - 5 < 0$
(12) $6x^2 + 5x + 1 > 0$

Q39. Solve the inequalities. Give answers correct to 3 s.f.

(1) $x^2 + 4x + 2 > 0$
(2) $3x^2 + 8x - 6 > 0$
(3) $2x^2 - 6x + 3 \geq 0$
(4) $3x^2 + 4x + 2 \leq 0$
(5) $-x^2 + 2x - 4 < 0$
(6) $4x^2 + 5x + 5 > 0$
(7) $-2x^2 + 5x - 1 \geq 0$
(8) $-3x^2 + 4x + 6 \leq 0$
(9) $-2x^2 + 7x - 4 < 0$
(10) $-4x^2 - 6x + 11 > 0$
(11) $-3x^2 - 4x + 6 \geq 0$
(12) $-4x^2 + 3x + 4 \leq 0$

(13) $0.21x^2 + 3.13x + 7.26 < 0$
(14) $0.813x^2 + 0.427x - 1.05 > 0$
(15) $-2.14x^2 + 1.02x + 0.982 \geq 0$
(16) $-3x^2 - 5x + 4 \leq 0$
(17) $-3.9x^2 + 6.21x - 5.2 < 0$
(18) $-1.03x^2 + 0.204x - 0.54 > 0$

Problems involving quadratics

Q40. The distance s kilmetres of a ship from the port is modelled by the equation $s = -2t^2 + 24t$, where $t \geq 0$ is the time in hours. The ship leaves the port at $t = 0$.

(i) At what time the ship comes back to the port?
(ii) At what time is the ship in furthest distance from the port?
(iii) What is the furthest distance of the ship from the port?

Q41. The distance s kilmetres of a ship from the port is modelled by the equation $s = -3t^2 + 24t + 27$, where $t \geq 0$ is the time in hours.

(i) What is the initial distance of the ship from the port?
(ii) At what time the ship comes back to the port?
(iii) At what time is the ship in furthest distance from the port?
(iv) What is the furthest distance of the ship from the port?

Q42. A boy standing on a roof of a building throws a ball upwards. The height h metres of the ball measured from the ground level is given by $h = 12 + 8t - 4.9t^2$, where $t \geq 0$ is the time in seconds.

(i) What is the initial height of the ball?

(ii) At what time the ball hits the ground?

(iii) At what time does the ball reach the largest height?

(iv) What is the largest height that the ball reaches?

Q43. A rectangular plot of land is adjacent from one side to the river. The river bank is a straight line. A fence 200 metres long is built along the other three sides of the plot. What are the dimensions of the plot if its area is greatest possible?

Q44. A sum of two numbers is 22. What are the numbers if their product is greatest possible? What is the product?

Q45. A sum of one number and twice the other is 22. What are the numbers if their product is greatest possible? What is the product?

Q46. A sum of three times one number and twice the other is 24. What are the numbers if their product is greatest possible? What is the product?

Q47. Sum of two shorter sides of a right angled triangle is 2. Find the largest possible area of the triangle.

Q48. Consider line given by $y = -\frac{1}{2}x + 5$.

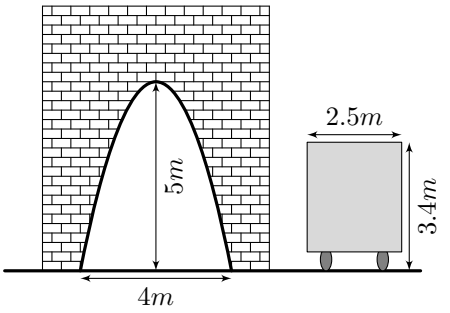
(i) Point A on the line has x-coordinate equal -2. Find y-coordinate of A.
Find the distance of A from the origin.

(ii) Point B on the line has x-coordinate equal -4. Find y-coordinate of B.
Find the distance of B from the origin.

(iii) Point C on the line has x-coordinate equal p. Find the value of p for which the distance of C from the origin is the smallest. Find the smallest distance.

Q49. Point A lies on the positive part of x-axis, point B lies on the positive part of y-axis. The sum of the distances of points A and B to the origin is 2. Let M be the midpoint of line segment AB. Find the smallest distance of M from the origin.

Q50. A tunnel's cross-section is in the shape of a parabola. Its height measured at the highest point is 5 metres and its width measured at the bottom is 4 metres (see the diagram). A truck is 3.4 metres high and 2.5 metres wide.

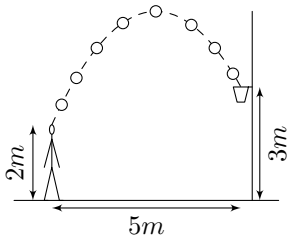


(i) Can the truck pass through the tunnel?
Justify your answer.

(ii) What is the greatest possible height of a truck 2.5 metres wide that can pass through the tunnel?
Give your answer correct to 1cm.

(iii) What is the greatest possible width of a truck 3.4 metres high that can pass through the tunnel?
Give your answer correct to 1cm.

Q51. A basketball player 2 metres high is standing 5 meters from the basket (the distance is measured horizontally, as shown in the diagram). The basket is suspended 3 metres above the ground level. The player throws a ball that lands inside the basket. The ball's trajectory is in the shape of a parabola. Let us assume that it starts from 2 metres above the ground and finishes exactly in the basket. Knowing that the greatest height that the ball reached is 5 metres find the distance between the highest position of the ball and the basket.



Q52. A climber kept a record of how long it took him to reach certain altitudes from the moment he had begun climbing. It turned out that the time t in hours can be modelled by the equation $t = 1.1(h - 2.4)^2$ where h is the altitude in kilometres.

(i) At what altitude did the climber begin climbing?

(ii) At what altitude was he after 8 hours of climbing?

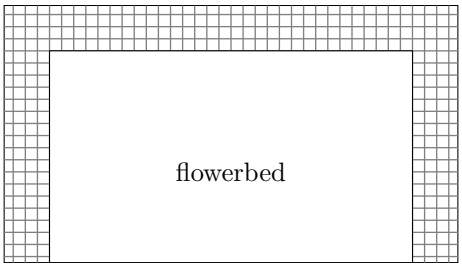
(iii) At what altitude was he after 10 hours of climbing?

(iv) After what time did he get to the altitude of 6000 metres?

(v) After what time did he get to the altitude of 7000 metres?

Give all answers correct to 10 metres / 10 minutes.

Q53. A pavement 1.5 metres wide is built along three sides of a rectangular flowerbed. The total area of the pavement is $43.5m^2$. Find the dimensions of the flowerbed knowing that its surface area is largest possible.



Q54. An ant-hill is in the shape of a *paraboloid*, a solid whose base is a circle and each vertical cross-section through the centre of the base has a shape of a part of a parabola (with vertex up). The radius of the base is 80 centimetres. The height of the ant-hill is 1.6 metre. What would be the area the cross-section obtained if the ant-hill was cut with a horizontal plane on the level of 1 metre above the ground? Give your answer correct to $100cm^2$.

Q55. A jar with jam is in the shape of a cylinder with the height of 7.5 centimetres and the diameter of the base 8.5 centimetres. Paul placed the jar on a table. He wanted to cover the jar with a bowl turned upside down. The bowl is in the shape of a *paraboloid*, i.e. all of its vertical cross-sections passing through the centre of the base are parabolas. The circular base of the bowl has the diameter of 20 centimetres. Its height is 8 centimetres. It turned out that the bowl is too small to cover the jar. What is the distance between the brim of the bowl and the table? Give your answer correct to 1 milimetre.

Q56. A fountain jet is in the shape of a parabola. The water is spouted from the ground level and it reaches the height of 80 centimetres after travelling 30 centimetres horizontally. The maximum height that it reaches is 1 meter. Find the horizontal distance that the water covers till it reaches the maximum height.

Q57. A fountain jet is in the shape of a parabola. The water is spouted from the point placed 20 cm above the ground level and it reaches its maximum height of 1 metre. Then it goes back to the ground level after travelling 180 centimetres horizontally. Find the horizontal distance that the water covers till it reaches the maximum height.

Q58. A ditch has a cross-section in the shape of a parabola. Its maximum depth is 1.5 meters and so is its maximum width. A man tries to hide a chest in a ditch. The chest is a cuboid with the dimensions 2 metres by 1 metre by 85 centimetres. Is it possible to hide the chest in the ditch? How should it be placed?

Investigating graphs of rational functions

Q59. For each question below:

1. Simplify the expression if possible.
2. Find x-intercepts and a y-intercept.
3. Find equations of vertical asymptotes.
4. Investigate the existence of a horizontal asymptote.
5. Check the values between x-intercepts and asymptotes to graph the curve. Compare with the graph on a GDC.

(1) $y = \frac{x-1}{x-2}$
(2) $y = \frac{2x+4}{x-1}$
(3) $y = \frac{2x-6}{x+2}$
(4) $y = \frac{x-2}{2x+4}$
(5) $y = \frac{x+1}{3x-3}$
(6) $y = \frac{2x-4}{x-2}$
(7) $y = \frac{x-3}{x^2-9}$
(8) $y = \frac{x+2}{x^2+5x+6}$
(9) $y = \frac{2x+1}{2x^2-3x-2}$
(10) $y = \frac{x^2-4}{x^2-4x+4}$
(11) $y = \frac{x^2+4x+3}{x^2+3x+2}$
(12) $y = \frac{4x^2+4x+1}{2x^2-x-1}$
(13) $y = \frac{x^2+x-6}{x^2-x-6}$
(14) $y = \frac{x^2-5x+6}{x^2+5x+6}$
(15) $y = \frac{2x^2+5x-3}{x^2-2x-3}$
(16) $y = \frac{2x^2+2x-4}{x^2+x-6}$
(17) $y = \frac{2x^2+x-15}{2x^2-7x-15}$
(18) $y = \frac{x^2-4}{x+1}$

Chapter 6

Trigonometry

6.1 Degrees and radians

Q1. Express the following angles in radians. Give exact answers in terms of π.

(1) $90°$
(2) $45°$
(3) $60°$
(4) $30°$
(5) $15°$
(6) $135°$
(7) $120°$
(8) $270°$
(9) $20°$
(10) $50°$
(11) $75°$
(12) $330°$
(13) $105°$
(14) $210°$
(15) $150°$

Q2. Express the following angles in radians. Give answers correct to 3 s.f.

(1) $180°$
(2) $90°$
(3) $80°$
(4) $12°$
(5) $100°$
(6) $57°$
(7) $78°$
(8) $108°$
(9) $200°$
(10) $18°$
(11) $72°$
(12) $292°$

Q3. The following angles are given in radians. Express them in degrees.

(1) $\frac{\pi}{6}$
(2) $\frac{2\pi}{3}$
(3) $\frac{5\pi}{4}$
(4) $\frac{\pi}{8}$
(5) $\frac{5\pi}{12}$
(6) $\frac{7\pi}{12}$
(7) $\frac{5\pi}{3}$
(8) $\frac{2\pi}{9}$
(9) $\frac{5\pi}{9}$
(10) $\frac{5\pi}{6}$
(11) 3
(12) 1.57
(13) 1
(14) 0.873
(15) 1.22

6.2 Trigonometric ratios

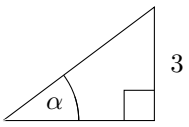
Q4. Fill in the table with exact values of trigonometric ratios.

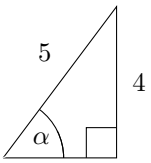
θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
	$30°$	$45°$	$60°$
$\sin\theta$			
$\cos\theta$			
$\tan\theta$			

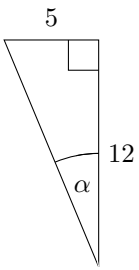
Q5. Use a calculator to find the values of all basic trigonometric ratios (sine, cosine, tangent) of the given angles. Note that some of the angles are given in degrees while the other in radians.

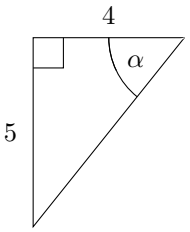
(1) $40°$
(2) $60°$
(3) $1°$
(4) 1
(5) $13.5°$
(6) 1.5
(7) 0.8
(8) $8°$
(9) 1.57
(10) 0.3
(11) 1.2
(12) $1.2°$

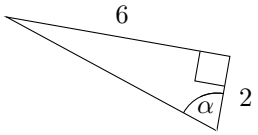
Q6. Find $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$. Give exact answers.

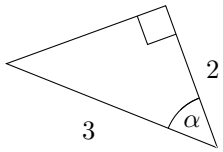
(1)


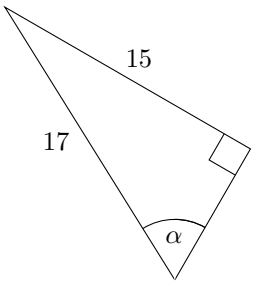
(2)


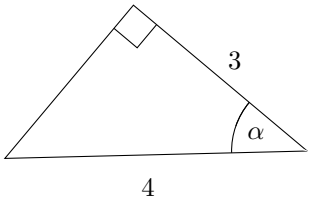
(3)


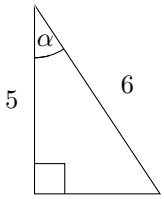
(4)


(5)


(6)


(7)


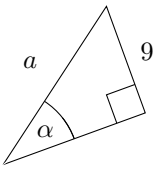
(8)


(9)


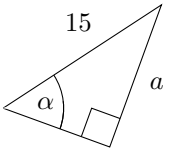
Q7. Find the other two basic trigonometric ratios. Give exact answers.

(1) $\sin\alpha = \frac{3}{5}$
(2) $\sin\alpha = \frac{3}{\sqrt{13}}$
(3) $\cos\alpha = \frac{2}{\sqrt{7}}$
(4) $\cos\alpha = \frac{40}{41}$
(5) $\tan\alpha = \frac{12}{5}$
(6) $\tan\alpha = 1$
(7) $\sin\alpha = \frac{2}{\sqrt{5}}$
(8) $\cos\alpha = \frac{2\sqrt{2}}{3}$
(9) $\tan\alpha = \frac{2}{\sqrt{5}}$
(10) $\sin\alpha = \frac{\sqrt{7}}{5}$
(11) $\cos\alpha = \frac{\sqrt{15}}{8}$
(12) $\tan\alpha = \frac{5}{4\sqrt{6}}$

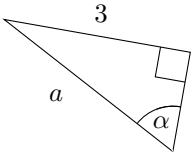
Q8. Find the value of a.

(1)


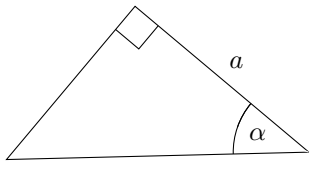
$\sin\alpha = 0.6$

(2)


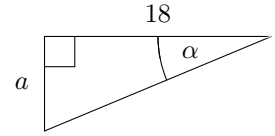
$\cos\alpha = 0.6$

(3)


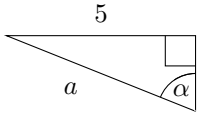
$\tan\alpha = 1.875$

(4)


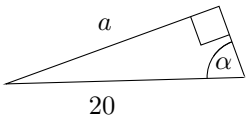
$\sin\alpha = \frac{\sqrt{7}}{4}$

(5)


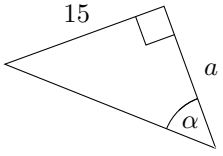
$\cos\alpha = \frac{12}{13}$

(6)


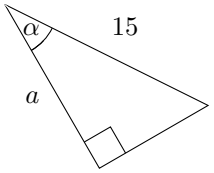
$\tan\alpha = 2.5$

(7)


$\tan\alpha = 3$

(8)


$\cos\alpha = \frac{2}{3}$

(9)


$\sin\alpha = \frac{\sqrt{11}}{6}$

Q9. Find the angle α. Give answer correct to 3 s.f. in both radians and degrees.

(1) $\sin\alpha = 0.68$ (4) $\sin\alpha = 0.303$ (7) $\sin\alpha = 0.777$ (10) $\sin\alpha = 0.503$
(2) $\cos\alpha = 0.135$ (5) $\cos\alpha = 0.0242$ (8) $\cos\alpha = 1.03$ (11) $\cos\alpha = 0.876$
(3) $\tan\alpha = 0.987$ (6) $\tan\alpha = 2.48$ (9) $\tan\alpha = 3$ (12) $\tan\alpha = 0.205$

Q10. One of the angles in a right-angled triangle is 73°. The side adjacent to the angle is 7 cm long. Find the lengths of the other sides.

Q11. One of the angles in a right-angled triangle is 64°. The side opposite is 8.4 cm long. Find the lengths of the other sides.

Q12. One of the angles in a right-angled triangle is 35°. The shortest side is 2.64 cm long. Find the lengths of the other sides.

Q13. One of the angles in a right-angled triangle is 42°. The longest side is 6.5 cm long. Find the lengths of the other sides.

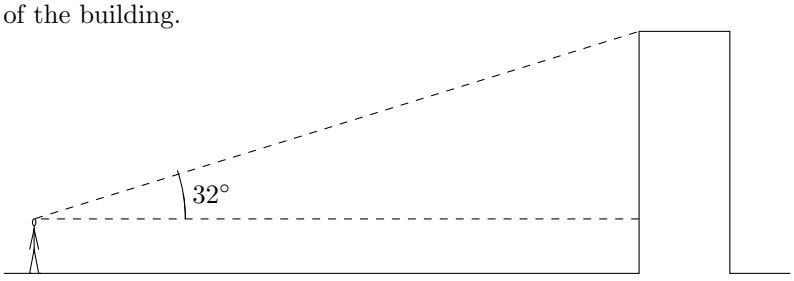
Q14. Given lengths of two sides a, b of a triangle and the measure of the angle between them θ find the area of the triangle. Note that some of the angles are given in degrees while the other in radians.

(1) $a = 5$, $b = 7$, $\theta = 30°$;
(2) $a = 4.25$, $b = 13.5$, $\theta = 0.946$;
(3) $a = 2.62$, $b = 1.53$, $\theta = 1.42$;
(4) $a = 0.118$, $b = 0.822$, $\theta = 1.27$;
(5) $a = 0.514$, $b = 9.4$, $\theta = 59.4°$;
(6) $a = 16.8$, $b = 16$, $\theta = 0.793$;
(7) $a = 20$, $b = 3.75$, $\theta = 33.2°$;
(8) $a = 12.5$, $b = 8.08$, $\theta = 0.975$;
(9) $a = 4.14$, $b = 14.2$, $\theta = 1.1$;
(10) $a = 11.6$, $b = 5.02$, $\theta = 1.08$;
(11) $a = 17.2$, $b = 4.9$, $\theta = 43°$;
(12) $a = 9.15$, $b = 16.9$, $\theta = 1.19$;
(13) $a = 3.1$, $b = 7.08$, $\theta = 30.7°$;
(14) $a = 9.04$, $b = 3.27$, $\theta = 64.4°$;
(15) $a = 10.1$, $b = 11.8$, $\theta = 14.7°$;
(16) $a = 16.4$, $b = 19.5$, $\theta = 1.47$;
(17) $a = 12.6$, $b = 0.294$, $\theta = 80°$;
(18) $a = 11.8$, $b = 6.27$, $\theta = 48.3°$.

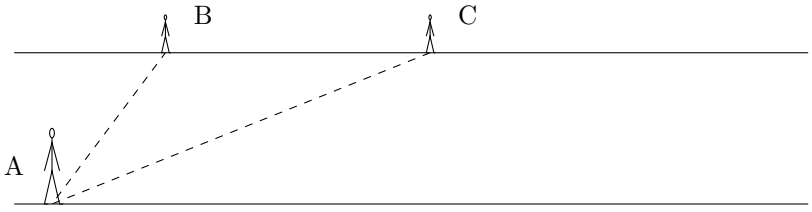
Q15. In triangle ABC the height CD is 18, the angle $B\hat{A}C = 59°$ and the line segment BD is 12. Find the area of the triangle.

Q16. In triangle ABC the height CD is 6, the angle $A\hat{C}B = 88°$ and the line segment BD is 4.2. Find the area of the triangle.[1]

Q17. A man 1.8 metre high stands on a horizontal ground. He watches the top of a building that stands 34 metres from the man. The man's line of sight makes an angle of 32° with the horizontal. Find the height of the building.



Q18. Three people (A, B and C) are standing on two parallel banks of the river. The distance between B and C is 15 metres. The line AB is perpendicular to the banks, while the angle BAC is 28°. Find the width of the river.



[1] You may like to find the angles $A\hat{B}C$ and $B\hat{A}C$ firstly.

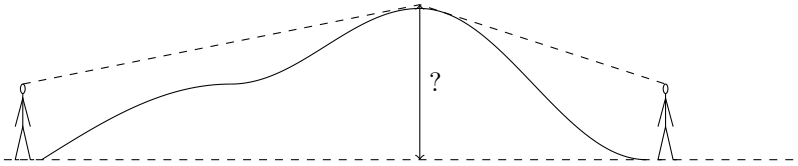
Q19. Find the area of an equilateral triangle with side length 6.

Q20. Find the area of an isosceles triangle with two angles of 42° and the length of side between them 6.

Q21. Find the area of an isosceles triangle with an angle 110° and the longest side of 8.4.

Q22. Find the area of a regular pentagon with side length 7.

Q23. Two people both 1.8 m high are standing on two sides of a hill. The top of the hill is exactly between them. Both people are watching the top. Line of sight of one of them makes 15° with the horizontal, while the other's 22°. The distance between them is 18 metres. Find the height of the hill.



Q24. Two people are standing on two sides of a hill. The top of the hill is exactly between them. Both people are watching the top. Line of sight of a man who is 1.8 m high makes 15° with the horizontal, while line of sight of a woman 1.6 m high makes 22° with the horizontal. The distance between the man and the woman is 18 metres. Find the height of the hill.

Q25. In a triangle ABC: $|AB| = 8.2$, $B\hat{A}C = 40°$ and $A\hat{B}C = 35°$. Find the area of the triangle.

6.3 Trigonometric functions

Q26. Find sine, cosine and tangent of the following angles. Give exact answers.

(1) $\frac{5\pi}{3}$ (9) $\frac{\pi}{2}$ (17) $-60°$ (25) $-\frac{7\pi}{4}$ (33) $\frac{7\pi}{4}$
(2) $-\frac{\pi}{4}$ (10) $\frac{5\pi}{6}$ (18) $210°$ (26) $-135°$ (34) $-\frac{5\pi}{4}$
(3) $-225°$ (11) $-\frac{3\pi}{4}$ (19) $135°$ (27) $120°$ (35) $-210°$
(4) $-\frac{7\pi}{6}$ (12) $225°$ (20) $-\frac{2\pi}{3}$ (28) $150°$ (36) $315°$
(5) $-\frac{5\pi}{6}$ (13) $-150°$ (21) $\frac{7\pi}{6}$ (29) $-\frac{5\pi}{3}$ (37) $\frac{5\pi}{4}$
(6) $-45°$ (14) $-30°$ (22) $-\frac{4\pi}{3}$ (30) $-315°$ (38) $-\frac{\pi}{3}$
(7) $\frac{3\pi}{4}$ (15) $-120°$ (23) π (31) $\frac{4\pi}{3}$ (39) $-\frac{\pi}{6}$
(8) $\frac{3\pi}{2}$ (16) $-240°$ (24) $\frac{2\pi}{3}$ (32) 0 (40) $330°$

Q27. Find the values of the other two basic trigonometric functions. Give exact answers.

(1) $\sin\alpha = \frac{3}{5}$, $\frac{\pi}{2} < \alpha < \pi$
(2) $\sin\alpha = -\frac{3}{\sqrt{13}}$, $\pi < \alpha < \frac{3\pi}{2}$
(3) $\cos\alpha = \frac{2}{\sqrt{7}}$, $\frac{3\pi}{2} < \alpha < 2\pi$
(4) $\cos\alpha = -\frac{40}{41}$, $\pi < \alpha < \frac{3\pi}{2}$
(5) $\tan\alpha = -\frac{12}{5}$, $\frac{\pi}{2} < \alpha < \pi$
(6) $\tan\alpha = -\frac{1}{1}$, $\frac{3\pi}{2} < \alpha < 2\pi$
(7) $\sin\alpha = -\frac{2}{\sqrt{5}}$, $\frac{3\pi}{2} < \alpha < 2\pi$
(8) $\cos\alpha = -\frac{2\sqrt{2}}{3}$, $\frac{\pi}{2} < \alpha < \pi$
(9) $\tan\alpha = \frac{2}{\sqrt{5}}$, $\pi < \alpha < \frac{3\pi}{2}$
(10) $\sin\alpha = -\frac{\sqrt{7}}{5}$, $\frac{3\pi}{2} < \alpha < 2\pi$
(11) $\cos\alpha = -\frac{\sqrt{15}}{8}$, $\pi < \alpha < \frac{3\pi}{2}$
(12) $\tan\alpha = -\frac{5}{4\sqrt{6}}$, $\frac{\pi}{2} < \alpha < \pi$

Q28. Find the angle θ such that $-\pi < \theta \leq \pi$ ($-180° < \theta \leq 180°$) given its sine and cosine. Give answers in both radians and degrees. Give exact answers.

(1) $\sin\theta = 1$ $\cos\theta = 0$
(2) $\sin\theta = -1$ $\cos\theta = 0$
(3) $\sin\theta = 0$ $\cos\theta = 1$
(4) $\sin\theta = -\frac{\sqrt{2}}{2}$ $\cos\theta = -\frac{\sqrt{2}}{2}$
(5) $\sin\theta = -\frac{1}{2}$ $\cos\theta = \frac{\sqrt{3}}{2}$
(6) $\sin\theta = -\frac{1}{2}$ $\cos\theta = -\frac{\sqrt{3}}{2}$
(7) $\sin\theta = \frac{1}{2}$ $\cos\theta = -\frac{\sqrt{3}}{2}$
(8) $\sin\theta = \frac{\sqrt{2}}{2}$ $\cos\theta = -\frac{\sqrt{2}}{2}$
(9) $\sin\theta = \frac{\sqrt{3}}{2}$ $\cos\theta = \frac{1}{2}$
(10) $\sin\theta = -\frac{\sqrt{3}}{2}$ $\cos\theta = \frac{1}{2}$
(11) $\sin\theta = -\frac{\sqrt{2}}{2}$ $\cos\theta = \frac{\sqrt{2}}{2}$
(12) $\sin\theta = -\frac{\sqrt{3}}{2}$ $\cos\theta = -\frac{1}{2}$
(13) $\sin\theta = \frac{\sqrt{3}}{2}$ $\cos\theta = -\frac{1}{2}$
(14) $\sin\theta = 0$ $\cos\theta = -1$

Q29. Find the angle θ such that $-\pi < \theta \leq \pi$ ($-180° < \theta \leq 180°$) given its sine and cosine. Give answers in both radians and degrees. Give answers correct to 3 s.f.

(1) $\sin\theta = 0.6621$ $\cos\theta = -0.7494$
(2) $\sin\theta = -0.5323$ $\cos\theta = 0.8466$
(3) $\sin\theta = -0.2937$ $\cos\theta = -0.9559$
(4) $\sin\theta = -0.7489$ $\cos\theta = -0.6627$
(5) $\sin\theta = -0.9089$ $\cos\theta = -0.417$
(6) $\sin\theta = -0.1141$ $\cos\theta = 0.9935$
(7) $\sin\theta = 0.3198$ $\cos\theta = 0.9475$
(8) $\sin\theta = 0.392$ $\cos\theta = 0.9199$
(9) $\sin\theta = 0.4745$ $\cos\theta = -0.8802$
(10) $\sin\theta = -0.2596$ $\cos\theta = 0.9657$
(11) $\sin\theta = 0.3653$ $\cos\theta = 0.9309$
(12) $\sin\theta = 0.6463$ $\cos\theta = -0.7631$
(13) $\sin\theta = -0.4508$ $\cos\theta = -0.8926$
(14) $\sin\theta = 0.5728$ $\cos\theta = 0.8197$
(15) $\sin\theta = -0.6366$ $\cos\theta = 0.7712$
(16) $\sin\theta = 0.5168$ $\cos\theta = 0.8561$
(17) $\sin\theta = 0.3023$ $\cos\theta = 0.9532$
(18) $\sin\theta = -0.981$ $\cos\theta = 0.194$
(19) $\sin\theta = -0.8779$ $\cos\theta = -0.4789$
(20) $\sin\theta = -0.6938$ $\cos\theta = 0.7202$
(21) $\sin\theta = -0.354$ $\cos\theta = -0.9352$
(22) $\sin\theta = -0.9538$ $\cos\theta = -0.3003$
(23) $\sin\theta = -0.4442$ $\cos\theta = 0.8959$
(24) $\sin\theta = 0.6381$ $\cos\theta = -0.77$

6.4 Trigonometric equations

Q30. Solve the equations over the given domain. Give exact answers.

(1) $2\sin x = 1$ for $0 \leq x \leq 2\pi$
(2) $2\cos x = 1$ for $0 \leq x \leq 2\pi$
(3) $\sin x = 1$ for $-2\pi \leq x \leq 2\pi$
(4) $\cos x = -1$ for $0 \leq x \leq 4\pi$
(5) $2\sin x = \sqrt{2}$ for $0 \leq x \leq 2\pi$
(6) $2\cos x = \sqrt{3}$ for $0 \leq x \leq 2\pi$
(7) $2\sin x + 1 = 0$ for $0 \leq x \leq 2\pi$
(8) $2\cos x + 1 = 0$ for $0 \leq x \leq 2\pi$
(9) $2\sin x + \sqrt{2} = 0$ for $0 \leq x \leq 2\pi$
(10) $2\cos x + \sqrt{3} = 0$ for $0 \leq x \leq 2\pi$
(11) $2\sin x + 1 = 0$ for $-2\pi \leq x \leq 3\pi$
(12) $2\cos x + 1 = 0$ for $-2\pi \leq x \leq 3\pi$
(13) $2\sin x = \sqrt{3}$ for $0 \leq x \leq 2\pi$
(14) $2\cos x = \sqrt{2}$ for $0 \leq x \leq 2\pi$
(15) $2\sin x + \sqrt{3} = 0$ for $0 \leq x \leq 2\pi$
(16) $2\cos x + \sqrt{2} = 0$ for $0 \leq x \leq 2\pi$
(17) $2\sin x = \sqrt{3}$ for $-4\pi \leq x \leq 0$
(18) $2\cos x = \sqrt{2}$ for $-4\pi \leq x \leq 0$

Q31. Solve the equations over the given domain. Give exact answers.

(1) $\tan x = 1$ for $0 \leq x \leq \pi$
(2) $\tan x = \sqrt{3}$ for $0 \leq x \leq \pi$
(3) $\tan x = \frac{1}{\sqrt{3}}$ for $0 \leq x \leq \pi$
(4) $\tan x = 0$ for $0 \leq x \leq \pi$
(5) $\tan x = -1$ for $0 \leq x \leq \pi$
(6) $\tan x = -\sqrt{3}$ for $0 \leq x \leq \pi$
(7) $\tan x = -\frac{1}{\sqrt{3}}$ for $0 \leq x \leq \pi$
(8) $\tan x = 1$ for $-2\pi \leq x \leq \pi$
(9) $\tan x = \sqrt{3}$ for $-2\pi \leq x \leq \pi$
(10) $\tan x = \frac{1}{\sqrt{3}}$ for $-2\pi \leq x \leq \pi$
(11) $\tan x = -1$ for $-2\pi \leq x \leq \pi$

Q32. Solve the equations over the given domain. Give exact answers.

(1) $\sin x = \frac{1}{2}$ for $0° \leq x \leq 360°$
(2) $\cos x = -\frac{1}{2}$ for $0° \leq x \leq 360°$
(3) $\tan x = 1$ for $0° \leq x \leq 360°$
(4) $\sin x = -\frac{\sqrt{3}}{2}$ for $0° \leq x \leq 360°$
(5) $\cos x = \frac{\sqrt{3}}{2}$ for $0° \leq x \leq 360°$
(6) $\tan x = -\sqrt{3}$ for $0° \leq x \leq 360°$
(7) $\sin x = 1$ for $0° \leq x \leq 360°$
(8) $\cos x = -1$ for $0° \leq x \leq 360°$
(9) $\tan x = 0$ for $0° \leq x \leq 360°$
(10) $\sin x = 0$ for $0° \leq x \leq 360°$
(11) $\cos x = 0.5$ for $0° \leq x \leq 360°$
(12) $\tan x = -\frac{1}{\sqrt{3}}$ for $0° \leq x \leq 360°$
(13) $\sin x = 2$ for $0° \leq x \leq 360°$
(14) $\cos x = 0$ for $0° \leq x \leq 360°$
(15) $\tan x = -\sqrt{3}$ for $0° \leq x \leq 360°$
(16) $\sin x = -1$ for $0° \leq x \leq 360°$
(17) $\cos x = -\frac{\sqrt{3}}{2}$ for $0° \leq x \leq 360°$
(18) $\tan x = \frac{1}{\sqrt{3}}$ for $0° \leq x \leq 360°$

Q33. Solve the equations over the given domain. Give answers correct to 3 s.f.

(1) $\sin x = 0.4$ for $0 \leq x \leq 2\pi$
(2) $\cos x = \frac{1}{3}$ for $0 \leq x \leq 2\pi$
(3) $\sin x = 0.42$ for $0 \leq x \leq 2\pi$
(4) $\cos x = 0.805$ for $0 \leq x \leq 2\pi$
(5) $\sin x = -0.496$ for $0 \leq x \leq 2\pi$
(6) $\cos x = -0.3$ for $0 \leq x \leq 2\pi$
(7) $\sin x = -0.749$ for $0 \leq x \leq 2\pi$
(8) $\cos x = -\frac{7}{25}$ for $0 \leq x \leq 2\pi$
(9) $\sin x = 0.71$ for $0 \leq x \leq 4\pi$
(10) $\cos x = 0.39$ for $0 \leq x \leq 4\pi$
(11) $\sin x = 0.63$ for $-2\pi \leq x \leq 2\pi$
(12) $\cos x = 0.85$ for $-2\pi \leq x \leq 2\pi$
(13) $\sin x = -0.43$ for $0 \leq x \leq 4\pi$
(14) $\cos x = -\frac{2}{9}$ for $0 \leq x \leq 4\pi$
(15) $\sin x = -0.39$ for $-2\pi \leq x \leq 2\pi$
(16) $\cos x = -0.81$ for $-2\pi \leq x \leq 2\pi$

Q34. Solve the equations over the given domain. Give answers correct to 3 s.f.

(1) $\tan x = 2$ for $0 \leq x \leq \pi$
(2) $\tan x = \sqrt{5}$ for $0 \leq x \leq \pi$
(3) $\tan x = \frac{1}{2}$ for $0 \leq x \leq \pi$
(4) $\tan x = -3$ for $0 \leq x \leq \pi$
(5) $\tan x = -0.3$ for $0 \leq x \leq \pi$
(6) $\tan x = -2.8$ for $0 \leq x \leq \pi$
(7) $\tan x = -\frac{11}{2}$ for $0 \leq x \leq \pi$
(8) $\tan x = 3$ for $-2\pi \leq x \leq \pi$
(9) $\tan x = 4.21$ for $-2\pi \leq x \leq \pi$
(10) $\tan x = \frac{2}{3}$ for $-2\pi \leq x \leq \pi$
(11) $\tan x = -1.55$ for $-2\pi \leq x \leq \pi$

Q35. Solve the equations over the given domain. Give answers correct to 3 s.f.

(1) $\sin x = 0.2$ for $0^\circ \leq x \leq 360^\circ$
(2) $\cos x = 0.3$ for $0^\circ \leq x \leq 360^\circ$
(3) $\tan x = 0.9$ for $0^\circ \leq x \leq 360^\circ$
(4) $\sin x = -0.4$ for $0^\circ \leq x \leq 360^\circ$
(5) $\cos x = -0.76$ for $0^\circ \leq x \leq 360^\circ$
(6) $\tan x = -3$ for $0^\circ \leq x \leq 360^\circ$
(7) $\sin x = 0.44$ for $-180^\circ \leq x \leq 180^\circ$
(8) $\cos x = 0.23$ for $-180^\circ \leq x \leq 180^\circ$
(9) $\tan x = 1.7$ for $-180^\circ \leq x \leq 180^\circ$
(10) $\sin x = -0.19$ for $-180^\circ \leq x \leq 180^\circ$
(11) $\cos x = -0.38$ for $-180^\circ \leq x \leq 180^\circ$
(12) $\tan x = -2.3$ for $-180^\circ \leq x \leq 180^\circ$

Q36. Solve the equations for $0 \leq x \leq 2\pi$. Give exact answers in terms of π whenever possible.

(1) $2\sin^2 x - \sin x = 0$
(2) $4\sin^2 x - 1 = 0$
(3) $\sin^2 x - 1 = 0$
(4) $4\sin^2 x - 3 = 0$
(5) $10\sin^2 x + \sin x - 2 = 0$
(6) $2\sin^2 x + \sin x - 1 = 0$
(7) $9\sin^2 x + 3\sin x - 2 = 0$
(8) $6\sin^2 x - 7\sin x + 2 = 0$
(9) $2\sin^2 x - 5\sin x - 3 = 0$
(10) $2\cos^2 x - \cos x = 0$
(11) $4\cos^2 x - 1 = 0$
(12) $\cos^2 x - 1 = 0$
(13) $4\cos^2 x - 3 = 0$
(14) $10\cos^2 x - \cos x - 2 = 0$
(15) $2\cos^2 x - \cos x - 1 = 0$
(16) $9\cos^2 x - 3\cos x - 2 = 0$
(17) $6\cos^2 x + 7\cos x + 2 = 0$
(18) $2\cos^2 x + 5\cos x - 3 = 0$
(19) $\tan^2 x - \tan x = 0$
(20) $\tan^2 x - 1 = 0$
(21) $\tan^2 x - 3 = 0$
(22) $3\tan^2 x - 1 = 0$
(23) $\tan^2 x - \tan x - 2 = 0$
(24) $2\tan^2 x - 3\tan x + 1 = 0$
(25) $\tan^2 x + 4\tan x + 3 = 0$

6.5 Trigonometry in geometry

All answers must be given exactly or correct to 3 significant figures.
Note. In this section, unless the question states otherwise, the lengths of sides BC, AC and AB of triangle ABC are a, b and c respectively. Angles $B\hat{A}C$, $A\hat{B}C$ and $A\hat{C}B$ will be denoted by A, B and C respectively.

Q37. Evaluate the missing lengths of sides in triangle ABC.

(1) $a = 5.2$, $A = 47°$, $B = 51°$
(2) $a = 5.2$, $A = 51°$, $B = 47°$
(3) $b = 2.67$, $B = 13°$, $A = 49°$
(4) $a = 7.87$, $A = 67°$, $C = 84°$
(5) $c = 16.75$, $C = 96°$, $A = 64°$
(6) $b = 6.78$, $B = 71°$, $C = 104°$
(7) $c = 9.61$, $C = 53°$, $B = 21°$
(8) $b = 4.05$, $B = 62°$, $A = 80°$
(9) $c = 5.73$, $C = 64°$, $A = 30°$
(10) $b = 3.15$, $B = 45°$, $C = 37°$
(11) $c = 5.05$, $C = 93°$, $B = 26°$
(12) $a = 11.54$, $A = 96°$, $B = 78°$
(13) $c = 6.26$, $C = 21°$, $B = 54°$
(14) $b = 6.45$, $B = 27°$, $A = 28°$
(15) $a = 6.1$, $A = 28°$, $C = 48°$
(16) $b = 6.81$, $B = 108°$, $A = 41°$
(17) $c = 14.3$, $C = 109°$, $A = 53°$
(18) $a = 14.36$, $A = 65°$, $B = 60°$
(19) $c = 4.37$, $C = 62°$, $B = 25°$
(20) $b = 6.4$, $B = 34°$, $C = 75°$

Q38. Evaluate the missing angles in triangle ABC.

(1) $a = 6.82$, $b = 7.38$, $A = 45°$
(2) $a = 13.41$, $b = 14$, $A = 57°$
(3) $b = 13.17$, $a = 6.4$, $B = 13°$
(4) $a = 10.5$, $c = 11.68$, $A = 12°$
(5) $c = 5.69$, $a = 8$, $C = 58°$
(6) $b = 13.39$, $c = 13$, $B = 54°$
(7) $c = 8.49$, $b = 5.7$, $C = 61°$
(8) $b = 10.08$, $a = 11.96$, $B = 45°$
(9) $c = 9.7$, $a = 12$, $C = 56°$
(10) $b = 6.32$, $c = 6$, $B = 76°$
(11) $c = 13.14$, $b = 20.9$, $C = 39°$
(12) $a = 7.28$, $b = 7$, $A = 72°$
(13) $c = 10.61$, $b = 11.54$, $C = 44°$
(14) $b = 13.16$, $a = 9.66$, $B = 42°$
(15) $a = 5.49$, $c = 7.01$, $A = 41°$
(16) $b = 8.8$, $a = 9$, $B = 74°$
(17) $c = 4.87$, $a = 5$, $C = 31°$
(18) $a = 13.31$, $b = 15$, $A = 55°$
(19) $c = 13.93$, $b = 9.05$, $C = 23°$
(20) $b = 14.4$, $c = 15$, $B = 57°$

Q39. Evaluate the length of the missing side in triangle ABC.

(1) $a = 4.98$, $b = 2.5$, $C = 131°$
(2) $c = 3.3$, $b = 2.1$, $A = 67°$
(3) $b = 7.56$, $a = 6.64$, $C = 25°$
(4) $a = 4.47$, $c = 6.78$, $B = 37°$
(5) $c = 6.6$, $a = 3.1$, $B = 57°$
(6) $b = 6.3$, $c = 5.47$, $A = 21°$
(7) $c = 4$, $b = 3.7$, $A = 118°$
(8) $b = 5.01$, $a = 6.78$, $C = 81°$
(9) $c = 6.3$, $a = 5.94$, $B = 53°$
(10) $b = 6.65$, $c = 4.94$, $A = 98°$
(11) $c = 7.63$, $b = 3.7$, $A = 28°$
(12) $a = 5.8$, $b = 4.2$, $C = 18°$
(13) $c = 5.18$, $b = 3.62$, $A = 14°$
(14) $b = 4.5$, $a = 2.2$, $C = 87°$
(15) $a = 6.89$, $c = 4.32$, $B = 15°$
(16) $b = 7.76$, $a = 4.7$, $C = 67°$
(17) $c = 3.4$, $a = 2.2$, $B = 85°$
(18) $a = 4.65$, $b = 6.3$, $C = 116°$
(19) $c = 4.21$, $b = 2.9$, $A = 19°$
(20) $a = 11.02$, $c = 12.1$, $B = 43°$

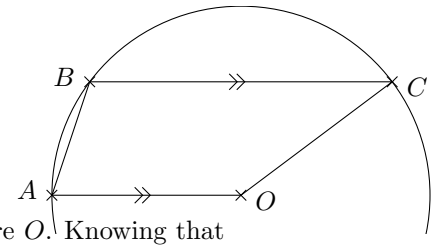
Q40. Evaluate the measure of the angle indicated in parenthesis in triangle ABC.

(1) $a = 3.04$, $b = 3.97$, $c = 3.8$, (A)
(2) $a = 3.2$, $b = 3.5$, $c = 3.45$, (C)
(3) $b = 3.46$, $a = 6.1$, $c = 5.2$, (A)
(4) $a = 4.3$, $c = 2.7$, $b = 5$, (B)
(5) $c = 6.6$, $a = 6.9$, $b = 2.65$, (B)
(6) $b = 5.7$, $c = 4.8$, $a = 0.96$, (C)
(7) $c = 5.7$, $b = 2.3$, $a = 4.48$, (A)
(8) $b = 6.24$, $a = 5.8$, $c = 5.8$, (C)
(9) $c = 2.6$, $a = 6.8$, $b = 8.3$, (B)
(10) $b = 5.6$, $c = 2.05$, $a = 6.5$, (C)
(11) $c = 5.1$, $b = 6.9$, $a = 2.35$, (A)
(12) $a = 6.7$, $b = 3$, $c = 9.2$, (B)
(13) $c = 2.73$, $b = 5.9$, $a = 4.1$, (C)
(14) $b = 5.26$, $a = 5.9$, $c = 5.5$, (B)
(15) $a = 4$, $c = 4.5$, $b = 3.2$, (A)
(16) $b = 5.53$, $a = 6.9$, $c = 5.69$, (B)
(17) $c = 6.63$, $a = 3.28$, $b = 7.52$, (A)
(18) $a = 3.83$, $b = 3.97$, $c = 0.2$, (C)
(19) $c = 3.2$, $b = 2.91$, $a = 3.3$, (A)
(20) $a = 8.1$, $b = 5.23$, $c = 3.3$, (B)

Q41. Evaluate the length of the missing side in triangle ABC. Consider all possibilities.

(1) $a = 13.1$, $b = 16.4$, $A = 25°$
(2) $c = 14.1$, $b = 4$, $C = 15°$
(3) $b = 10.8$, $a = 21.98$, $B = 46°$
(4) $a = 19.38$, $c = 21.3$, $A = 64°$
(5) $c = 8.8$, $a = 9.4$, $C = 50°$
(6) $b = 13.84$, $c = 15.3$, $B = 13°$
(7) $c = 6.91$, $b = 14.8$, $C = 97°$
(8) $b = 13.06$, $a = 10$, $B = 75°$
(9) $c = 12.57$, $a = 18.2$, $C = 11°$
(10) $b = 18.08$, $c = 23.2$, $B = 46°$
(11) $c = 22.62$, $b = 9.8$, $C = 30°$
(12) $a = 11.4$, $b = 11.7$, $A = 71°$
(13) $c = 15.8$, $b = 16.5$, $C = 72°$
(14) $b = 9.79$, $a = 16.91$, $B = 90°$
(15) $a = 8.2$, $c = 6.8$, $A = 42°$
(16) $b = 12.9$, $a = 21.3$, $B = 32°$
(17) $c = 17.4$, $a = 10.4$, $C = 89°$
(18) $a = 12.63$, $b = 13.08$, $A = 74°$
(19) $c = 18.49$, $b = 32$, $C = 18°$
(20) $a = 5.58$, $c = 6.2$, $A = 56°$

Q42. Points A, B and CC are placed on the circle with centre O and radius 5 in such a way that AO is parallel to BC. Knowing that angle $O\hat{A}B$ is 75° find:



(i) the area of trapezium $AOCB$.
(ii) the perimeter of trapezium $AOCB$.

Q43. Trapezium $ABCD$ is inscribed in a circle with radius 6 and centre O. Knowing that $A\hat{O}B = B\hat{O}C = C\hat{O}D = 30°$ find:

(i) the area of the trapezium.
(ii) the perimeter of the trapezium.

Q44. The lengths of sides AB, BC, CD and DA of a convex quadrilateral $ABCD$ are 4.2, 7.9, 6.25 and 3.7 respectively. Knowing that $A\hat{B}C = 99.5°$ find the size of angle $A\hat{D}C$.

Q45. Find the length of a diagonal of a regular pentagon with side of length equal 4.

Q46. Port A is on a bearing of 067° and in the distance of 41 kilometres from ship S. Port B is on a bearing of 166° and in the distance of 57 kilometres from the ship. Find:

(i) the bearing of B from A,
(ii) the bearing of A from B,
(iii) the distance between A and B.

Q47. The ship sailed from the port on a bearing of 115°. After 50 kilometres it changed the direction to a bearing of 062° from the point of change of direction. How far should it sail to be exactly east from its starting position? What will be its distance from the starting position?

Q48. Four ships: Albatross, Bluebird, Cockatoo and Dove left the port in the morning. At noon it turned out that Bluebird is 38 kilometres on a bearing 016° from Albatross, Cockatoo is 25 kilometres on a bearing 142° from Albatross and Dove is 43 kilometres on a bearing of 248° from Albatross. Find:

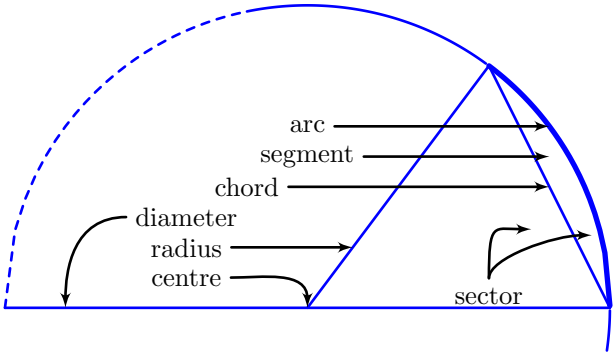
(i) the distance between Bluebird and Cockatoo,
(ii) the distance between Cockatoo and Dove,
(iii) the distance between Bluebird and Dove,
(iv) a bearing of Bluebird from Dove,
(v) a bearing of Dove from Cockatoo,

Q49. The ship sailed from the port on a bearing of 288°. After 38 kilometres it changed the direction to east. How far did it sail then if its distance from the port was 20 kilometres?

Q50. Two ships left a port simultaneously and both sailed with constant speeds along straight lines. The first ship's bearing from the port was 353° while the second's 028°. After some time they were 40 kilometres apart. Find the distance that each of them covered if:

(1) their speeds were the same,
(2) the first ship was twice faster than the second.

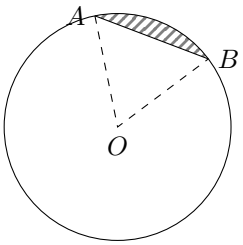
6.6 Arcs, sectors, segments



Q51. Consider a circle with centre O and radius r. The points A and B are on the circle. The area of a sector AOB is given by S. Find the length of arc AB.

(1) $r = 6.3$, $S = 32.1$
(2) $r = 6$, $S = 55$
(3) $r = 2.3$, $S = 6.99$
(4) $r = 11.7$, $S = 97$
(5) $r = 19.9$, $S = 204.5$
(6) $r = 5.04$, $S = 30.1$
(7) $r = 3.2$, $S = 8.1$
(8) $r = 5.9$, $S = 45.2$

Q52. Find the area of the shaded segment (see: diagram) if the circle's radius is r and the angle $A\hat{O}B = \alpha$ for:



(1) $r = 5$, $\alpha = 47°$
(2) $r = 6$, $\alpha = 52°$
(3) $r = 11.4$, $\alpha = 93°$
(4) $r = 5.04$, $\alpha = 128°$
(5) $r = 14.09$, $\alpha = 11°$
(6) $r = 7.5$, $\alpha = 23°$

Q53. The length of arc AB of a circle with centre O and radius r is equal l. Find the area of the segment bounded by the arc and by chord AB.

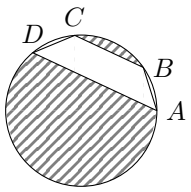
(1) $r = 6.2$, $l = 6.2$
(2) $r = 11.3$, $l = 20$
(3) $r = 2$, $l = 3.5$
(4) $r = 1$, $l = 1.57$
(5) $r = 8.3$, $l = 10.2$
(6) $r = 4.7$, $l = 3.3$

Q54. Given the area of the shaded segment (see: diagram above) and the radius of the circle r find the size of angle $A\hat{O}B$. Give answers in degrees.

(1) area =2.19, $r = 2.4$
(2) area =3, $r = 5$
(3) area =4.84, $r = 3.5$
(4) area =52.1, $r = 20.6$
(5) area =14.01, $r = 19.9$
(6) area =0.34, $r = 4.1$

Q55. An angle at the centre of a circle is such that the perimeter of a sector with this angle is four times the length of arc inside the angle. Give the answer in both radians and degrees.

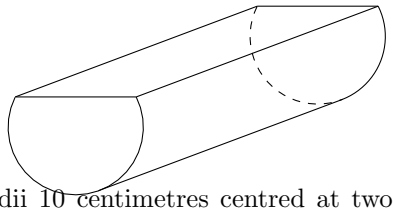
Q56. Trapezium $ABCD$ is inscribed in the circle with radius 5 as shown in the diagram. Sides AD and BC are parallel. The lengths of sides AB and BC are 3 and 4 respectively. Find the ratio of areas of the larger shaded segment to the smaller shaded segment.



Q57. In a circle with radius 6 a chord AB shorter than a diameter divides the circle into two segments.

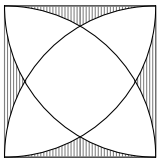
(i) When $AB = 8$ what is the ratio of areas of the major segment to the minor segment?
(ii) If the area of the major segment is twice the area of the minor segment, what is the length of AB?

Q58. A gutter is in the shape of a cylinder cut in such a way that its ends are congruent segments of a circle with diameter 12 centimetres. The edges of the gutter (as seen from the top) form a rectangle with dimensions 10.5 centimetres by 2 metres. Evaluate the amount of water that the gutter can contain. Give your answer correct to 0.01 litre.



Q59. Consider a square with side length 10 centimetres. Two arcs with radii 10 centimetres centred at two opposite vertices of the square are drawn inside a square. Evaluate the area of the region bounded by the two arcs.

Q60. Consider a square with side length 10 centimetres. Four arcs with radii 10 centimetres centred at vertices of the square are drawn inside a square as shown in the diagram. Find the area of the non-shaded region.



Q61. (1) A pen for a goat is in the shape of a rectangle with sides of 7 and 4 metres. A goat that is inside the pen is tied with a rope 5 metres long to one of the vertices of the rectangle. A goat has eaten all the grass that it has had access to. What is the area of the part of the pen that is not accessible for the goat?

(2) A pen for a goat is in the shape of a rectangle with sides of 7 and 4 metres. A goat that is outside the pen is tied with a rope 5 metres long to one of the vertices of the rectangle. A goat has eaten all the grass that it has had access to. What is the area of the grass eaten?

(3) A pen for a goat is in the shape of a right-angled triangle with sides adjacent to the right angle of 7 and 4 metres. A goat that is outside the pen is tied with a rope 5 metres long to the vertex of the smallest angle. A goat has eaten all the grass that it has had access to. What is the area of the grass eaten?

(4) A pen for a goat is in the shape of a right-angled triangle with sides adjacent to the right angle of 7 and 4 metres. A goat that is outside the pen is tied with a rope 5 metres long to the vertex of the right angle. A goat has eaten all the grass that it has had access to. What is the area of the grass eaten?

6.7 Chapter review

Q1. (non-calculator) Solve the equation for $-\pi \leq x \leq 2\pi$.

$$4\sin^3 x + 4\sin^2 x - 3\sin x - 3 = 0$$

Q2. (non-calculator) Solve the equation for $-180^\circ \leq x \leq 360^\circ$.

$$2\sin^2 x + 3\sqrt{3}\sin x - 6 = 0$$

Q3. (calculator) Solve the equation for $0 \leq x \leq 2\pi$. Give answers in terms of π whenever possible or correct to 3 s.f. otherwise.

$$6\sin^2 x + 5\sin x - 6 = 0$$

Q4. (non-calculator) Solve the equation for $-180^\circ \leq x \leq 360^\circ$.

$$2\cos^2 x + \cos x - 1 = 0$$

Q5. (non-calculator) Solve the equation for $-2\pi \leq x \leq \pi$.

$$2\cos^2 x - \cos x - 1 = 0$$

Q6. (calculator) Solve the equation for $0 \leq x \leq 2\pi$. Give answers in terms of π whenever possible or correct to 3 s.f. otherwise.

$$4\cos^2 x + 5\cos x + 1 = 0$$

Q7. (non-calculator) Solve the equation for $0 \leq x \leq 2\pi$.

$$3\tan^2 x - 1 = 0$$

Q8. (calculator) Solve the equation for $0 \leq x \leq 2\pi$. Give answers in terms of π whenever possible or correct to 3 s.f. otherwise.

$$\sqrt{3}\tan^2 x - 8\tan x - 3\sqrt{3} = 0$$

Q9. (non-calculator) Express $\cos x$ in terms of $\sin x$ and hence solve the equation for $-\pi < x \leq \pi$.

$$2\cos^2 x + 7\sin x + 1 = 2\sin x$$

Q10. (calculator) Express $\sin x$ in terms of $\cos x$ and hence solve the equation for $-360° < x \leq 0°$.

$$10\sin^2 x + \cos x = 8$$

Q11. (non-calculator) A function f is defined by

$$f(x) = \sqrt{3}\cos x - \sin x$$

Points $A = (\frac{5\pi}{6}, a)$ and $B = (b, 0)$ lie on the graph of f.

(i) Find the value of a.

(ii) **(*)** Find the value of b knowing that $0 \leq b \leq \frac{\pi}{2}$.

Q12. (non-calculator) Find the values of all other basic trigonometric ratios (sine, cosine, tangent) in the domain given.

(1) $\sin A = \frac{2}{3}$, $\frac{\pi}{2} < A < \frac{3\pi}{2}$

(2) $\sin A = -\frac{\sqrt{11}}{6}$, $-\frac{\pi}{2} < A < \frac{\pi}{2}$

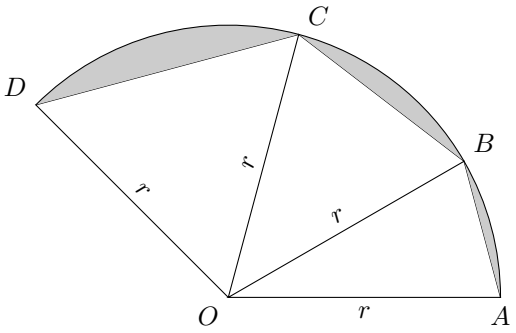
(3) $\cos A = \frac{1}{3}$, $-\pi < A < 0$

(4) $\cos A = -0.4$, $-\frac{\pi}{2} < A < \pi$

(5) $\tan A = \sqrt{2}$, $-\pi < A < 0$

(6) $\tan A = -\sqrt{2}$, $0 < A < \pi$

Q13. The diagram shows theree sectors of a circle with centre O and radius $r = 4cm$. Points A, B, C and D are on the circumference and the angles $A\hat{O}B$, $B\hat{O}C$ and $C\hat{O}D$ are 30°, 45° and 60° respectively.



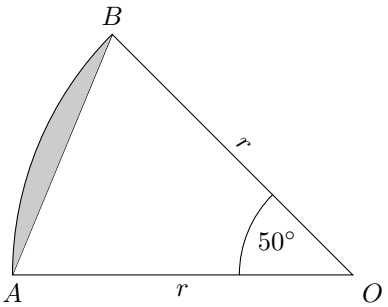
(i) (paper 1) Find the area of the shaded shape consisting of three segments.

(ii) (paper 1) Find the perimeter of the sector $OABCD$.

(iii) (paper 2) Find the perimeter of the shaded shape.

Q14. (calculator)

The diagram shows a sector of a circle with centre O. Points A and B are on the circumference and the angle $B\hat{O}A$ is 50°.
The area of the shaded segment is 2.4.



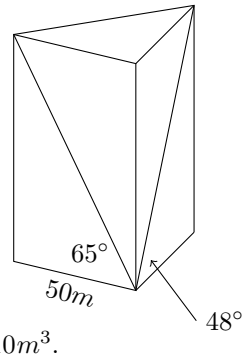
(i) Find the length of r, the radius of the circle.

(ii) Find the perimeter of the sector OAB.

(iii) Find the perimeter of the shaded segment.

Q15. (calculator)

A building is in the shape of a triangular prism. Its base is a scalene triangle (i.e. all sides have different lengths). All rectangular faces are perpendicular to the base. One of the sides of the base is 50 m long. The angle that the diagonal of this face makes with the ground level is 65°.
The angle that the diagonal of the second face makes with the ground level is 48°.
The angle that the diagonal of the third face (not shown in the diagram) makes with the ground level is 42°.

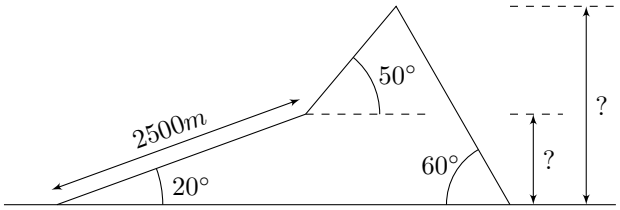


(i) Find the height of the building.

(ii) Find the perimeter of the base.

(iii) Find the capacity (the volume) of the building giving your answer correct to $10m^3$.

Q16. (calculator)



A tourist made a trip up and down a hill. His route consisted of three phases. The first two were ascents up the hill and the last one was decent down the hill as shown in the diagram. The first ascent was at an angle of 20° from the horizontal and it was 2500 m long. This second was at an angle of 50° from the horizontal. The decent was at an angle of 60° from the horizontal. The final position of a tourist is on the same altitude as the initial one. The whole route was 6 kilometers long.

(i) Find the altitude of the endpoint of the first ascent (measured vertically from the initial point).

(ii) Find the height of the hill (measured vertically from the initial point).

Q17. (calculator) In triangle ABC the length of side BC is 8.2. The angle $A\hat{B}C$ is 57°. The area of the triangle is 22. Find the lengths of the sides AB and AC.

Q18. (calculator) In triangle ABC the lengths of sides BC and AB are 16.1 and 11.8 respectively. The angle C is 47°. Find the measure of the angle B.

Q19. (calculator) In an isosceles triangle the longest side is 20% longer than any of the two other equal sides. Find the angles in the triangle.

Q20. (calculator) A man walked 8 km on a bearing of 237°. Then he changed the direction and walked 6 km on a bearing of 348°.

(i) What is his distance from the initial position?

(ii) On what bearing is the initial position?

Chapter 7

Statistics

*All numerical answers in this chapter must be given **exactly** or correct to **3 s.f.** (three significant figures).*

7.1 Types of data

Q1. Are the following data qualitative or quantitative?

(1) Colors of cars in the street.
(2) Number of cars passing a point in a street in a given time period.
(3) Heights of students.
(4) Shoe sizes of girls in a school.
(5) Weights of boys in a school.
(6) Winners of 3000m run in Olimpic Games.
(7) The world records in 3000m run over last 50 years.
(8) Usage share of a certain web browser.
(9) Types of web browsers.
(10) Average speeds of cars.
(11) Areas of a randomly marked plot of land.
(12) Lengths of randomly cut pieces of string / rope.
(13) Amount of water in a number of cups.

Q2. Are the following data discrete or continuous?

(1) Number of cars passing a point in a street in a given time period.
(2) Heights of students.
(3) Shoe sizes of girls in a school.
(4) Weights of boys in a school.
(5) The world records in 3000m run over last 50 years.
(6) Usage share of a certain web browser.
(7) Average speeds of cars.
(8) Areas of a randomly marked plot of land.

(9) Lengths of randomly cut pieces of string / rope.
(10) Amount of water in a number of cups.
(11) The number of goals scored by football teams on Wednesday.
(12) The weights of sugar in 1kg packs.
(13) The numbers of chairs in classrooms.
(14) The number of trees in the woods.
(15) The heights of trees in a wood.
(16) The sizes of ladies dresses in a store.
(17) Air temperature.

7.2 Averages, range, quartiles

Q3. Without a graphic calculator find:

(i) the mean
(ii) the mode
(iii) the median
(iv) the lower quartile
(v) the upper quartile
(vi) the range
(vii) the interquartile range (I.Q.R.)

for the following sets of data.

(1) $140, 141, 141, 141, 142, 143, 145$
(2) $0.995, 0.995, 0.998, 1.002, 1.003, 1.005$
(3) $7, 7, 8, 9, 9.5, 10.5$
(4) $998, 998, 998, 999, 1000, 1000, 1002, 1003$
(5) $1, 2, 2, 3, 4, 4, 4, 6$
(6) $72, 78, 81, 83, 86, 87, 90, 90$
(7) $0, 0, 0, 1, 2, 2, 3, 3, 3, 3, 3$
(8) $1, 1, 2, 2, 3, 3, 3, 4, 5, 7$

Q4. Each of the sets of data in question 3 describes one of the following situations. Find the set that fits each description.

(1) number of times Tina was late for her school bus in a few consecutive weeks
(2) shoe sizes
(3) heights of boys in a middle school class
(4) scores from a fair 6-sided die
(5) end of term grades in a group of IB students
(6) areas (in square metres) of pieces of wood that are supposed to be approximately $1m^2$
(7) weights in grams of 1kg packs of sugar
(8) weights of adult men

Q5. The mean height of 8 students is 169cm. A boy 173cm high joins the group. What is the new mean height?

Q6. The mean weight of 7 students is 63kg. A girl that weighs 52kg joins the group. What is the new mean weight?

Q7. The mean height of 8 basketball players is 186cm. Two more players, 188cm and 196cm high joined the team. What is the new mean height?

Q8. The mean height of 10 students is 169cm. A boy 173cm high leaves the group. What is the new mean height?

Q9. The mean weight of 11 students is 63kg. A girl that weighs 52kg leaves the group. What is the new mean weight?

Q10. The mean number of matches in 6 boxes is 56.5. What would be the mean number of matches in the 6 boxes if:

(1) one match was taken out of each box?
(2) two matches were taken out of two boxes (i.e. two from each) and to each of the other four boxes one match was added?

(3) all 59 matches from one of the boxes were used?

Q11. The mean record time in 1km run of a team of four athletes is 2:24 (minutes: seconds). One of the team members, whose record time was 2:28 was replaced by a new one, whose record time was 2:20. What is the new mean record time of the team?

Q12. The mean height of 8 girls is 154.5cm while the mean height of 12 boys is 166cm. What is the mean height of all 20 children?

Q13. The mean weight of students in a group of 24 is 61.5kg. The mean weight of 14 girls in the group is 54.5kg. What is the mean weight of the boys?

Q14. For the set of numbers:

165 187 174 169 170 182 168 173 166 171

find

(1) the mean (2) the median (3) the range

Q15. Find all the averages possible in all cases. Which one is the most sensible? Why?

(1) marks (out of 100) obtained in a mock exam:
11, 18, 79, 82, 82, 83, 84, 84, 85, 86, 88, 88, 89

(2) times (in seconds) to complete a simple puzzle:
10, 12, 18, 19, 20, 20, 20, 20, 21, 21, 23, 25, 25, 25, 26, 28, 55, 134, 195, 215

(3) types of shoes that people on a bus wear:
25 - trainers,
11 - sandals,
9 - high-heeles

Q16. For the numbers:

61	62	64	65	67
68	69	70	72	72
76	79	80	88	90

find:

(1) the median
(2) the lower quartile
(3) the upper quartile.

Q17. In the following set of data:

164 ? 172 164 164 ? 172 166 172 174

2 numbers are missing.

The mode is 164 while the mean is 169.

Find the two missing numbers.

Q18. Find the missing *natural* numbers in the following sets of data.

(1) $3 \leq a \leq b \leq 5 \leq 6$, median= 5, $Q_1 = 3$
(2) $5 \leq a \leq b \leq 8 \leq 10$, median= 8, $Q_1 = 6$
(3) $a \leq 5 \leq 7 \leq 7 \leq 8 \leq b$, range= 6, mean= 6.5
(4) $1 \leq 2 \leq a \leq b \leq 8 \leq 10$, median= 4.5, no mode
(5) $11 \leq a < b < c \leq d \leq e$, median= 15, $Q_1 = 13$, $Q_3 = 17$, range= 8

Q19. The stem and leaf diagram below shows prices of a can of a refreshing drink in different shops.

1.9	5 5 5 9 9 9 9 9	(8)
2.0	0 5 5 9 9	(5)
2.2	5 9 9	(3)
2.4	9 9 9 9	(4)

where

2.4	9	denotes 2.49zł.

Find the mode, the median and the mean price.

Q20. The stem and leaf diagram below shows results of a class in two tests. The results are in $0 - 100$ scale.

	test 1		test 2	
(4)	6 5 1 1	2		(0)
(3)	9 6 3	3	2	(1)
(2)	4 1	4	3 6 6	(3)
(0)		5	1 8	(2)
(2)	8 4	6	2 4 7	(3)
(3)	7 6 4	7	4 5 9	(3)
(2)	5 3	8	1 2 7	(3)
(1)	2	9	3 4	(2)

Find:
(1) range
(2) mode
(3) median
(4) quartiles
(5) I.Q.R.
(6) mean
for both tests.

7.3 Grouped data, frequencies

Q21. Complete the table and hence find the mean height of students.

height (h)	frequency (f)	$h \times f$
152	4	
153	2	
157	3	
160	3	
163	6	
165	3	
168	3	

Q22. The table shows the number of siblings in 40 families. The median number is 1.5. Find the values of a and b.

no of siblings	0	1	2	3	4
frequency	8	a	10	6	b

Q23. A standard, six-sided die was rolled several times. A number of dots seen at the top was noted. How many times a "4" was seen if the mean number is 3.6?

no of dots	1	2	3	4	5	6
frequency	4	6	3	x	7	4

Q24. In a block of flats there are flats that have from 1 to 5 rooms, as shown in the table.

no of rooms	1	2	3	4	5
frequency	12	16	x	18	9

How many 3-room flats are there if the mean number of rooms is 2.95?

Q25. The table shows shoe sizes of a team of 20 rugby players. The mode size is 11. Find the values of a and b and the median size.

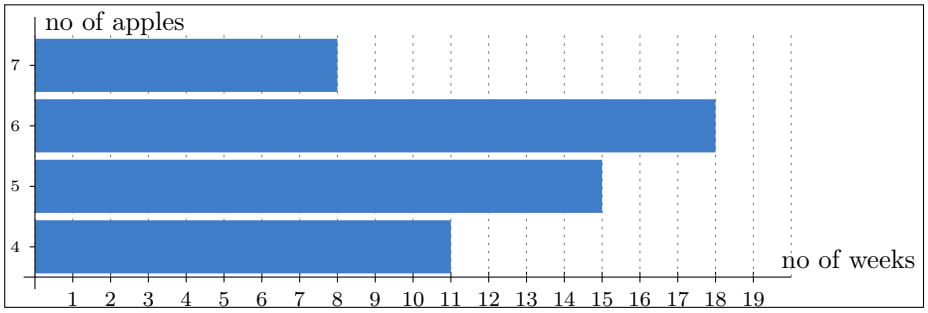
shoe size	9	10	11	12
no of players	4	a	6	b

Q26. The table shows the number of customers of a shop in the first 5 minutes from opening noted on a number of days.

no of customers	1	2	3	5	6	7
no of days	2	a	b	7	4	4

Knowing that the median number of customers is 4 and that the mean number of customers is 4.1 find the values of a and b.

Q27. A grandma buys 1kg of apples every week. As the fruits differ in size, she buys different number of apples that weigh 1kg. The diagram below shows the number of apples she bought in 52 consecutive weeks.

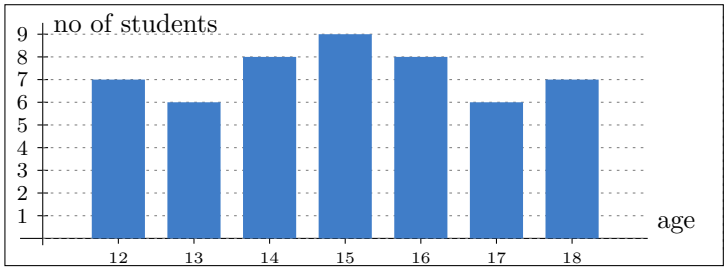


Find:

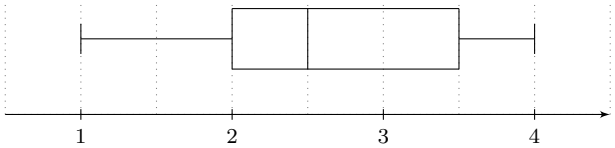
(1) range
(2) mode
(3) median
(4) quartiles
(5) I.Q.R.
(6) mean

Q28. Using the diagram below find:

(1) range
(2) mode
(3) median
(4) quartiles
(5) I.Q.R.
(6) mean



Q29. A fair 4-sided die with faces numbered 1 to 4 was rolled a number of times. The resuls are shown in the box plot and in the table below. Find the values of a and b.



result	1	2	3	4
frequency	8	10	a	b

Q30. Consider the heights of student shown below:

139	140	141	142	143	144	145	146	147	147
148	148	149	149	150	150	151	151	152	152
153	153	153	154	154	154	155	155	155	156
156	156	157	157	157	158	158	158	158	159
159	159	159	160	160	160	160	161	161	161
161	162	162	162	162	163	163	163	163	164
164	164	165	165	165	166	166	166	167	167
167	168	168	168	169	169	169	170	170	170
171	171	172	172	173	173	174	174	175	175
176	176	177	178	179	180	181	182	183	184

(1) Complete the tables and use them to answer the questions below.

class	f	h (approx.)	$h \times f$
$130 < h \leq 140$			
$140 < h \leq 150$			
$150 < h \leq 160$			
$160 < h \leq 170$			
$170 < h \leq 180$			
$180 < h \leq 190$			

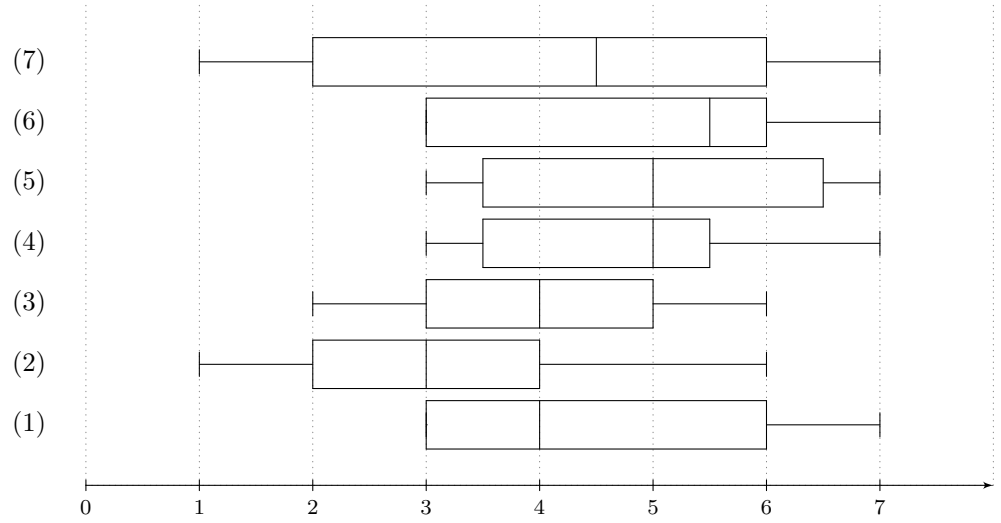
h	$c.f.$
$h \leq 140$	
$h \leq 150$	
$h \leq 160$	
$h \leq 170$	
$h \leq 180$	
$h \leq 190$	

(2) Write down the class with the median height.

(3) Locate the classes with quartiles.

(4) Estimate the mean height.

Q31. The box plots below show the distribution of IB grades in 7 groups of students. The IB grades are integers from 1 to 7.



(1) In which groups the number of students must be even?

(2) Which of the groups has the largest range of results? What is the range?

(3) Which of the groups has the largest interquartile range of results? What is the I.Q.R.?

(4) In which group the top 25% of students got the best results?

(5) In which group the better half of students got the best results?

(6) If there were 16 students in group (7):

(a) what would be the maximum number of students with 6's?

(b) what would be the minimum number of students with 6's?

(c) what would be the maximum number of students with 7's?

(d) what would be the minimum number of students with 7's?

(7) Can the number of students in group (5) be 7?

(8) Can the number of students in group (5) be 8?

(9) If there were 12 students in group (7):

(a) what would be the maximum number of students with 6's?

(b) what would be the minimum number of students with 6's?

(c) what would be the maximum number of students with 7's?

(d) what would be the minimum number of students with 7's?

Q32. A hundred children in school A and a hundred children in school B were asked, how many sandwiches they ate in a lunch break. The table below shows the results of the survey.

number of sandwiches (x)	number of children (f)	cumulative frequency
0	6	6
1	21	27
2	41	
3	17	
4	9	
5	6	

number of sandwiches (x)	number of children (f)	cumulative frequency
0	8	
1	17	
2	34	
3	16	
4	13	
5	12	

(1) Find the mean numbers of sandwiches in both schools.

(2) Write down the mode for both schools.

(3) Complete cumulative frequencies in the table above.

(4) Hence complete the table below:

	A	B
median		
Q_1		
Q_3		

(5) Draw box plots for both schools.

(6) Looking at the box plots only and not refering to the original data state whether the following statements are true. Give separate answers for schools A and B.

(a) At least 50% of children ate not more than 2 sandwiches.

(b) At least 50% of children ate more than 2 sandwiches.

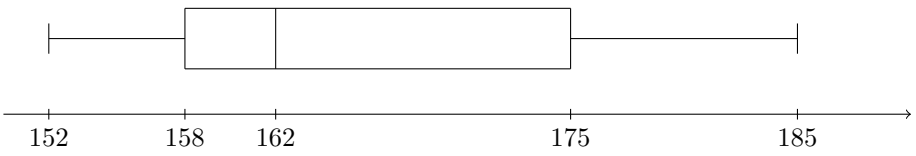
(c) A child in top 25% ate more than 3 sandwiches.

(d) All children in top 25% ate more than 3 sandwiches.

(e) A child in the bottom 25% ate less than 2 sandwiches.

(f) A child in middle 50% ate 2 or 3 sandwiches.

Q33. The box plot below shows the spread of heights of a group of students.



Looking at the diagram above state whether the following statements are true or false:

(a) At least 50% of the students are higher than 162cm.

(b) Exactly 50% of the students are higher than 162cm.

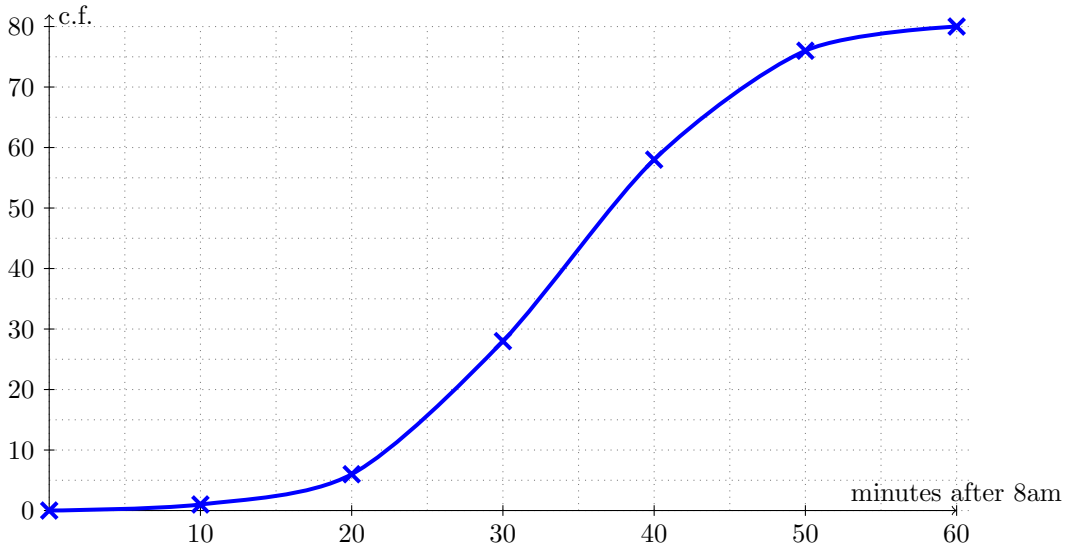
(c) At least 50% of the students are shorter than 162cm.

(d) There are more students in a group of 25% highest than in a group of 25% shortest students.

Q34. A school's doorman decided to collect the data on when pupils come to school. He started his observations at 8am and put a tick in his notebook every time a pupil entered the school building. Every 10 minutes the doorman counted the ticks. The table below shows the numbers he got.

minutes after 8am	number of ticks
10	1
20	6
30	28
40	58
50	76
60	80

The data can be shown in the followig diagram.



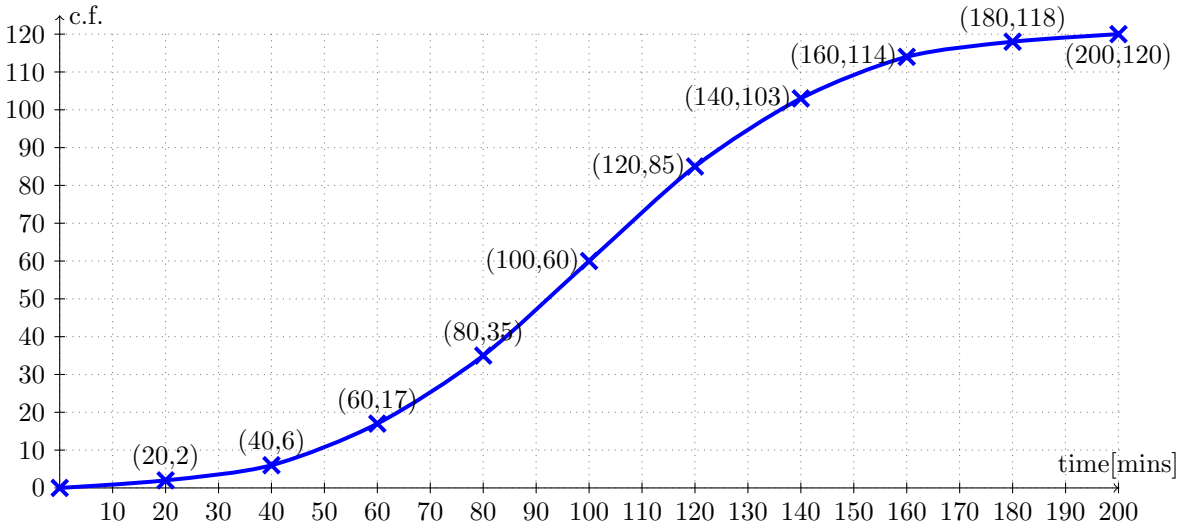
Assuming that the lessons start at 8.45am:

(1) how many students were more than 5 minutes late?

(2) how many students arrived to school before the beginning of the lessons?

(3) how many students arrived to school between 8.25 and 8.35am?

Q35. The cumulative frequency curve shows how much time students spent logged in a popular social networking service on a single chosen day.



(1) Estimate the interquartile range.

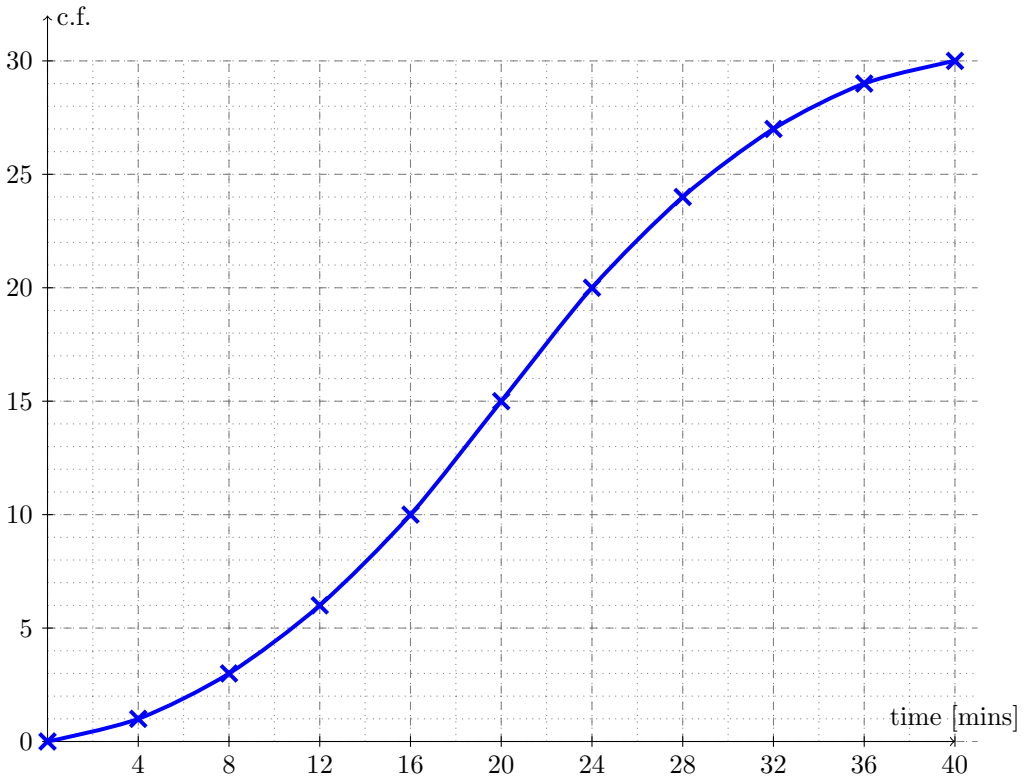
(2) How many students spent an hour or less logged in?

(3) How many students spent more than 2 hours on the social network?
(4) How many students spent between one and two hours logged in?
(5) Estimate the 30^{th} and the 70^{th} percentiles.
(6) Estimate the number of students who spent less than 50 minutes on the social network.
(7) The top[1] 30 students spent more than a minutes logged in. What is the value of a?
(8) The bottom 30 students spent not more than b minutes logged in. Find the value of b?
(9) What is the percentage of students who spent between a and b minutes online?
(10) Copy and complete the frequency table below.

time	no of students
$0 < t \leq 20$	2
$20 < t \leq 40$	4
$\vdots$	$\vdots$
$180 < t \leq 200$	

(11) Estimate the mean time.

Q36. A teacher asked the students to write a short essay. He gave them 40 minutes for the task. Majority of the students finished writing earlier, which is shown in the cumulative frequency curve below.

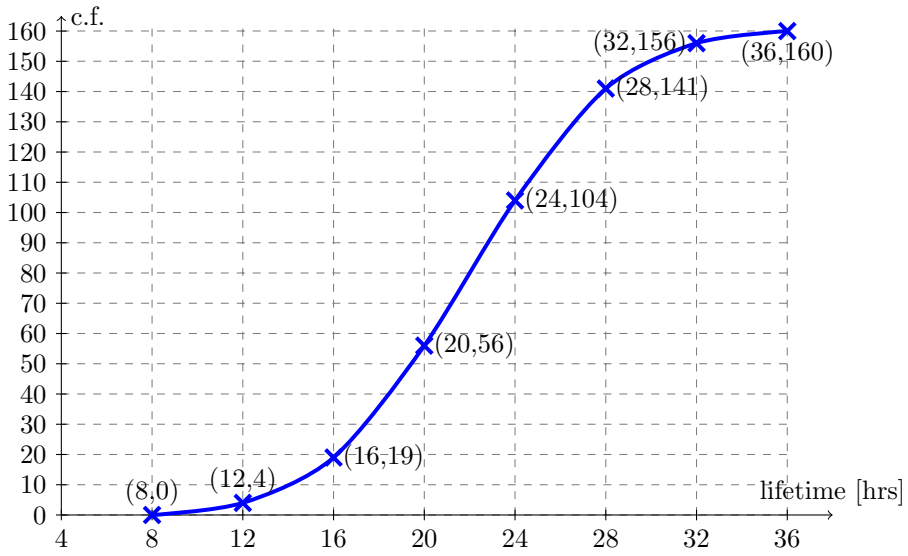


(1) What is the median time?
(2) How many students needed between 20 and 30 minutes to complete the task?
(3) 20 students needed less than x minutes to finish writing. What is the value of x?
(4) 20 students needed more than y minutes to finish writing. What is the value of y?
(5) Copy and complete the following frequency table. Hence estimate the mean time.

time	no of students
$0 < t \leq 4$	1
$4 < t \leq 8$	2
$\vdots$	$\vdots$

[1] "top" means with longest time

Q37. A firm tested a sample of batteries. The cumulative frequency graph shows the data on lifetime of batteries collected.



(1) Estimate the median lifetime.

(2) Estimate the interquartile range.

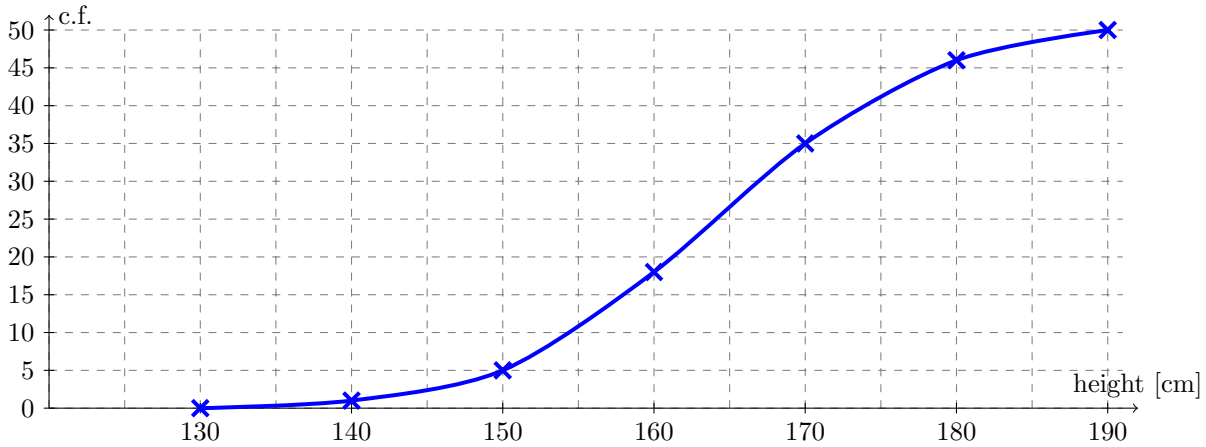
(3) Estimate the 15^{th} and the 65^{th} percentiles.

(4) The top (i.e. best) 12.5% of batteries live longer than a hours. Find the value of a.

(5) The bottom (i.e. worst) 12.5% of batteries live not longer than b hours. Find the value of b.

(6) The middle 75% of batteries live between c and d hours. Find the values of c and d.

Q38. The heights of a number of men is shown in the cumulative frequency graph.



(1) Estimate the median height.

(2) Estimate the interquartile range.

(3) The top 20% of men are higher than a cm. Find the value of a.

(4) The bottom 20% of men are shorter than b cm. Find the value of b.

(5) The middle 60% of men are between c and d cm. Find the values of c and d.

7.4 Miscellaneous problems

Q39. The mean height of 20 students before the summer break was 168 centimetres. The range of heights was 12 cm. What were the mean height and the range ater the summer break if:

(1) each student grew by 3 cm,

(2) 10 students grew by 3 cm each and the other 10 by 5 cm each,

(3) 15 students grew by 3 cm each and the other 5 by 5 cm each.

Q40. 160 cakes were baked in a bakery. The mean volume of the cakes before they were placed in the oven was 480 ml. After the cakes were taken out of the oven it turned out that each of them was twice larger. What was the mean volume then?

Q41. Water was poured into 8 buckets. They contained $6.5l$ on average. How would the mean volume change if:

(1) half of water was poured out of each bucket,

(2) 2 litres were poured out of each bucket.

Q42. Eight kids kept some pocket money in their pockets. They had 26.50 zł on average. How would the mean change if:

(1) two boys joined the group, one with 30 zł in his pocket and the other with 22 zł,

(2) one kid bought a sandwich for 12 złand the other a drink for 8 zł?

Q43. Let $a \leq b \leq c \leq d$ be integers. Find the numbers knowing that the mode is 7, the median is 7, the range is 4 and the mean is 7.5.

Q44. Let $a \leq b \leq c \leq d$ be integers. Find the numbers knowing that the mode is 8, the median is 7 and the mean is 6.5.

Q45. Let $a \leq b \leq c \leq d \leq e \leq f \leq g$. Find the numbers knowing that $Q_1 = 8$ and so is the median, there is no mode, $Q_3 = 9$ and the range is 3.

Q46. The following table shows the shoe sizes of 30 people. The mode size is 9. Find the values of x and y.

size x	frequency f
8	4
8.5	5
9	x
9.5	7
10	y

Q47. The following table shows the shoe sizes of a group of people. The mean size is 7.5. Find the value of x.

size x	frequency f
6.5	2
7	7
7.5	8
8	x
8.5	4

Q48. The stem and leaf diagram below shows results in 60 metres run.

8	8 8 9 9 9	(5)
9	2 2 2 2 3 4 4 4 6 6 7 8 8 8 9 9 9	(17)
10	0 0 1	(3)

where

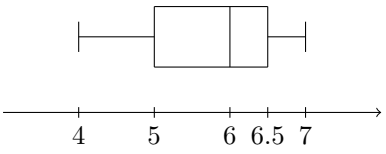
9	1	denotes 9.1 seconds.

(1) Copy and complete the following frequency table.

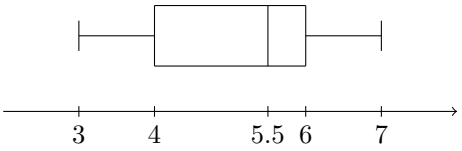
time	frequency
8.8	
8.9	
⋮	

(2) Find the median, the quartiles, range and interquartile range, mode and mean.

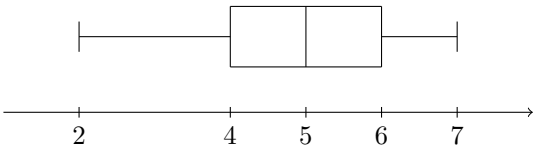
Q49. IB grades are integers from 1 to 7. The box-plot below shows IB grades of 5 students. Find the grades.



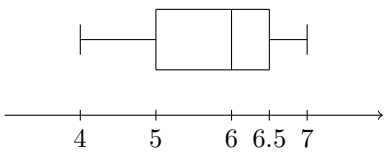
Q50. IB grades are integers from 1 to 7. The box-plot below shows IB grades of 6 students. Find the grades.



Q51. IB grades are integers from 1 to 7. The box-plot below shows IB grades of 7 students. The mean grade is 5. Find the grades.

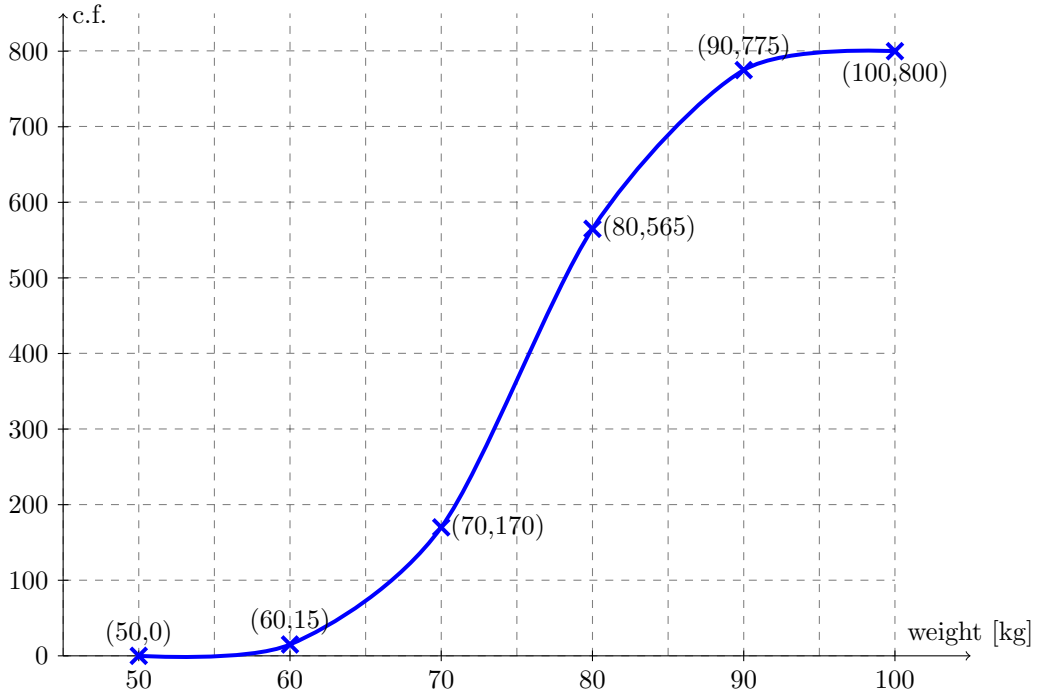


Q52. IB grades are integers from 1 to 7. The box-plot below shows IB grades of 9 students. Fill in the table with the possible number of grades.



grade	min no	max no
4		
5		
6		
7		

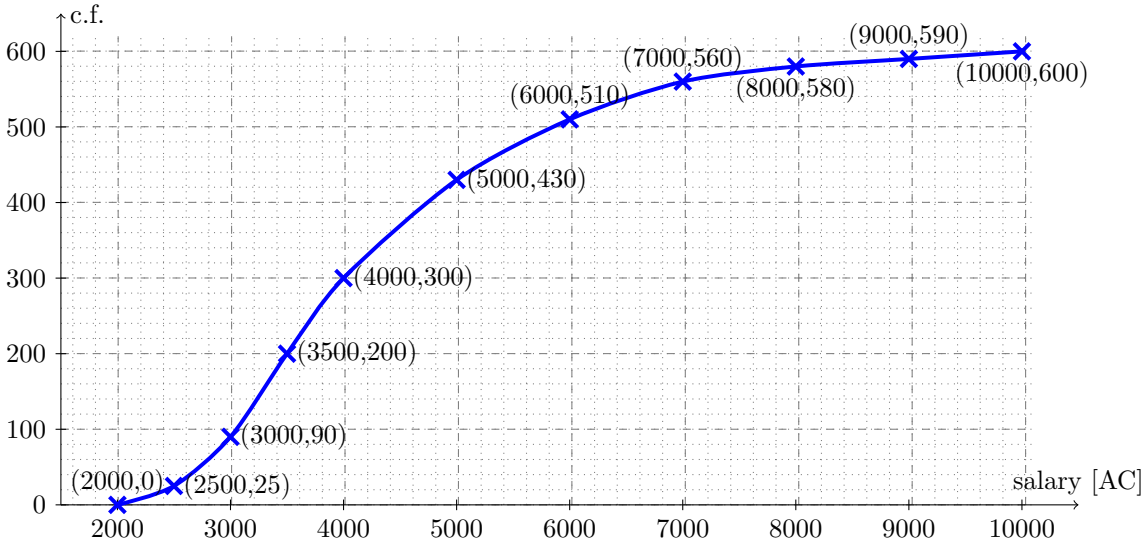
Q53. The cumulative frequency graph shows the data on weights of 800 people.



(1) Estimate the median weight.
(2) Estimate the interquartile range.
(3) Estimate the 30^{th} and the 70^{th} percentiles.
(4) The top (i.e. heaviest) 12.5% of the group weigh more than a kg. Find the value of a.
(5) The bottom (i.e. lightest) 12.5% of the group weigh less than b kg. Find the value of b.
(6) The middle 75% of the group weigh between c and d kilograms. Find the values of c and d.
(7) Fill in the fraquency table below and use it to estimate the mean weight.

weight $w[kg]$	frequency
$50 < w \leq 60$	
$60 < w \leq 70$	
$70 < w \leq 80$	
$80 < w \leq 90$	
$90 < w \leq 100$	

Q54. The cumulative frequency graph shows the data on monthly earnings (in AC - artificial currency) of 600 employees of a huge company.



(1) Estimate the median salary.
(2) Estimate the interquartile range.
(3) Estimate the 30^{th} and the 70^{th} percentiles.
(4) The top (i.e. with highest salaries) 10% of the group earn more than a AC. Find the value of a.
(5) The bottom (i.e. with lowest salaries) 10% of the group earn less than b AC. Find the value of b.
(6) The middle 80% of the group earn between c and d AC. Find the values of c and d.
(7) Fill in the fraquency table below and use it to estimate the mean salary.

sallary $s[AC]$	frequency
$2000 < s \leq 3000$	
$3000 < s \leq 4000$	
$4000 < s \leq 5000$	
$5000 < s \leq 6000$	
$6000 < s \leq 7000$	
$7000 < s \leq 8000$	
$8000 < s \leq 9000$	
$9000 < s \leq 10000$	

Chapter 8

Geometry

All answers must be given exactly or correct to 3 significant figures.

8.1 Polygons

Q1. Which of the following statements are true and which are false?

(1) Every square is a rectangle.
(2) Every square is a parallelogram.
(3) Every square is a rhombus.
(4) Every parallelogram is a rhombus.
(5) Every rectangle is a parallelogram.
(6) Every rectangle is a rhombus.
(7) Every parallelogram is a rectangle.
(8) Every rhombus is a rectangle.
(9) Every parallelogram is a trapezium.
(10) A circle can be inscribed in every isosceles trapezium.
(11) A circle can be circumscribed on every isosceles trapezium.
(12) A circle can be circumscribed on no parallelogram.
(13) If diagonals of a quadrilateral intersect at a right angle then it is a rhombus.
(14) If diagonals of a parallelogram intersect at a right angle then it is a rhombus.

Q2. Perimeter of a rhombus is 20 centimetres. One of its diagonals is twice the other. Find the area of a rhombus.

Q3. Find the area of a trapezium with lengths of consecutive sides equal to 15, 16, 15 and 40 centimetres.

Q4. Find the area of an equilateral triangle with height $4\sqrt{6}$.

Q5. Diagonals of an isosceles trapezium intersect at a right angle. Knowing that the height of the trapezium is 8 centimetres find its area.

8.2 Circles

Q6. Triangle ABC is inscribed in a circle. Tangents to the circle at points A and B intersect at point S. Angle $A\hat{S}B$ is 46°. Evaluate the angle $A\hat{C}B$.

Q7. An isosceles trapezium with perimeter 52 is circumscribed on a circle with radius 6. Find the area of the trapezium.

Q8. Find the area of the equilateral triangle circumscribed on a circle with radius 3 centimetres.

Q9. One of the diagonals of a parallelogram is equal to one of the sides. A circle with radius 3 is inscribed in the parallelogram. Find the area of the parallelogram.

Q10. The length of side of a rhombus is 5.2 decimetres. The ratio of lengths of the diagonals of the rhombus is $12 : 5$. The diagonals cut the rhombus in four triangles. A circle is inscribed in each one of them. Find the area of a quadrilateral whose vertices are centres of the circles.

Q11. A quadrilateral $ABCD$ is inscribed in a circle with radius 6. Diagonal AC bisects the angle $B\hat{C}D$. The angle $B\hat{A}D$ is $60°$ and one of the other angles in the quadrilateral is a right angle. Find the area of the quadrilateral.

Q12. A quadrilateral $DEFG$ is inscribed in a circle. The diagonal EG makes an angle of $33°$ with side DG and an angle of $80°$ with a tangent to the circle at point G. Find the size of angles $E\hat{F}G$ and $E\hat{D}G$. Consider all possibilities.

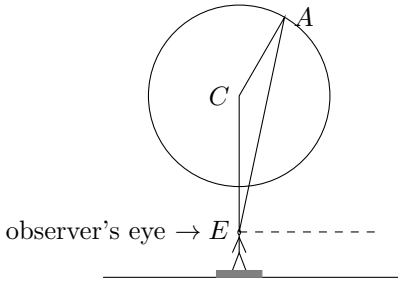
Q13. Triangle ABC is inscribed in a circle. Angle $A\hat{C}B$ is $73°$. What is the angle between the tangents to the circle at points A and B?

Q14. Triangle ABC is inscribed in a circle. The tangents to the circle at points A and B intersect at point P. Angle $B\hat{A}P$ is $58°$. Evaluate the angles in the triangle if the bisector of angle $B\hat{A}C$ is perpendicular to side BC.

Q15. A circle is circumscribed on a right angle triangle ABC (angle at C is the right angle). A tangent to the circle at C intersects a line AB at a point which is closer to B than to A. The angle at which the lines intersect is $28°$. Evaluate the angles in the triangle.

Q16. Evaluate the area of a circle inscribed in an isosceles trapezium knowing that the trapeziums perimeter is 60 centimetres and that its non-parallel side is 2.5 times longer than the shortest side.

Q17. The centre C of a Ferris wheel is located 8 metres above the ground level. Its radius is 4 metres. The Observer's eye E is placed directly below the centre of the wheel and 2 metres above the ground level. When point A placed on the wheel is 11.5 metres above the ground the Observer watches the point. Find the size of the angle that the Observer's line of sight makes with the horizontal.



Q18. An acute angle in a parallelogram is $45°$. The parallelogram is circumscribed on a circle with radius 4. Find the area of the parallelogram.

Q19. An acute angle in a parallelogram is $60°$. The parallelogram is circumscribed on a circle with area $9\pi cm^2$. Find the area of the parallelogram.

Q20. An acute angle in a parallelogram is $30°$. The parallelogram is circumscribed on a circle with circumference $3\pi cm$. Find the area of the parallelogram.

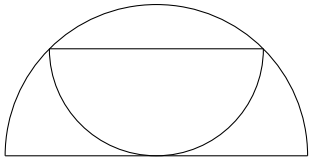
Q21. An angle in an isosceles trapezium is $45°$. The trapezium is circumscribed on a circle with radius $\sqrt{2}cm$. Find the area of the trapezium.

Q22. An angle in an isosceles trapezium is $60°$. The trapezium is circumscribed on a circle with radius $2\sqrt{3}cm$. Find the area of the trapezium.

Q23. An acute angle in a right trapezium is $30°$. The trapezium is circumscribed on a circle with radius $4cm$. Find the area of the trapezium.

Q24. A trapezium is inscribed in a circle with radius 6. One of the trapeziums bases is the circles diameter while the length of the other is 6. Find the area of the trapezium.

Q25. A semicircle is inscribed in the other semicircle in such a way that two diameters are parallel and have no common points (see the diagram). Find the ratio of areas of the larger semicircle to the smaller one.



Q26. Points A and B are points on a circle such that AB is a diameter. C is a point on line AB outside the circle. The distance from C to B is equal to the circles radius. A line through C is tangent to the circle at point S. Find the angles in triangle ABS.

Q27. *[Alternate segment theorem]* Consider a triangle and its circumcircle. A tangent is drawn at one of the vertices of the triangle. Prove that the angle between the tangent and chord (side of the triangle that touches the tangent) equals the angle in triangle in the alternate segment.

8.3 Similarity

Q28. (1) Quadrilateral $ABCD$ is inscribed in a circle. Its diagonals AC and BD intersect at point P. Knowing that $AP = 3$, $BP = 4$ and $CP = 6$ find the lenght DP.

(2) Point P lies inside circle C. A secant AB of the circle is drawn through point P. Prove that the product $AP \times BP$ is independent of the choice of secant.

Q29. In an obtuse-angled triangle ABC the lengths AC and BC are 13 and 15 respectively. The height CD of a triangle is 12. A line parallel to AB is drawn in the distance of 4 from AB. The line intersects sides BC and AC at points E and F respectively. Find the area of the trapezium $ABEF$.

Q30. Points A and B are points on a circle such that AB is a diameter. C is a point on line AB outside the circle. The distance from C to B is equal to the circles diameter. A line through C is tangent to the circle at point S. Find the exact length AS in terms of r, the radius of the circle.

Q31. Points P and Q are points on a semicircle such that PQ is a diameter. A and B are points on line PQ outside the semicircle. The lengths PA and QB are both equal to r, the semicircles radius. Lines tangent to the semicircle through A and through B intersect at point C. Find the exact height CD of triangle ABCin terms of r.

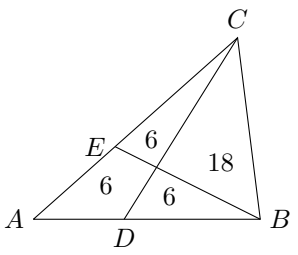
Q32. (1) Point P lies outside circle C. A tangent to C through P meets C at point S. A secant of C through P meets C at points A and B. Prove that $AP \times BP = PS^2$.

(2) Point P lies outside circle C. A secant AB of the circle is drawn through point P. Prove that the product $AP \times BP$ is independent of the choice of secant.

Q33. In a trapezium $ABCD$ (AB parallel to CD) the diagonals intersect at point P at a right angle. The lines BC and AD intersect at point S. Knowing that $AC = 9.9$, $DP = 2.2$, $AP = 6.6$ and $BS = 11$ find the area of triangle ABS.

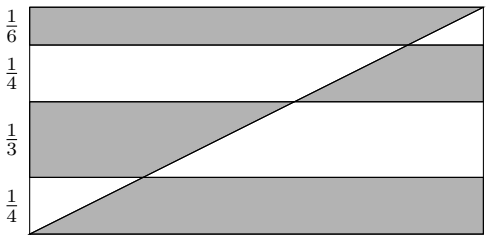
Q34. In a convex quadrilateral midpoints of sides are vertices of the second quadrilateral. The midpoints of sides of this second quadrilateral are vertices of the third quadrilateral. Find the ratio of areas of the first, largest quadrilateral to the third, smallest one.

Q35. The diagram shows triangle ABC. Points D and E are on sides AB and AC respectively. Line segments BE and CD split the triangle into four regions with areas show.
Find the area of triangle ADE.



Q36. In a right angled triangle the sides adjacent to the right angle are 8 and 16 centimetres long. A square is inscribed in the triangle in such a way that two of its sides are on the sides adjacent to the right angle and one of the vertices is on a hypotenuse. Find the length of a side of the square.

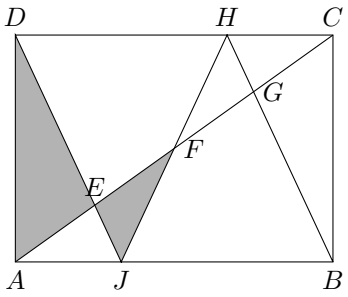
Q37. The diagram shows a rectangle with height 1. The rectangle is split into four smaller rectangles with heights shown. The rectangles are cut with the diagonal of the largest rectangle and some of the regions are shaded. Is the area of two shaded regions above the diagonal larger, smaller or equal to the area of the two shaded regions below the diagonal? Justify your answer.



Q38. The diagram show a rectangle $ABCD$. Points H and J are such points on sides CD and AB respectively that:

$$AJ = CH = \frac{1}{3}AB$$

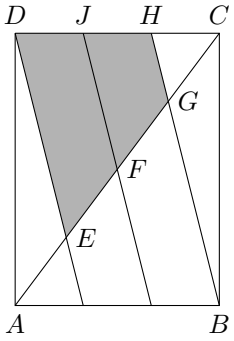
Points E, F and G are intersections of the diagonal AC with segments DJ, HJ and BH respectively.
What is the ratio of areas of triangles AED to EFJ?



Q39. The diagram show a rectangle $ABCD$ with sides $AB = 6$ and $BC = 8$. The diagonal AC was cut with three parallel segments that cut sides AB and CD into equal parts. Bok AB ma dugo 6cm, a bok BC ma dugo 8cm.

(i) Find the area of the shaded quadrilateral $DEGH$.

(ii) Find the ratio of areas of quadrilaterals $DEFJ$ and $FGHJ$.



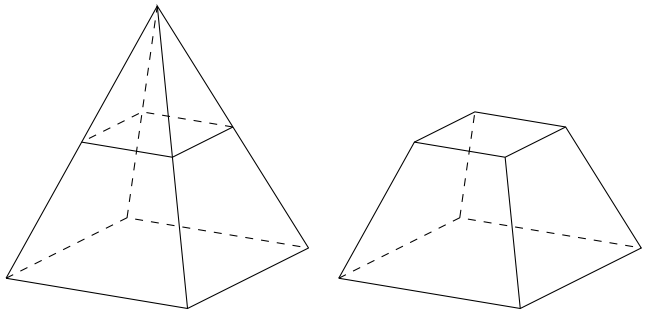
Q40. *[Angle bisector theorem]* In a triangle ABC the angle bisector of angle C intersects side AB at point D. Prove that $\frac{AD}{BD} = \frac{AC}{BC}$.

8.4 Solid geometry

Q41. A right pyramid has a square base with a side of 3. A cross-section of the pyramid through the apex and two opposite vertices of the base has a shape of a right triangle. Find the volume and the total surface area of the pyramid.

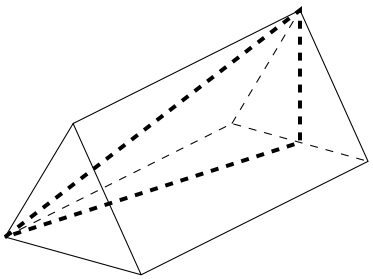
Q42. A right pyramid has a square base. A cross-section of the pyramid through the apex and two opposite vertices of the base has a shape of an equilateral triangle. The volume of the pyramid is $36\sqrt{6}$. Find its total surface area.

Q43. The diagram on the left shows a regular pyramid with a square base. The side of the base is 8 centimetres. The height of the pyramid is 12 centimetres. The pyramid is cut in half of its height with a plane parallel to its base. The diagram on the right shows frustum of the pyramid - i.e. a solid formed "below the cut". Find the volume of the frustum.

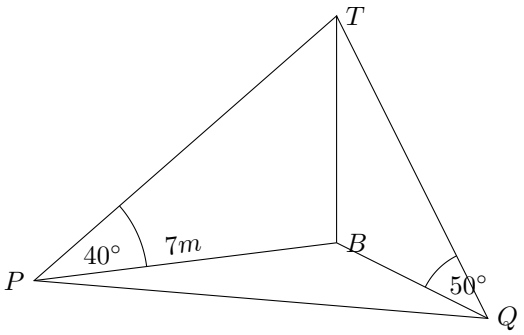


Q44. A tent is in the shape of a right triangular prism. Its entrance face is an equilateral triangle with base 1.6 metres. The angle that a diagonal of a side face makes with the horizontal is 30°. Find the volume of the tent.

Q45. A tent is in the shape of a right triangular prism. Its entrance face is an isosceles right triangle (with right angle at the top). The side faces are squares and the volume of the tent is $4m^3$. Find the height of the tent (measured from the bottom to the highest point) correct to 1 centimetre.



Q46. The diagram shows a vertical mast BT. Points P and Q are on the same level as B, the base of the mast. The distance of P from B is 7 metres. The angles of elevation of T, the top of the mast, measured from P and Q are $40°$ and $50°$ respectively. Point T is connected with points P and Q with a straight rope, so are points P and Q. Knowing that the segments BP and BQ meet at a right angle find the total length of the rope giving your answer correct to 10 centimetres.



Q47. A cube is inscribed in a sphere which is inscribed in a larger cube. Find the ratio of volumes of the larger to the smaller cube.

Q48. A sphere is inscribed in a cube which is inscribed in a larger sphere. Find the ratio of volumes of the larger to the smaller sphere.

Q49. The centres of faces of a cube are vertices of a new polyhedron. What is the ratio of its volume to the volume of the cube?

Q50. The centres of faces of a cube are vertices of a new polyhedron. Two spheres are inscribed into both polyhedrons: in the cube and in new polygon. What is the ratio of the volumes of the smaller sphere to the larger?

Q51. The midpoints of edges of a regular octahedron are vertices of a new polyhedron. What is the ratio of its volume to the volume of the octahedron?

Q52. The centres of faces of a regular octahedron are vertices of a new polyhedron. Name the new polyhedron. What is the ratio of its volume to the volume of the octahedron?

Q53. Find the volume and the total surface area of the regular tetrahedron wih edge length 2.

Q54. The midpoints of edges of a regular tetrahedron are vertices of a new polyhedron. Name the new polyhedron. What is the ratio of its volume to the volume of original tetrahedron?

Q55. The centres of faces of a regular tetrahedron are vertices of a new polyhedron. Name the new polyhedron. What is the ratio of its volume to the volume of original tetrahedron?

Q56. A sphere is inscribed in a regular tetrahedron. What percentage of the volume of the tetrahedron makes the volume of the sphere?

Q57. A regular tetrahedron is inscribed in a sphere. What percentage of the volume of the sphere makes the volume of the tetrahedron?

Q58. The length of an edge of a regular tetrahedron is 2.

(i) Find the volume of the smalest cube in which the tetrahedron is contained.

(ii) The tehrahedron is reflected in its centre of mass and a new tatrahedron is formed. Find the volume of the intersection of both tetrahedrons.

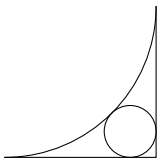
8.5 Miscellaneous problems

Q59. Find the radius of a circle inscribed in the triangle with sides lengths: 6, 8 and 10.

Q60. The longest side AB of isosceles right-angled triangle ABC is $18\sqrt{2}$ long. Point P is the centre of the circle inscribed in the triangle. M is the midpoint of AB. Find the area of triangle AMP.

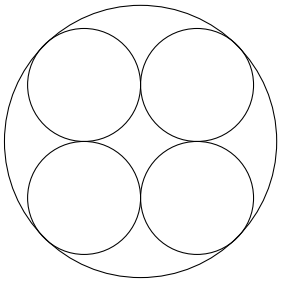
Q61. (1) A circle with radius $\sqrt{2}$ is inscribed in a square. Another circle is externally tangent to the given circle and internally tangent to the square. Find the other circle's radius. Give **exact** answer.

(2) A circle with radius R is inscribed in a square. Another circle is externally tangent to the given circle and internally tangent to the square. Show that the other circle's radius is given by $R(3-2\sqrt{2})$.



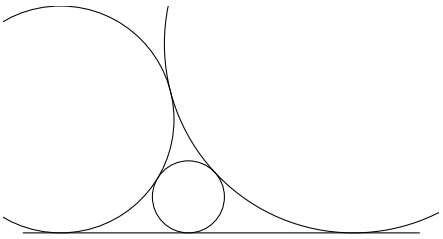
Q62. A circle is inscribed in an equilateral triangle. Another circle is externally tangent to the given circle and internally tangent to the triangle. Show that larger circle's radius is three times the smaller circle's radius.

Q63. Four circles with the same radius are inscribed in a larger circle in such a way that each two smaller circles are externally tangent and each smaller circle is internally tangent to the larger one. Find the ratio of the area of the larger circle to the sum of areas of smaller circles. Give the answer as a decimal approximated to 3 significant figures.



Q64. Three circles with the same radius are inscribed in a larger circle in such a way that each two smaller circles are externally tangent and each smaller circle is internally tangent to the larger one. Find the ratio of the area of the larger circle to the sum of areas of smaller circles. Give the answer as a decimal approximated to 3 significant figures.

Q65. (1) Two circles with radii 4 are externally tangent. Each of them is tangent to the same straight line. The third circle is externally tangent to both circles and to the line. Find the radius of the third circle.



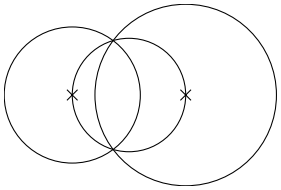
(2) Two circles with radii 4 and 9 are externally tangent. Each of them is tangent to the same straight line. The third circle is externally tangent to both circles and to the line. Find the **exact** radius of the third circle.

(3) Two circles with radii 3 and 5 are externally tangent. Each of them is tangent to the same straight line. The third circle is externally tangent to both circles and to the line. Find the radius of the third circle. Give your answer correct to 5 significant figures.

(4) Two circles with radii r and R are externally tangent. Each of them is tangent to the same straight line. The third circle is externally tangent to both circles and to the line. Find the radius of the third circle. Give your answer as simplified expression in terms of r and R.

Q66. Three circles with radius 2 each are tangent one to another. The fourth circle is inscribed in the region bounded by the three circles and it is tangent to each of the three.
Find the radius of the fourth circle.

Q67. Two circles with radii 6 and 8 intersect at two points in such a way that the third circle is drawn through the two points of intersection and through the centres of both circles.
Find the radius of the third circle.



Q68. An isosceles triangle has a base of 10 and the area of 60. Find the radius of the circle inscribed in the triangle.

Q69. The lenghts of sides of a quadrilateral inscribed in a circle are 16, 25, 60 and 63. Two interior angles of the quadrilateral are right angles. Find the length of the diameter of the circle.

Q70. In a triangle ABC segments AD and BE are the heights of the triangle. Show that $B\hat{E}D = B\hat{A}D$.

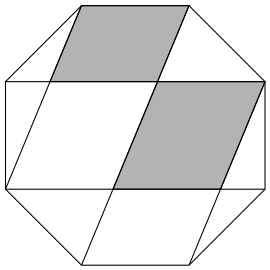
Q71. A quadrilateral $ABCD$ is inscribed in a circle with radius 6. A tangent to the circle at point D is parallel to AB while a tangent to the circle at point B is parallel to AD. Find the area of the quadrilateral knowing that an angle at A is 60° and that $CD = CB$.

Q72. The heights h_a, h_b and h_c of a triangle are perpendicular to the sides a, b and c respectively. A circle with radius r is inscribed in the triangle. Show that

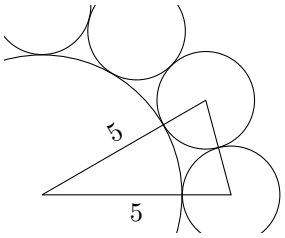
$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

Q73. A triangle with sides 15, 20 and 25 is cut with its shortest height into two smaller triangles. A circle is inscribed in each of the two smaller triangles. Find the distance between the centres of the two circles.

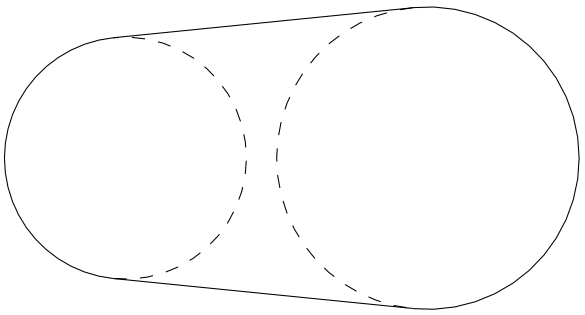
Q74. The diagram shows a regular octagon. In the octagon some of its diagonals have been drawn. They form triangles and parallelograms. What is the ratio of the area of the smaller shaded parallelogram to the area of the larger one?



Q75. Twelve circles with the same radius are drawn in a way that each of them is tangent to one larger circle and to two other of twelve smaller ones. (A part of the drawing is shown in the diagram.) The distance between the centre of the larger circle and a centre of a smaller circle is 5. Find the area of a single smaller circle.



Q76. A playground was designed in the following way. Firstly two circles were drawn, with radii 4 metres and 5 metres respectively. The distance between centres of the circles is 10 metres. Then two tangents to both circles were drawn in a way that *convex* shape was bounded by the arcs of the circles and segments of the tangents. The shape created is a playground. What is its area? Give answer correct to $0.1m^2$.



Q77. The *centroid* of a triangle is the intersection of the three *medians* of the triangle (each median connecting a vertex with the midpoint of the opposite side). A line segment joining a vertex of a tetrahedron with the centroid of the opposite face is called a *median*. The four medians of a tetrahedron intersect at the *centroid* of the tetrahedron.
A regular tetrahedron is reflected in its centroid. What percentage of the volume of the original tetrahedron makes the volume of the intersection of the two tetrahedrons (the original one and its reflection)?

Chapter 9

Numbers II

9.1 Factorials, Pascal's triangle and binomial theorem

Definition 1 (factorial).
For any positive integer n ($n \in \mathbb{Z}^+$) we define $n!$ (read: "n factorial") as a product of all consecutive integers from 1 to n:

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$$

To extend the definition to all non-negative integers we have to additionally define:

$$0! = 1$$

Factorials can be defined also in the following recursive way:

$$n! = \begin{cases} 1 & \text{for } n = 0 \\ n \times (n-1)! & \text{for } n \in \mathbb{Z}^+ \end{cases}$$

Example 2 (factorial).

$$\begin{aligned} 7! &= 7 \times 6! = \\ &= 7 \times 6 \times 5! = \\ &= 7 \times 6 \times 5 \times 4! = \\ &= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \\ &= 5040 \end{aligned}$$

Example 3 (simplifying factorials).

$$\begin{aligned} \frac{9!}{6!} &= \frac{9 \times 8 \times 7 \times 6!}{6!} = \\ &= 9 \times 8 \times 7 = 504 \end{aligned}$$

Definition 4 (binomial coefficient).
*The **binomial coefficient** is defined for $n \in \mathbb{Z}^+$ and $r \in \{0, 1, 2, 3, \ldots, n\}$ as follows:*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\binom{n}{r}$ is read as "n-c-r" or "n-choose-r" and sometimes denoted as ${}^nC_r$.
The binomial coefficients can be arranged in a graphical pattern called the **Pascal's triangle** .

$$\begin{array}{ccccccccccc}
 & & & & \binom{0}{0} & & & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & &
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & 1 & & & & & & \\
 & & & 1 & & 1 & & & & & \\
 & & 1 & & 2 & & 1 & & & & \\
 & 1 & & 3 & & 3 & & 1 & & & \\
1 & & 4 & & 6 & & 4 & & 1 &
\end{array}$$

PROPERTIES OF BINOMIAL COEFFICIENTS AND PASCAL'S TRIANGLE.

$\binom{n}{r} = \binom{n}{n-r}$ *(symmetry of Pascal's triangle)*

$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ *(each number is a sum of the two above it)*

$\binom{n}{0} = \binom{n}{n} = 1$

$\binom{n}{1} = \binom{n}{n-1} = n$

$\binom{n}{2} = \binom{n}{n-2} = \frac{n\times(n-1)}{2}$

$\binom{n}{3} = \binom{n}{n-3} = \frac{n\times(n-1)\times(n-2)}{1\times2\times3}$ *etc.*

THEOREM 5 (binomial theorem).
For any numbers a and b and a positive integer n:
$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$
or

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

Questions

Q1. Evaluate without a calculator.

(1) 3! (2) 4! (3) 5! (4) 6! (5) 7!

Q2. Simplify not using a calculator.

(1) $\frac{7!}{6!}$ (2) $\frac{8!}{7!}$ (3) $\frac{12!}{11!}$ (4) $\frac{11!}{9!}$ (5) $\frac{20!}{18!}$

Q3. Solve the equations without a calculator.

(1) $\frac{n!}{(n-1)!} = 7$ (2) $\frac{n!}{(n-2)!} = 20$ (3) $\frac{(n+1)!}{(n-1)!} = 72$

Q4. Evaluate without a calculator.

(1) $\binom{3}{2}$ (4) $\binom{5}{2}$ (7) $\binom{6}{1}$ (10) $\binom{6}{4}$ (13) $\binom{7}{4}$
(2) $\binom{4}{1}$ (5) $\binom{5}{3}$ (8) $\binom{6}{2}$ (11) $\binom{7}{2}$ (14) $\binom{8}{5}$
(3) $\binom{4}{2}$ (6) $\binom{5}{4}$ (9) $\binom{6}{3}$ (12) $\binom{7}{3}$ (15) $\binom{8}{6}$

Q5. Simplify not using a calculator.

(1) $\binom{n+1}{2}$ (3) $\binom{n}{n-2}$ (5) $\binom{2n}{2n-2}$
(2) $\binom{3n}{2}$ (4) $\binom{n+1}{n-1}$ (6) $\binom{n+2}{3}$

Q6. Expand and simplify.

(1) $(a+b+c)^2$
(2) $(a^2-b+2c)^2$
(3) $(a^2+ab+b^2)^2$
(4) $(a^2+a-1)^2$
(5) $(2x-3y+5xy)^2$
(6) $(3ab+2ac-bc)^2$
(7) $(2a^2b+3ab^2+1)^2$
(8) $(2s-3st+2t^2)^2$
(9) $(2+\sqrt{2}+\sqrt{3})^2$
(10) $(2+\sqrt{2}-\sqrt{3})^2$
(11) $(\sqrt{5}+\sqrt{2}+\sqrt{3})^2$
(12) $(\sqrt{6}-\sqrt{2}+\sqrt{3})^2$

Q7. Expand and simplify.

(1) $(a-c)^3$
(2) $(a^2+2b)^3$
(3) $(a^2-ab)^3$
(4) $(a^2+2a)^3$
(5) $(2x^2y+3xy^2)^3$
(6) $(3x^2y-2xy^2)^3$
(7) $(3+\sqrt{3})^3$
(8) $(\sqrt{3}+\sqrt{2})^3$
(9) $(\sqrt{3}-\sqrt{2})^3$
(10) $(2\sqrt{3}-3\sqrt{2})^3$
(11) $(\sqrt{6}+\sqrt{3})^3$
(12) $(\sqrt{6}-\sqrt{2})^3$

Q8. Expand and simplify.

(1) $(a-c)^4$
(2) $(x+2y)^4$
(3) $(3-\sqrt{2})^4$
(4) $(a-c)^5$
(5) $(x+2y)^5$
(6) $(3-\sqrt{2})^5$
(7) $(\sqrt{3}+\sqrt{2})^6$
(8) $(2ab-3a)^6$

Q9. Expand and evaluate.

(1) $\sum_{k=0}^{5} 2^k$
(2) $\sum_{r=1}^{10}(-1)^r$
(3) $\sum_{r=1}^{5} 2r$
(4) $\sum_{k=5}^{8}(3k-5)$
(5) $\sum_{n=1}^{4}(\frac{1}{2})^n$
(6) $\sum_{r=10}^{13}(30-2r)$
(7) $\sum_{r=0}^{4}(-2)^r$
(8) $\sum_{r=3}^{8} 4$

Q10. Express the following sums using sigma notation.

(1) $1+2+3+\cdots+20$
(2) $2+4+6+\cdots+44$
(3) $1+2+3+\cdots+n$
(4) $1+3+5+\cdots+51$
(5) $1+2+4+\cdots+1024$
(6) $3+1+\frac{1}{3}+\cdots+\frac{1}{729}$
(7) $3+1-1+\cdots-75$
(8) $3+7+11+\cdots+79$
(9) $10+7+4+\cdots-35$
(10) $\frac{2}{3}+1+\frac{4}{3}+\cdots+6$
(11) $1-2+4-8+\cdots-2048$
(12) $3-9+27-81+243-729+2187-6561$
(13) $1-\frac{1}{2}+\frac{1}{4}-\cdots+\frac{1}{4096}$
(14) $1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{20}$
(15) $1+4+9+\cdots+625$
(16) $1-8+27-\cdots+1331$
(17) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{99}{100}$
(18) $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\cdots-\frac{1}{52}$
(19) $2+6+12+20+\cdots+462$
(20) $\frac{2}{1\times3}+\frac{4}{3\times5}+\frac{6}{5\times7}+\cdots+\frac{36}{35\times37}$

Q11. Find the coefficient of x^3 in the expansion $(2x-3)^6$.

Q12. Find the coefficient of x^2 in the expansion $(\frac{1}{2}x-2)^5$.

Q13. Find the coefficient of x^4 in the expansion $(\frac{2}{3}x+3)^7$.

Q14. Find the independent term in the expansion $(4x+\frac{1}{2x})^6$.

Q15. Find the coefficient of x in the expansion $(\frac{2}{3x}-3x)^7$.

Q16. Find the independent term in the expansion $(3x-\frac{2}{x^2})^9$.

Q17. Find the coefficient of x in the expansion $(2x-\frac{1}{x^2})^7$.

Q18. Find the coefficient of x^4 in the expansion $(2x-\frac{1}{x^2})^7$.

Q19. Find the coefficient of x^7 in the expansion $(3x^2-\frac{1}{x})^5$.

Q20. Find the constant term in the expansion $(4x^2-\frac{2}{x})^6$.

Q21. Find the first three terms in the **decending** powers of x of the expansion $(2x-\frac{3}{2})^{11}$.

Q22. Find the first three terms in the **ascending** powers of x of the expansion $(\frac{1}{2}x-2)^9$.

Q23. Find the first three terms in the ascending powers of x of the expansion $(x-1)(x-2)^3$.

Q24. Find the first three terms in the ascending powers of x of the expansion $(2x+1)(x-1)^5$.

Q25. Find the first three terms in the ascending powers of x of the expansion $(3x-1)(2x+1)^4$.

Q26. Find the first three terms in the ascending powers of x of the expansion $(3x-2)^4(2x+\frac{1}{2})^3$.

Q27. Find the coefficient of x^3 in the expansion $(x^2-3x+1)(x-2)^5$.

Q28. Find the coefficient of x^2 in the expansion $(x-3)^3(3x-\frac{1}{3})^4$.

Q29. Find the first three terms in **ascending** powers of x in the following expansions:

(1) $(x+1)^4(1-2x)^3$
(2) $(2+x)^2(2x-1)^4$
(3) $(x-4)(2x+1)^4$
(4) $(x-1)^2(\frac{1}{2}x+1)^6$

Q30. Find the first three terms in **descending** powers of x in the following expansions:

(1) $(x+1)^4(1-2x)^3$
(2) $(2+x)^2(2x-1)^4$
(3) $(x-4)(2x+1)^4$
(4) $(x-1)^2(\frac{1}{2}x+1)^6$

Q31. Find the value(-s) of a for which the coefficient of x in the expansion $(ax-\frac{1}{x})^5$ is 80.

Q32. Find the value(-s) of a for which the coefficient of x^2 in the expansion $(2x^2-\frac{1}{ax})^4$ is $\frac{8}{3}$.

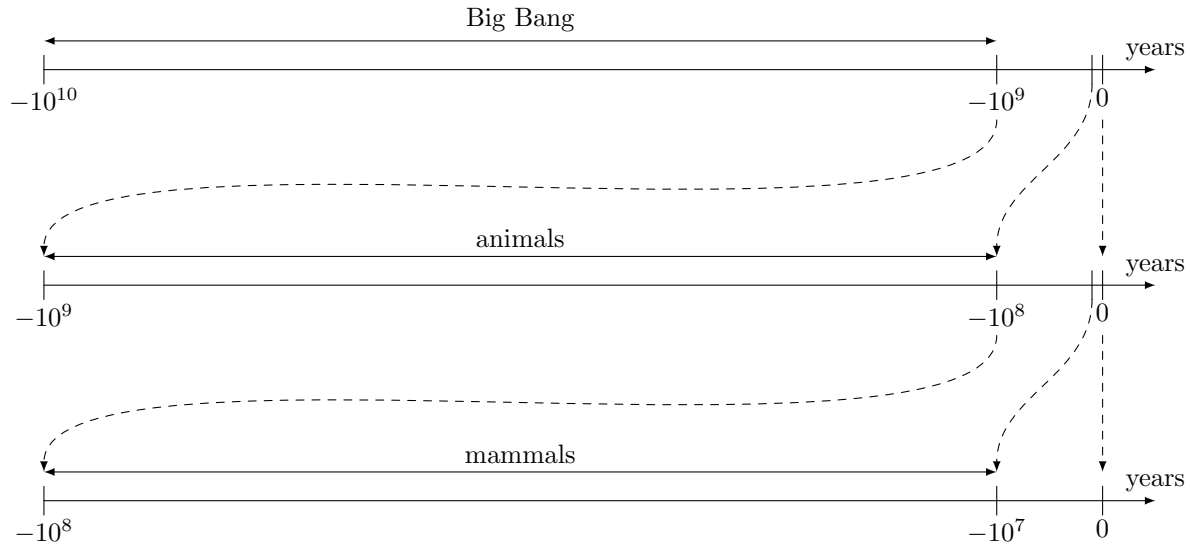
Q33. Find the value(-s) of a for which the coefficient of x^2 in the expansion $(x-a)(3x+a)^4$ is 84.

9.2 Logarithms

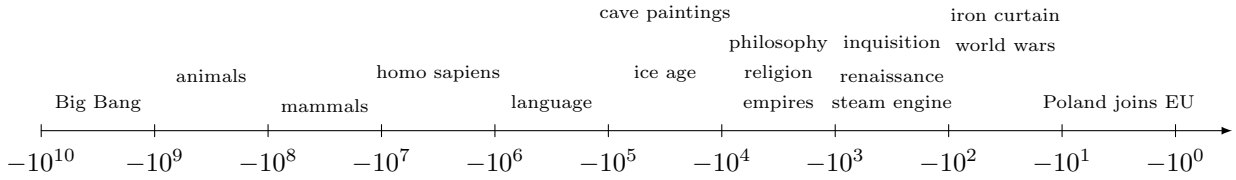
The table below shows a non-linear time line with examples of various events. Some of them are of huge importance for us, people living in the world, in Europe, in Poland. Some of them are of smaller significance. The table shows natural tendency to place recent events more accurately than the older ones. We can observe easily that the accuracy of description of time decreases with the increase of amount of time that has passed since an event.

years ago	**i.e. approximately**	**actions**
10^{-6} to 10^{-5}	a few minutes ago	you are looking at this handout
10^{-5} to 10^{-4}	less than an hour ago	...
10^{-4} to 10^{-3}	a few hours ago	...
10^{-3} to 10^{-2}	a few days ago	...
10^{-2} to 10^{-1}	a month ago	...
10^{-1} to 10^{0}	a few months ago	...
10^{0} to 10^{1}	last decade	Poland joins EU
10^{1} to 10^{2}	last century	world wars, iron curtain
10^{2} to 10^{3}	last millenium, middle ages	inquisition, renaissance, steam engine
10^{3} to 10^{4}	...	religion, philosophy, empires
10^{4} to 10^{5}	...	ice age, cave paintings
10^{5} to 10^{6}	...	language
10^{6} to 10^{7}	...	homo sapiens
10^{7} to 10^{8}	...	mammals
10^{8} to 10^{9}	...	animals
10^{9} to 10^{10}	...	Big Bang

If we wanted to show all these actions graphically on a time line it would not be easy, as each of the periods considered is 10 times longer than the previous one.



etc.
We can create a different time line instead:



The scale on the axis above is obviously non-standard, not **linear**. It is **logarithmic**.

Algebra of logarithms

PROPERTIES: RULES OF POWERS.
For $a, b, m, k \in \mathbb{R}$ and $a, b \neq 0$:

- $a^m \times a^k = a^{m+k}$
- $a^k \times b^k = (ab)^k$
- $a^m \div a^k = a^{m-k}$
- $a^k \div b^k = (\frac{a}{b})^k$
- $(a^m)^k = a^{mk}$

DEFINITION 6 (logarithm).
For $a, b, x \in \mathbb{R}$ such that $a \neq 1$ and $a, b > 0$
*a **logarithm** of b to base a is the exponent x to which a must be raised to produce b, i.e.*

$$\log_a b = x \;\Leftrightarrow\; a^x = b$$

NOTATION.

- $\log x = \log_{10} x$

PROPERTIES: RULES OF LOGARITHMS.
For $a, b, c, s, x \in \mathbb{R}$, $a, b, c, s > 0$, $a, s \neq 1$:

- $\log_a b + \log_a c = \log_a bc$
- $\log_a b - \log_a c = \log_a(\frac{b}{c})$
- $\log_a n^s = s \log_a n$
- $\log_a b = \frac{\log_s b}{\log_s a}$ *(change of base)*
- $\log_b b^a = a$ $\qquad b^{\log_b a} = a$

Questions

Q34. Evaluate. Do not use a calculator.

(1) $\frac{3^5 9^4}{27^4}$
(2) $\frac{16^6 8^{-4}}{4^7}$
(3) $\frac{(4\sqrt{2})^5 8^{-3}}{4^2}$
(4) $\frac{(8^6+4^{10})^2}{2^{37}}$
(5) $4^5+4^5+4^5+2^{10}$
(6) $3^{n+1}+3^{n+2}-3^n$

Q35. Evaluate. Do not use a calculator.

(1) $\log_2 8$
(2) $\log_4 8$
(3) $\log_8 4$
(4) $\log_2 2\sqrt{2}$
(5) $\log_4 \sqrt{2}$
(6) $\log_{\frac{1}{2}} \sqrt{2}$
(7) $\log_{\frac{1}{2}} \sqrt{8}$
(8) $\log_{\frac{1}{3}} 27$
(9) $\log_4 \frac{\sqrt{2}}{16}$
(10) $\log_{\frac{1}{2}} \frac{\sqrt{8}}{32}$
(11) $\log_3 \frac{27}{\sqrt{3}}$
(12) $\log_9 \frac{81}{\sqrt{27}}$
(13) $\log_{\sqrt{3}} \frac{\sqrt{3}}{27}$
(14) $\log_{\frac{1}{\sqrt{3}}} \frac{\sqrt{27}}{81}$
(15) $\log_{0.2} 25$
(16) $\log_4 0.125$
(17) $\log_{4\sqrt{2}} \sqrt{8}$
(18) $\log_{32} 4\sqrt{2}$
(19) $\log_{2\sqrt{2}}(\frac{1}{16})$
(20) $\log_{27} 9\sqrt{3}$
(21) $\log_{25} 5\sqrt{5}$
(22) $\log_{0.25} 4\sqrt{8}$
(23) $0.01 \times 0.001^{\log 5}$
(24) $(9\sqrt{3})^{\frac{\log_3 4-1}{5}}$
(25) $(2\sqrt{2})^{\frac{\log_4 3+2}{3}}$

Q36. Knowing that $a = \log_2 3$ and $b = \log_2 5$ express the following in terms of a and b.

(1) $\log_2 25$
(2) $\log_2 15$
(3) $\log_2 45$
(4) $\log_2 40$
(5) $\log_2 60$
(6) $\log_2 3000$
(7) $\log_2 27$
(8) $\log_2 72$

Q37. Knowing that $a = \log_3 5$ and $b = \log_9 2$ express the following in terms of a and b.

(1) $\log_3 10$
(2) $\log_3 100$
(3) $\log_9 10$
(4) $\log_9 100$
(5) $\log_3 40$
(6) $\log_3 8$
(7) $\log_9 80$
(8) $\log_9 25$

Q38. Knowing that $a = \log_2 b$ express the following in terms of a.

(1) $\log_2 \frac{1}{b}$
(2) $\log_2(b^2)$
(3) $\log_b 2$
(4) $\log_b 8b$
(5) $\log_4 b$
(6) $\log_8 b$
(7) $\log_8 4b^5$
(8) $\log_{b^2} 32$

Q39. Let $a = \log x$ and $b = \log y$. Express the following in terms of a and b.

(1) $\log(\frac{1}{xy^3})$
(2) $\log(\frac{\sqrt{x}}{y})$
(3) $\log(\frac{x}{\sqrt{y}})$
(4) $\log(\frac{10}{x^4\sqrt{y^3}})$

Q40. Let $a = \log x$, $b = \log y$ and $c = \log z$. Express the following in terms of a, b and c.

(1) $\log(\frac{x}{y^2\sqrt{z}})$
(2) $\log(\frac{x^3\sqrt{y}}{z^2})$
(3) $\log(\frac{x^5}{zy\sqrt{y}}\frac{1}{x^2\sqrt{x}})$
(4) $\log(\frac{1}{x^4\sqrt{y^5}\sqrt[3]{z^4}})$

Q41. Find integer values of a and b such that:

(1) $\log 400 = a\log 2 + b$
(2) $\log 75 = a\log 5 + b\log 3$
(3) $\log 160 = a\log 2 + b$
(4) $\log_2 1000 = a\log_2 5 + b$
(5) $\log_2 250 = a\log_2 5 + b$
(6) $\log_3 144 = a\log_3 2 + b$

Q42. (1) Knowing that $\log_4 98 = a$ express $\log_4 343$ in terms of a.
(2) Knowing that $\log_{49} 28 = a$ express $\log_4 98$ in terms of a.
(3) Knowing that $\log_{25} 75 = a$ express $\log_{15} 125$ in terms of a.
(4) Knowing that $\log_{40} 160 = a$ express $\log_8 250$ in terms of a.

Q43. Simplify.

(1) $2^{\log_4 3}$
(2) $4^{\log_2 5}$
(3) $10^{3\log 2}$
(4) $4^{2\log_8 3}$
(5) $10^{\frac{1}{2}\log 8}$
(6) $(0.1)^{2\log 3}$
(7) $3^{-\log_9 16}$
(8) $1000^{2\log\sqrt{2}}$

Q44. Let $a = 2^x$, $b = 3^x$ and $c = 5^x$. Express the following in terms of a, b and c.

(1) 2^{x+1}
(2) 2^{x-1}
(3) 2^{x+5}
(4) 2^{2x+3}
(5) 3^{3x-1}
(6) 4^x
(7) 8^x
(8) 9^{-x}
(9) 25^x
(10) 16^x
(11) 16^{-x}
(12) 27^x
(13) 4^{x-2}
(14) 5^{-x}
(15) 2^{4-x}
(16) 9^{3-x}
(17) 25^{2x}
(18) 8^{2x-1}
(19) $\frac{1}{16^{3x-2}}$
(20) $\frac{81}{27^{3-2x}}$
(21) $\frac{3\times 9^{2x+1}}{81^{1+x}}$
(22) $\frac{625^{x-1}}{125^{3x-2}}$
(23) 10^x
(24) 18^{2x-1}

Logarithmic equations

Q45. Solve the equations.

(1) $\log_{\frac{\sqrt{3}}{2}} x = 6$
(2) $\log_{3\sqrt{3}} x = \log 100$
(3) $\log_8 x + \log_4 x = 2.5$
(4) $\log_{27} x + \log_9 x - \frac{5}{3} = 0$
(5) $\log_{25\sqrt{5}} x^2 + 3\log_5 x = 19$
(6) $\log_4 \log_{\frac{1}{\sqrt{3}}} \log_2 x = 0.5$
(7) $\log_{x^2} 4 + \log_{x^3} 64 - 9 = 0$
(8) $\log_{x^2} 64 = \log_{x^8} 256 - 2$
(9) $4\log^2_{x^2} 5 + \log_3 7 = \log_9 4$
(10) $\log_{\frac{\pi}{4}} x + \log_{\frac{\pi}{4}} x^2 + 9 = 0$
(11) $\log_{\frac{\sqrt{2}}{2}} x = 6$
(12) $\log^2_5 x = 4\log_{x^2}\sqrt{x}$

Q46. Solve the equations.

(1) $(\frac{1}{4})^{x-1} = \frac{6}{2^{2x-1}}$
(2) $4^{x+1} = 8^{2x-3}$
(3) $0.2^{3-5x} = 25^{3x-5}$
(4) $27^{3-x} = (\frac{1}{\sqrt{3}})^{2x-1}$
(5) $(\frac{1}{\sqrt{2}})^{3x-1} = (4\sqrt{2})^{3x+2}$
(6) $16 \times 2^{x-3} \times (\frac{1}{4})^{3-2x} = 256^5$
(7) $\left(\frac{4}{9}\right)^{4-6x} \times 1.5^{8x} = \frac{64}{729}$
(8) $12.5^x \div (0.2\sqrt{2})^{x^2} = 0$

Q47. Solve the equations.

(1) $(4\sqrt{2})^{2x-1} = \frac{1}{8}(2\sqrt{2})^{3-2x}$
(2) $\log_{3\sqrt{3}} x^2 - \log_{9\sqrt{3}} x^3 = 0.4$
(3) $\log_{x^2} 3 + \log_{x^3} 27 - 3 = 0$
(4) $0.2^{3-5x} = 25^{3x-5}$

Aplications

Q48. The number of **thousands** of radioactive atoms N of a particular material present at time t years may be written in the form $N = 5000 \times 2^{-kt}$, where k is a positive constant. It is found that after 6 years there was 2.5 milion atoms of the material present.

(i) What was the initial number of atoms of radioactive material?
(ii) Determine the value of k.
(iii) When the number of atoms is reduced to 50 thousand?

Q49. A liquid leaks from a container at such a pace that in every moment the amount left in the container is 2% smaller than the amount a minute ago. The amount A litres can be therefore modelled by the equation $A = 15 \times 0.98^t$, where $t \geq 0$ is time in minutes.

(i) What is the initial amount of liquid in the container?
(ii) When the amount of liquid is below half of the initial amount?

(iii) When the amount of liquid is below 1% of the initial amount?

Q50. An animal body is found to be infected with approximately a million viral particles. Under the systematic treatment the number of particles decreases. The number N of viral particles in an animal body in thousands is modelled by the equation $N = 1000 \times 2.7^{-kt}$, where $t \geq 0$ is time measured in days while k is a positive real constant. It is known that after 4 days there are approximately 383 thousand viral particles in the body.

(i) Find the value of k.

(ii) How many particles are eliminated in the first two weeks?

An animal is considered healthy when the number of viral particles goes below a thousand.

(iii) After how many full days is the animal considered healthy?

9.3 Absolute value equations and inequalities

Q51. Solve the equations.

(1) $|x| = 1 - |x|$
(2) $|x| = |3x - 2|$
(3) $|2x + 2| = 2|2x - 1|$
(4) $|2x - 2| = 4 - |x|$
(5) $|2x + 1| = |4x - 2|$
(6) $|2x + 1| = 3|1 - x|$
(7) $|x + 1| = |\frac{1}{2}x - 1|$
(8) $\frac{1}{2}|x + 2| = 2|3 - x|$
(9) $\frac{1}{2}|x - 2| = |\frac{3}{2}x + 6|$

Q52. Solve the equations.

(1) $|x + 1| + |3x - 4| = 9$
(2) $|x + 2| + 4 = 2|1 - x|$
(3) $|2x - 1| + |2 - x| = 6$
(4) $2|x - 1| + 3 - 3|2 - x| = 0$
(5) $2|x + 2| + 3|2 - x| = 10$
(6) $|2x + 3| + |1 - 2x| = 4$
(7) $|2x + 1| - |1 - 2x| + 1 = 0$
(8) $|3x + 1| + 2|2x - 1| = 8$

Q53. Solve the following inequalities.

(1) $|x + 2| > 0$
(2) $|2x - 3| > 1$
(3) $|3x + 1| \geq 5$
(4) $|-2x + 3| \geq 0$
(5) $|4x + 5| < 4$
(6) $|5x - 4| < -2$
(7) $|2x + 3| \leq 2$
(8) $|3x - 6| \leq -2$
(9) $|2x + 4| > -3$
(10) $|-2x + 4| < 5$
(11) $|6x + 3| \leq 6$
(12) $|-2x + 3| \geq 6$
(13) $|3x - 4| > 2$
(14) $|2x + 5| < 0$
(15) $|2x - 5| \leq 0$
(16) $|-3x + 1| \geq -2$
(17) $|2x - 5| \geq 3$
(18) $|3x - 5| \leq 1$
(19) $|2x + 3| > 4$
(20) $|-3x + 2| < 6$

Q54. Solve the inequalities.

(1) $|3x - 1| > |2x + 1|$
(2) $|x - 1| > |2x + 1|$
(3) $|x + 1| \geq |2x + 1|$
(4) $|3x - 2| \leq |2x + 3|$
(5) $\left|\frac{4x-2}{x+4}\right| > 1$
(6) $\left|\frac{x+4}{4x-2}\right| > 1$
(7) $\left|\frac{2x-1}{2x+1}\right| \geq 2$
(8) $\left|\frac{3x-2}{x+1}\right| < 2$
(9) $\left|\frac{5x-2}{x+1}\right| \geq 3$
(10) $|5 - 2x| - \frac{1}{2}|x + 6| \leq 0$

Q55. Solve the inequalities.

(1) $|5 - 2x| + 2 \leq \frac{1}{2}|x + 6|$
(2) $\frac{1}{2}|3x - 8| + |\frac{1}{2}x + 2| > 5$
(3) $\frac{5}{2}|-x - 2| + 1 < \frac{1}{2}|x - 6|$
(4) $|2x - 3| \geq \frac{1}{2}(3 + |6 - x|)$

9.4 Complex numbers

Q56. Evaluate. Give answer in the form $a + ib$, where $a, b \in \mathbb{R}$.

(1) $(2 + i)(3 + 4i)$
(2) $(3 - i)(2 + 5i)$
(3) $(4 + 2i)(4 - 2i)$
(4) $(-2 + 3i)(3 - 2i)$
(5) $(-1 + 4i)(1 + 2i)$
(6) $(5 + 3i)(2 - 3i)$
(7) $(4 - 3i)(3i)$
(8) $(1.5 + 2i)(2 - 2.5i)$
(9) $(-2i)(4 - 3i)$
(10) $(-1 - 3i)(-2 - 4i)$
(11) $(2 + 3i)(2 - 3i)$
(12) $(3 - 2i)(2 - i)$

Q57. Evaluate. Give answer in the form $a + ib$, where $a, b \in \mathbb{R}$.

(1) $\frac{2+i}{3+4i}$
(2) $\frac{3-i}{2+5i}$
(3) $\frac{4+2i}{4-2i}$
(4) $\frac{-2+3i}{3-2i}$
(5) $\frac{-1+4i}{1+2i}$
(6) $\frac{5+3i}{2-3i}$
(7) $\frac{4-3i}{3i}$
(8) $\frac{1.5+2i}{2-2.5i}$
(9) $\frac{-2i}{4-3i}$
(10) $\frac{-1-3i}{-2-4i}$
(11) $\frac{2+3i}{2-3i}$
(12) $\frac{3-2i}{2-i}$

Q58. Evaluate. Give answer in the form $a + ib$, where $a, b \in \mathbb{R}$.

(1) $(2 - i)^3$
(2) $(1 + 2i)^3$
(3) $(1 + i)^4$
(4) $(2 - 2i)^4$
(5) $(1 + i\sqrt{3})^4$
(6) $(\sqrt{2} - i\sqrt{2})^5$
(7) $(-1 - i\sqrt{3})^5$
(8) $(\frac{1}{2} - i\frac{\sqrt{3}}{2})^6$

Q59. Find $a, b \in \mathbb{R}$ such that:

(1) $(2a + 1) - i(3a + b) = 5 + 3i$
(2) $b^2 + a + 2ai = 7 + 6i$
(3) $(b - 3a) + i(2a - b) = 7 - 5i$
(4) $2b - bi - 3a + 3ai = -17 + 13i$
(5) $a^2 + b^2 - b^2i + a^2i = 5 + 3i$
(6) $4a - 3b - 4 + (4 - 7a)i - 2bi = -1 + 4i$
(7) $3a + 2b - 4 + 3ai - i(b + 3) = 4i$
(8) $(ai)^2 + b^2 + (b + 8)i = i + a^2i - 5$

Q60. Solve the equations in complex numbers (i.e. $x \in \mathbb{C}$).

(1) $2x + 3i = ix - 4$
(2) $7x = 3 + 21i + ix$
(3) $2x + i = 23 + 7xi$
(4) $2x + 4 = (3x - 7)i$
(5) $5x - 13 = (11 - 2x)i$
(6) $2(3x - 8) = (7 + 5x)i$
(7) $(x + 20)i = 10 - 7x$
(8) $3x - 23 = (7x - 15)i$

9.5 Mathematical induction

Prove all statements in this section by mathematical induction.

Q61. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for $n \in \mathbb{Z}^+$

Q62. $2 + 4 + 6 + \cdots + 2n = n(2n + 1)$ for $n \in \mathbb{Z}^+$

Q63. $1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$ for $n \in \mathbb{N}$

Q64. $4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n+5)}{2}$ for $n \in \mathbb{Z}^+$

Q65. $1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \in \mathbb{Z}^+$

Q66. $1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for $n \in \mathbb{Z}^+$

Q67. $(1)(2) + (2)(3) + (3)(4) + \ldots (n)(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$ for $n \in \mathbb{Z}^+$

Q68. $1 + 3 + 9 + \cdots + 3^n = \frac{3^n-1}{2}$ for $n \in \mathbb{N}$

Q69. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ for $n \in \mathbb{Z}^+$

Q70. $\frac{2}{3} + 1 + \frac{3}{2} + \cdots + (\frac{3}{2})^n = \frac{3^{n+1}}{2^n} - \frac{4}{3}$ for $n \in \mathbb{Z}^+$

Q71. $1-2+4-8+\cdots+(-2)^n=\frac{1-(-2)^n}{3}$ for $n\in\mathbb{N}$

Q72. $1\times 2+2\times 3+3\times 4+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$ for $n\in\mathbb{Z}^+$

Q73. $\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}$ for $n\in\mathbb{Z}^+$

Q74. $\sum_{k=2}^{n}\frac{1}{(k-1)(k+1)}=\frac{(n-1)(3n+2)}{4n(n+1)}$ for $n\in\mathbb{N}, n\geq 2$

Q75. $\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$ for $n\in\mathbb{Z}^+$

Q76. $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$ for $n\in\mathbb{Z}^+$

Q77. $\sum_{k=1}^{n}\frac{k}{3^k}=\frac{1}{4}(3-\frac{2n+3}{3^k})$ for $n\in\mathbb{Z}^+$

Q78. $\sum_{k=1}^{n}k2^k=2^{n+1}(n-1)+2$ for $n\in\mathbb{Z}^+$

Q79. Prove that 6^n-1 is divisible by 5 for $n\in\mathbb{N}$.

Q80. Prove that 5^n-2^n is divisible by 3 for $n\in\mathbb{N}$.

Q81. Prove that $5^{2n}-3^{2n}-1$ is divisible by 3 for $n\in\mathbb{Z}^+$.

Q82. Prove that 16^n+9^{2n+1} is divisible by 5 for $n\in\mathbb{N}$.

Q83. Prove that $4^{2n+1}+6^n$ is divisible by 5 for $n\in\mathbb{N}$.

Q84. Prove that $7^{2n+1}-2^{2n+1}-2$ is divisible by 3 for $n\in\mathbb{N}$.

Q85. Prove that $4^n-(-1)^n$ is divisible by 5 for $n\in\mathbb{Z}^+$.

Q86. Prove that $3^n+5^n+2\times(-1)^n$ is divisible by 6 for $n\in\mathbb{Z}^+$.

Q87. Prove that when n lines are drawn in a plane then the maximum number of points formed by intersections of lines is $\frac{n(n-1)}{2}$.

Q88. Prove that when n lines are drawn in a plane then the maximum number of regions that the plane is cut by the lines is $\frac{n^2+n+2}{2}$.

Chapter 10

Quadratics, circles and polynomials

10.1 Vieta's formulae for quadratics

Q1. State the number of solutions of the following equations. What are their signs (i.e. are they both positive, both negative or of different signs)?

(1) $4x^2 - x - 3 = 0$
(2) $3x^2 + 40x + 10 = 0$
(3) $x^2 - x + 4 = 0$
(4) $0.4x^2 + 2.4x + 3.6 = 0$
(5) $3.7x^2 + 4.5x + 1 = 0$
(6) $1.9x^2 + 4.7x - 4.1 = 0$
(7) $3x^2 - 6x + 3 = 0$
(8) $-0.72x^2 + 3.85x - 0.6 = 0$
(9) $2.5x^2 - 2.8x + 3.1 = 0$
(10) $4.77x^2 - 3.45x + 0.1 = 0$
(11) $4.142x^2 - 2.087x + 0.25 = 0$
(12) $-2.45x^2 + 3.07x + 1.12 = 0$
(13) $2.6x^2 + 1.8x + 0.1 = 0$
(14) $-6.39x^2 + 4.26x - 0.71 = 0$
(15) $-4.9x^2 + 2.1x - 0.2 = 0$
(16) $-4.89x^2 - 4.52x + 4.12 = 0$
(17) $4.254x^2 - 13.08x + 2.736 = 0$
(18) $-0.93x^2 - 4.89x - 2.72 = 0$
(19) $-0.9x^2 - 3.8x + 0.9 = 0$
(20) $1.8x^2 + 7.4x + 5 = 0$

Q2. For what values of parameter m the two different solutions of $mx^2 + (m-3)x + m = 0$ are both positive?

Q3. For what values of parameter m the solutions of $mx^2 + (m+2)x + m + 2 = 0$ have different signs?

Q4. For what values of parameter m the two different solutions of $mx^2 + (m+2)x + m + 2 = 0$ are both positive?

Q5. Find the values of parameter m for which the sum of real solutions of the equation $2x^2 + (4m+2)x + (2m+1) = 0$ is smaller than 4?

Q6. Find the values of parameter m for which the solutions of the equation $(3m-1)x^2 + 3mx + (3m-1) = 0$ are positive?

Q7. Find the values of parameter m for which the solutions of the equation $mx^2 + (2m+1)x + (3m-1) = 0$ are negative?

Q8. Find the values of parameter m for which the solutions of the equation $2mx^2 + (2m-1)x + (m+3) = 0$ are positive?

Q9. Find the values of parameter m for which the solutions of the equation $(m+2)x^2+(m+1)x+(2m+1)=0$ have different signs.

Q10. Find the values of parameter m for which the solutions of the equation $(2m-1)x^2+3mx+(m-2)=0$ are negative.

Q11. Find the values of parameter m for which the equation $(3m-2)x^2+(m+2)x+(m-2)=0$ has solutions whose product does not exceed their sum.

Q12. Find the values of parameter m for which equation $(2m-3)x^2-(m-2)x+(2m-3)=0$ has solutions whose sum is greater than their product.

Q13. Consider the equation $4x^2-(3m+1)x+(3m-1)=0$. Find the values of parameter m for which:

(i) the sum of squares of the roots of the equation exceeds $\frac{9}{16}$,

(ii) the sum of reciprocals of squares of the roots of the equation exceeds $\frac{9}{4}$.

Q14. Consider the equation $3x^2+(m+2)x+(2m+1)=0$. Find the values of parameter m for which:

(i) the sum of squares of the roots of the equation exceeds 2,

(ii) the sum of reciprocals of squares of the roots of the equation is greater than 7.

10.2 Algebraic fractions

Q15. Simplify the following fractions.

(1) $\frac{x^2+5x+6}{2x^2+7x+3}$
(2) $\frac{2x^2-7x+6}{x^2-4}$
(3) $\frac{9x^2-12x+4}{6x^2+11x-10}$
(4) $\frac{4x^2-4x-3}{4x^2-9}$
(5) $\frac{6x^2-x-2}{4x^2+8x+3}$
(6) $\frac{6x^2+11x-10}{4x^2+4x-15}$
(7) $\frac{5x^2+11x+2}{5x^2+9x-2}$
(8) $\frac{4x^2+8x-5}{10x^2-x-2}$
(9) $\frac{16x^2-20x-6}{2x^2-7x+6}$
(10) $\frac{18x^2-27x+10}{6x^2-x-2}$
(11) $\frac{10x^2-21x-10}{25x^2+20x+4}$
(12) $\frac{-6x^2+17x-12}{4x^2-4x-3}$

Q16. Evaluate and give answers in simplest form.

(1) $\frac{2x^2-2x-12}{2x-3}\times\frac{6-4x}{3x^2-11x+6}$
(2) $\frac{9x^2-3x-2}{6x-4}\times\frac{15-6x}{6x^2-13x-5}$
(3) $\frac{10x^2+16x-8}{3x^2+10x+8}\times\frac{12x^2+13x-4}{60x^2-39x+6}$
(4) $\frac{6x^2-16x+8}{3x+2}\times\frac{12x^2+5x-2}{5x-10}$
(5) $\frac{8x+2}{12x^2+4x-1}\times\frac{18x-3}{12x^2-x-1}$
(6) $\frac{3x^2-8x+5}{4x^2+5x+1}\times\frac{8x^2-2x-1}{6x^2-7x-5}$
(7) $\frac{6x^2-13x+6}{3-6x}\div\frac{6x^2-x-2}{4x-2}$
(8) $\frac{2x^2+3x-2}{6x^2-13x+6}\div\frac{2x^2-5x+2}{6x^2+5x-6}$
(9) $\frac{1-4x^2}{9x^2-6x+1}\div\frac{4x^2+4x+1}{1-9x^2}$

Q17. Express as a single fraction in simplest terms.

(1) $\frac{2}{x+2}+\frac{3}{x+3}$
(2) $\frac{3}{2x-3}-\frac{2}{x-2}$
(3) $\frac{4}{3x-2}+\frac{4}{3x+2}$
(4) $\frac{3}{2x+1}-\frac{3}{2x-3}$
(5) $\frac{5}{3x-4}-\frac{2}{2x-1}$
(6) $\frac{4}{3x-2}+\frac{2}{2x+5}$
(7) $\frac{2}{5x+1}+\frac{3}{x+2}$
(8) $\frac{1}{2x+5}+\frac{3}{2x-1}$
(9) $\frac{3}{6x+2}-\frac{2}{3-2x}$
(10) $\frac{3}{6x-5}-\frac{1}{3x-2}$
(11) $\frac{2}{2x-5}-\frac{3}{5x+2}$
(12) $\frac{3}{4-3x}+\frac{4}{2x-3}$

Q18. Express as a single fraction in simplest terms.

(1) $\frac{x+2}{2x^2+7x+3}+\frac{3x-1}{x^2+x-6}$
(2) $\frac{3}{x^2-4}+\frac{x+2}{x^2+x-6}$
(3) $\frac{3x-2}{6x^2+11x-10}-\frac{3x+1}{6x^2-19x+10}$
(4) $\frac{2x+1}{4x^2-9}-\frac{3x+2}{6x^2-13x+6}$
(5) $\frac{3x-2}{4x^2+8x+3}+\frac{4x-1}{4x^2-4x-3}$
(6) $\frac{3x-2}{4x^2+4x-15}+\frac{3x+1}{10x^2+23x-5}$
(7) $\frac{5x+1}{5x^2+9x-2}-\frac{x-3}{3x^2+8x+4}$
(8) $\frac{2x+5}{10x^2-x-2}+\frac{x-2}{6x^2-3x}$
(9) $\frac{8x+2}{2x^2-7x+6}-\frac{4x}{6x^2-13x+6}$
(10) $\frac{6x-5}{6x^2-x-2}-\frac{3x-3}{9x^2-3x-2}$
(11) $\frac{2x-5}{25x^2+20x+4}-\frac{5x}{15x^2+6x}$
(12) $\frac{-3x+4}{4x^2-4x-3}+\frac{2}{10x^2-11x-6}$

10.3 Equation of a circle

Q19. Find the centre and the radius of the following circles.

(1) $x^2 - 4x + y^2 - 2y - 20 = 0$
(2) $x^2 + y^2 + 4y - 12 = 0$
(3) $x^2 - 6x + y^2 = 0$
(4) $x^2 - 8x + y^2 + 8y + 16 = 0$
(5) $x^2 - 3x + y^2 + 5y - 0.5 = 0$
(6) $x^2 + 10x + y^2 - 4y - 20 = 0$
(7) $x^2 + 6x + y^2 - 2y - 8 = 0$
(8) $x^2 - 2x + y^2 - 3y - 44.75 = 0$
(9) $x^2 + 6x + y^2 - 3y - 6.75 = 0$
(10) $x^2 + 3x + y^2 + y - 5.5 = 0$
(11) $x^2 - 4x + y^2 - y - 70.75 = 0$
(12) $x^2 - x + y^2 + 6y - 30.75 = 0$

Q20. Find the points of intersection of the circle and the line.

(1) $x^2 - 4x + y^2 - 2y - 20 = 0$, $x - 7y + 30 = 0$
(2) $x^2 + y^2 + 4y - 21 = 0$, $3x - y - 17 = 0$
(3) $x^2 - 6x + y^2 - 41 = 0$, $3x + y - 19 = 0$
(4) $x^2 - 8x + y^2 + 8y - 53 = 0$, $x - 4y + 14 = 0$
(5) $x^2 - 3x + y^2 + 5y - 41.5 = 0$, $3x + y + 8 = 0$
(6) $x^2 + 10x + y^2 - 4y - 36 = 0$, $2x - 3y - 3 = 0$
(7) $x^2 + 6x + y^2 - 2y - 55 = 0$, $x + 2y + 16 = 0$
(8) $x^2 - 2x + y^2 - 3y - 46.75 = 0$, $4x - 2y - 31 = 0$
(9) $x^2 + 6x + y^2 - 3y - 38.75 = 0$, $x - 2y + 11 = 0$
(10) $x^2 + 3x + y^2 + y - 82.5 = 0$, $3x - 5y - 49 = 0$
(11) $x^2 - 4x + y^2 - y - 60.75 = 0$, $3x + 2y = 20$
(12) $x^2 - x + y^2 + 6y - 75.75 = 0$, $4x - y - 39 = 0$
(13) $x^2 + 4x + y^2 + 4y - 17 = 0$, $4x + 3y - 11 = 0$
(14) $x^2 + 6x + y^2 - 4y - 4 = 0$, $4x - y - 3 = 0$
(15) $x^2 - 2x + y^2 - 8y - 36 = 0$, $2x + 7y + 23 = 0$
(16) $x^2 - 10x + y^2 - 25 = 0$, $7x - y + 15 = 0$

Q21. Find the points of intersection of the two circles.

(1) $x^2 - 8x + y^2 + 4y - 5 = 0$ and $x^2 - 18x + y^2 - 6y + 85 = 0$
(2) $x^2 - 7x + y^2 + 6y - 10 = 0$ and $x^2 - 31x + y^2 - 12y + 170 = 0$
(3) $x^2 + 2x + y^2 - 12y - 28 = 0$ and $x^2 - 8x + y^2 + 8y - 8 = 0$
(4) $x^2 + 14x + y^2 - 6y + 8 = 0$ and $x^2 - 4x + y^2 - 16 = 0$

Q22. A circle is circumscribed on a right triangle ABC, where $A = (5, -3)$, $B = (3, 1)$ and $C = (11, 5)$. Find the equation of the circle.

Q23. A circle is circumscribed on a right triangle ABC, where $A = (1, -5)$, $B = (3, 1)$ and $C = (-6, 4)$. Find the equation of the circle.

Q24. A circle is circumscribed on triangle ABC, where $A = (-3, -1)$, $B = (8, -3)$ and $C = (3, 7)$. Find the equation of the circle.

Q25. A circle is circumscribed on triangle ABC, where $A = (-2, -2)$, $B = (10, -2)$ and $C = (-8, 4)$. Find the equation of the circle.

Q26. A circle with diameter 5 passes through points $(-3, 2)$ and $(1, 4)$. Find the possible positions of the centre of the circle.

Q27. A circle with diameter $\sqrt{130}$ passes through points $(-2, 1)$ and $(0, -5)$. Find the possible positions of the centre of the circle.

Q28. Find the values of parameter r for which the the circles $x^2 + (y + 5)^2 = 52$ and $(x + 6)^2 + (y - 4)^2 = r^2$ are tangent.

Q29. Find the values of parameter b for which the the circles $c_1 : \ (x + 2)^2 + (y + 1)^2 = 2$ and $c_2 : \ (x - 1)^2 + (y - b)^2 = 8$ are tangent.
For these values of b, what value (other than $\sqrt{2}$) can the radius of c_1 take for the circles to be tangent?

Q30. Find the values of parameter a for which the the circles $c_1 : \ (x + 4)^2 + (y - 4)^2 = 5$ and $c_2 : \ (x - a)^2 + (y - 1)^2 = 20$ are tangent.
For these values of a, what value (other than $\sqrt{5}$) can the radius of c_1 take for the circles to be tangent?

Q31. Find the values of parameter m for which the line $y = mx$ is tangent to the circle $(x - 5)^2 + (y - 5)^2 = 5$.

Q32. Find the values of parameter m for which the line $y = mx+1$ is tangent to the circle $(x+5)^2+(y-6)^2 = 10$.

Q33. Find the values of parameter m for which the line $y = 2x+m$ is tangent to the circle $(x+3)^2+(y+2)^2 = 20$.

Q34. Find the points of intersection of the circle and the parabola.

(1) $x^2 + (y + 2)^2 = 25$, $y = x^2 - 15$
(2) $x^2 + (y + 2)^2 = 25$, $y = x^2 - 7$
(3) $x^2 + (y + 2)^2 = 25$, $y = x^2 - 6x + 3$

Q35. Triangle ABC is bounded by the straight lines: $y = 2x$, $y = -\frac{1}{2}x$, and $y = -x + 6$.
Find the equation of the circumcircle of the triangle.

10.4 Polynomials

Q36. Solve the following inequalities.

(1) $x^2(x - 1) > 0$
(2) $x^2(x - 1) \leq 0$
(3) $3x(x - 2)^2 < 0$
(4) $(x + 1)^2x \geq 0$
(5) $(x + 1)x(x - 3) \leq 0$
(6) $(x + 1)(1 - x)(x - 1) < 0$
(7) $(x + 3)(x + 1)(1 - x) \geq 0$
(8) $(3x - 1)(2x - 1)^2 > 0$
(9) $(3x + 1)(2 - 3x)^2 < 0$
(10) $(3x + 2)^2(1 - 3x) > 0$
(11) $(x + 1)(1 - 2x)(x - 2) \leq 0$
(12) $(2 - 3x)(2x - 3)^2 \geq 0$
(13) $(2 - 3x)(3 - 2x)^2 < 0$
(14) $(5x - 3)^2(3 - 4x) \leq 0$
(15) $(x + 2)(1 - 4x)(x - 3) > 0$
(16) $(x + 5)(5x - 7)^2 \leq 0$
(17) $(3x - 7)^2(5x - 12) \geq 0$
(18) $(2x + 3)(3x + 2)(2x - 3) < 0$

Q37. Solve the following inequalities.

(1) $5x^3 - x^2 > 0$
(2) $5x^3 - 2x^2 \leq 0$
(3) $12x^3 - 12x^2 + 3x < 0$
(4) $x^3 + 4x^2 + 4x \geq 0$
(5) $x^3 - x^2 - 2x \leq 0$
(6) $-x^3 + x < 0$
(7) $-x^3 - 4x^2 - 3x \geq 0$
(8) $4x^3 - 4x^2 + x > 0$
(9) $-9x^3 + 3x^2 + 2x < 0$
(10) $-9x^3 - 3x^2 + 2x > 0$
(11) $2x^3 + x^2 - x \leq 0$
(12) $-6x^3 + 13x^2 - 6x \geq 0$
(13) $-4x^3 + 12x^2 - 9x < 0$
(14) $-20x^3 + 27x^2 - 9x \leq 0$
(15) $-4x^3 - 7x^2 + 2x > 0$
(16) $5x^3 + 18x^2 - 35x \leq 0$
(17) $9x^3 + 9x^2 - 70x \geq 0$
(18) $4x^3 - 9x < 0$

Q38. Solve the following inequalities.

(1) $x^3 - 2x^2 - 9x + 18 > 0$
(2) $x^3 - x^2 - 4x + 4 \leq 0$
(3) $-16x^3 + 4x^2 + 4x - 1 < 0$
(4) $8x^3 + 4x^2 - 18x - 9 \geq 0$
(5) $4x^3 - 4x^2 - 25x + 25 \leq 0$
(6) $-18x^3 + 27x^2 + 50x - 75 < 0$
(7) $x^3 + 2x^2 - 16x - 32 \geq 0$
(8) $-18x^3 + 9x^2 + 8x - 4 > 0$
(9) $18x^3 - 9x^2 - 8x + 4 < 0$
(10) $-27x^3 + 9x^2 + 3x - 1 > 0$
(11) $-20x^3 + 8x^2 + 5x - 2 \leq 0$
(12) $8x^3 + 12x^2 - 18x - 27 \geq 0$
(13) $-8x^3 + 12x^2 + 18x - 27 < 0$
(14) $64x^3 + 48x^2 - 36x - 27 \leq 0$
(15) $80x^3 - 16x^2 - 5x + 1 > 0$
(16) $75x^3 - 100x^2 - 147x + 196 \leq 0$
(17) $18x^3 - 27x^2 - 98x + 147 \geq 0$
(18) $8x^3 - 12x^2 - 18x + 27 < 0$

Q39. Divide polynomial $P(x)$ by $D(x)$.

(1) $P(x) = x^4 + x^3 - 8x^2 + 3x + 3$, $D(x) = x - 2$
(2) $P(x) = 2x^4 - x^3 - 5x^2 + 4x + 2$, $D(x) = 2x + 1$
(3) $P(x) = 6x^5 - 8x^4 - 7x^3 + 21x^2 + 3x + 3$, $D(x) = 3x - 1$
(4) $P(x) = -6x^5 + 11x^4 - 3x^3 + 4x^2 - 12x + 4$, $D(x) = -2x + 3$
(5) $P(x) = -2x^5 - 2x^4 + 4x^3 - 2x - 5$, $D(x) = 2x + 2$
(6) $P(x) = -4x^5 + 6x^4 + 6x^3 - 13x^2 + 8x - 7$, $D(x) = 2x - 3$
(7) $P(x) = x^5 - x^4 - 4x^3 + 5x^2 + 6x - 7$, $D(x) = x^2 - 2$
(8) $P(x) = x^5 + x^4 - 2x^3 + 9x^2 - 5x + 9$, $D(x) = x^2 - x + 2$
(9) $P(x) = 2x^6 + 2x^5 - x^4 - 3x^3 + 5x^2 - 2x + 2$, $D(x) = 2x^2 + 1$
(10) $P(x) = 2x^6 + 3x^5 + 14x^3 + 2x^2 - 7x + 6$, $D(x) = x^2 - x + 3$
(11) $P(x) = 9x^6 + 8x^4 - 7x^3 + 13x^2 - 2x + 1$, $D(x) = 3x^2 - x + 1$
(12) $P(x) = 4x^6 + x^4 + 5x^3 + x + 9$, $D(x) = 2x^2 - x + 3$

Q40. Express the following fractions in the form $Q(x) + \frac{R(x)}{D(x)}$, where $\deg R < \deg D$.

(1) $\frac{3x^5+9x^4-8x^3-9x^2+6x+3}{3x^2-2}$
(2) $\frac{4x^4+2x^3-7x^2+10x-7}{x^2+x-2}$
(3) $\frac{2x^5-4x^4-7x^3+5x^2-2x-5}{x^2-2x-3}$
(4) $\frac{6x^5-7x^4-4x^3+7x^2+x-5}{3x^2+x-2}$
(5) $\frac{3x^5-14x^4+25x^3-24x^2+13x}{x^2-3x+2}$
(6) $\frac{2x^5-8x^4+13x^3-9x^2+x+5}{x^2-2x+1}$
(7) $\frac{2x^5-15x^4+14x^3+17x^2+4x+4}{2x^2-3x-2}$
(8) $\frac{-6x^5+7x^4-x^3+2x^2-x-7}{-3x^2+2x-1}$
(9) $\frac{x^5-x^4+2x^3+5x^2+8x+26}{x^2-3x+5}$
(10) $\frac{4x^5-11x^4+20x^3-14x^2+7x+2}{x^2-2x+3}$
(11) $\frac{4x^5+2x^4-4x^3+7x^2+9x}{2x^2+2x-1}$
(12) $\frac{x^5+x^4-x^3+2x^2+4x-3}{x^2+x-1}$

Q41. Find the remainder when polynomial $P(x)$ is divided by $D(x)$. Give exact answers.

(1) $P(x) = x^4 + 2x^3 - 3x^2 + 2x - 5$, $D(x) = x - 1$
(2) $P(x) = x^3 + 2x^2 - 4x - 5$, $D(x) = x + 2$
(3) $P(x) = 2x^3 - 5x^2 + 3x - 6$, $D(x) = x - 3$
(4) $P(x) = 2x^3 + x^2 - 3x + 2$, $D(x) = 2x + 1$
(5) $P(x) = 9x^4 + 6x^3 - 3x^2 + 2x + 1$, $D(x) = 3x - 1$
(6) $P(x) = 4x^5 - 6x^4 - 7x^3 - x^2 - 3x + 2$, $D(x) = 2x - 5$
(7) $P(x) = 5x^3 - 4x^2 + 3x + 2$, $D(x) = 5x + 2$
(8) $P(x) = 9x^4 - 6x^3 + 2x^2 + 5x - 2$, $D(x) = 3x + 2$
(9) $P(x) = 2x^6 - x^5 + 4x^4 - 2x^3 + 3x^2 + x - 2$, $D(x) = 4x - 2$
(10) $P(x) = 16x^4 + 8x^3 - 7x^2 - 3x + 2$, $D(x) = 4x + 1$
(11) $P(x) = 8x^4 - 4x^3 - 3x^2 + 2x - 1$, $D(x) = 4x - 3$
(12) $P(x) = 18x^4 + 12x^3 - 6x^2 - 3x + 2$, $D(x) = 3x - 2$

Q42. Consider $P(x) = 2x^3 - x^2 - 5x - 2$.

(i) Check which of the numbers: $-\frac{1}{2}$, $\frac{1}{5}$ or $-\frac{2}{3}$ is a zero of $P(x)$.
(ii) Hence factorise $P(x)$ fully.

Q43. Consider $P(x) = 3x^3 - 2x^2 - 19x - 6$.

(i) Check which of the numbers: $-\frac{1}{3}$, 1 or $-\frac{1}{2}$ is a zero of $P(x)$.
(ii) Hence factorise $P(x)$ fully.

Q44. Consider $P(x) = 8x^3 - 36x^2 + 54x - 27$.

(i) Check which of the numbers: $\frac{3}{2}$, 1 or $-\frac{3}{4}$ is a zero of $P(x)$.
(ii) Hence factorise $P(x)$ fully.

Q45. Consider $P(x) = 4x^3 - 28x^2 + 65x - 50$.

(i) Check which of the numbers: 2, $\frac{1}{2}$ or $\frac{3}{2}$ is a zero of $P(x)$.

(ii) Hence factorise $P(x)$ fully.

Q46. Consider $P(x) = 3x^3 - 7x^2 - 56x + 80$.

(i) Check which of the numbers: -4, $\frac{2}{3}$ or 1 is a zero of $P(x)$.
(ii) Hence factorise $P(x)$ fully.

Q47. Consider $P(x) = 9x^3 + 12x^2 - 35x - 50$.

(i) Check which of the numbers: 2, -1 or $-\frac{1}{2}$ is a zero of $P(x)$.
(ii) Hence factorise $P(x)$ fully.

Q48. Consider $P(x) = 32x^3 - 14x + 3$.

(i) Check which of the numbers: $-\frac{3}{4}$, $\frac{2}{3}$ or $-\frac{1}{2}$ is a zero of $P(x)$.
(ii) Hence factorise $P(x)$ fully.

Q49. Consider $P(x) = 27x^3 + 27x^2 - 18x - 8$.

(i) Check which of the numbers: $-\frac{4}{3}$, 2 or $-\frac{3}{4}$ is a zero of $P(x)$.
(ii) Hence factorise $P(x)$ fully.

Q50. Consider circle c defined by $x^2 + (y-5)^2 = 25$ and parabola p defined by $y = x^2 + 6x + 10$.

(i) Show that the vertex of p lies on c.
(ii) Hence find all other intersections of p and c.

Q51. Consider:

- circle c defined by $(x-2)^2 + (y-6)^2 = 20$,
- parabola p defined by $y = \frac{1}{2}x^2 - 4x + 10$
- and line k defined by $x + y = 10$.

(i) Find the coordinates of points of intersection of k and c.
(ii) Show that the points lie on p, too.
(iii) Hence find the coordinates of one more point of intersection of p and c.
(iv) What can you say about the way that c and p meet at this point?

Q52. Consider circle c defined by $x^2 - 7x + y^2 - 7.5y + 6 = 0$ and parabola p defined by $y = -\frac{1}{2}x^2 + 2x + 6$. Line k is perpendicular to line $y = \frac{2}{3}x + 7$ and passes through point $(6, 0)$.

(i) Find the equation of k.
(ii) Find the coordinates of points of intersection of k and p.
(iii) Show that the points lie on c, too.
(iv) Hence find the coordinates of all other intersections of p and c.

Q53. Find all points of intersection of a parabola $y = \frac{1}{2}x^2 - x + 1$ and a hyperbola $y = -\frac{3}{x+1} + 2$.

Q54. Consider a parabola p defined by $y = \frac{1}{3}x^2 - \frac{5}{3}x + 3$ and a hyperbola h defined by $y = -\frac{3}{x-2} + 4$.

(i) Show that p and h meet at $(-1, 5)$.
(ii) Find all other points of intersection of p and h.
(iii) Find the equation of the circle passing through all points of intersection of p and h.
(iv) Find all other points of intersection of the circle and h.

10.5 Chapter Review

Q1. (non-calculator) Consider the function

$$f(x) = \frac{1}{2}x^2 - 3x - \frac{7}{2}$$

(i) Express $f(x)$ in the form $f(x) = a(x-h)^2 + v$ and hence wirte down the coordinates of the vertex of the graph of f.
(ii) Solve the equation $f(x) = 0$.
(iii) Graph $y = f(x)$ showing clearly the position of the vertex, the y- and x-intercepts. Write down the equation of the line of symmetry.

Q2. Give examples of **quadratic** inequalities whose solutions are as follows:

(1) $x \in \emptyset$
(2) $x \in \mathbb{R}$
(3) $x \in \mathbb{R}, x \neq 2$
(4) $x = 5$
(5) $-3 < x < 5$
(6) $x \leq 4$ or $x \geq 6$

Q3. (1) The equation $ax^2 + 5x - 3 = 0$ has two difrerent roots with different signs. What is the sign of a?

(2) The equation $-3x^2 + bx + c = 0$ has two difrerent negative roots. What are the signs of b and c?

(3) The equation $ax^2 + bx + 2 = 0$ has two difrerent positive roots. What are the signs of a and b?

Q4. (1) The equation $(m+1)x^2 - 4x + (m-2) = 0$ has two difrerent positive roots. Find the set of possible values of m.

(2) The equation $(m-2)x^2 - 4x + (m+1) = 0$ has two roots with different signs. Find the set of possible values of m.

Q5. (non-calculator) Consider the functions defined by

$$f(x) = \frac{x-5}{x^2-2x-3} + \frac{x-2}{x^2-4x+3},$$
$$g(x) = \frac{2x^2-5x+2}{4x^2-1},$$
$$h(x) = \frac{2x^2-5x+3}{2x-1-x^2}$$

(i) Show that $f(x)$ can be expressed in the form: $f(x) = \frac{2x-1}{(x-1)(x+1)}$.

(ii) Simplify $g(x)$ and $h(x)$.

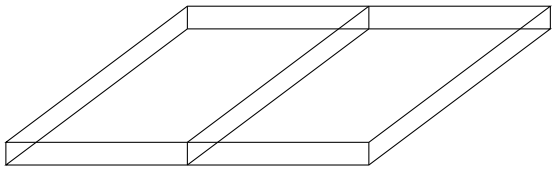
(iii) Sketch the graphs of $y = f(x)$, $y = g(x)$ and $y = h(x)$ showing clearly the position of all asymptotes and the points of intersections with coordinate axes.

Q6. (non-calculator) Solve the following inequalities. Show clearly your workings (e.g. by graphing appropriate polynomials).

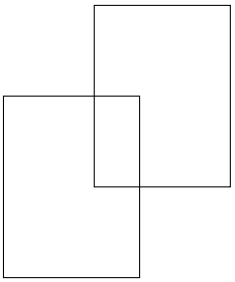
(1) $x(2-x)^2(1+x) > 0$
(2) $x^2(2-x)(1+x) > 0$
(3) $(x-2)^2(3-x)^2(1-x) \leq 0$
(4) $(x-2)^2(3-x)(1-x)^2 \leq 0$
(5) $(x-2)^2(3-x)(1-x) \leq 0$
(6) $x^3 - 4x^2 + 3x < 0$
(7) $x^4 - 4x^3 + 3x^2 < 0$
(8) $25x^3 - 20x + 4x \geq 0$
(9) $x^3 - x^2 + x - 1 \geq 0$
(10) $3x^3 - 2x^2 - 6x + 4 > 0$

Q7. The longest side of a right angled triangle is 10. Its perimeter is 24. Find the lengths of the other two sides.

Q8. A plot of land is in the shape of a rectangle. The plot is to be surrounded by the fencing. Additional line segment of fencing is to be built to split the plot into two equal rectangles. The total length of fencing is 600 metres. Find the dimensions of the plot of land given that its area is largest possible.

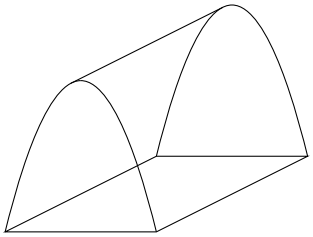


Q9. Two identical rectangles are drawn in a way that their intersection is in the shape of a smaller rectangle. Its width is a third of the width of the initial rectangles while its height is a half of the heigth of the initial rectangles.
The perimeter of the whole shape created is 60 cm.
Find the dimensions of the initial rectangles for which the area of the shape is largest possible.



Q10. (calculator)

An office building is built on a base of a rectangle 20 m wide and 40 m long. Its cross section parallel to the width is in the shape of a parabola (see the diagram). The height of the building in its heighest point is 30 m.
An architect wants to fit an office block in the shape of cuboid inside the building. He wants the block to consist of 8 floors each with the area of $400m^2$. The minimum height of a room in an office building is 2.8 m.
Is it possible to fit such a block inside the building?



Q11. Consider the circle and the line defined by:

$$x^2 + y^2 - 8x + 8y + 22 = 0,$$
$$x - y - 4 = 0$$

Let C be the centre of the circle. Let O be the origin and A, B be the intersections of the line and the circle.

(i) Find the centre and the radius of the circle.

(ii) Find the perimeter and the area of quadrilateral $OACB$.

Q12. Find the set of common solutions of the following inequalities.

(1) $\begin{cases} -\frac{1}{2}x^2 + 2x + 2 & \geq \frac{1}{2}x + 2 \\ 4x - 5 & \leq 2x^2 - 4x + 3 \end{cases}$ (2) $\begin{cases} \frac{1}{2}x^2 - 2x + 3 & > -\frac{1}{2}x^2 + 3x - 1 \\ 2x^2 - 4x + 2 & \leq 2x + 11 \end{cases}$

Q13. Find all *complex* solutions of the following equations:

(1) $2x^3 + 5x^2 + 2x - 21 = 0$ knowing that $x = 1.5$ is one of the solutions,

(2) $3x^3 - 16x^2 + 27x + 26 = 0$ knowing that either $x = -1.5$ or $x = -\frac{2}{3}$ is one of the solutions,

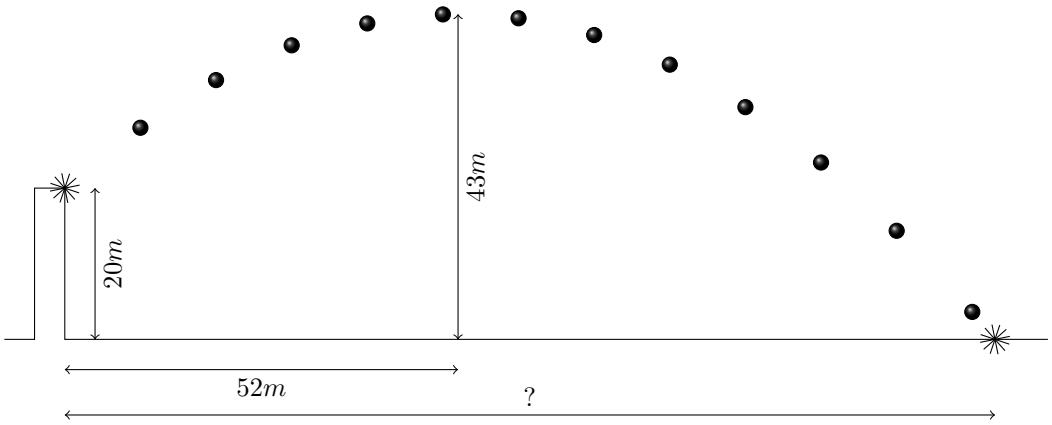
(3) $6x^3 + 15x^2 + 18x + 45 = 0$,

(4) $20x^3 - 4x^2 + 5x - 1 = 0$.

Q14. Last year the Magic Calculator cost PLN 600 and 500 calculators were sold. A survey found that more customers will buy the calculator if its cost is reduced. To be precise it was found that for every PLN 20 reduction in price, 40 more customers will buy the calculators.
Let x denote the integer number of reductions of the price by PLN 20.
Let $f(x)$ denote the income (revenue) in thousands of PLN.
For example, if $x = 2$, the price of a single calculator is PLN$(600 - 2(20))$ = PLN 560, the number of customers who buy the calculators $(500 + 2(40) = 580)$, so the total income (revenue) is PLN $560 \times 580 =$ PLN $324,800$.
Find an expression for $f(x)$ and hence find the price of the calculator for which the income (revenue) is largest.

Q15. For what values of parameter m the equation $mx^2 - (2m - 1)x + 4 = 0$ does not have real roots?

Q16. A cannon is placed on the top of the tower 20 metres above the ground level. Let us assume that the trajectory of the cannonball is in the shape of a parabola. Let us also assume that the cannonball reaches its largest altitude of 43 metres at the point which is in horizontal distance of 52 metres from the tower (see the diagram).



Find the distance from the tower to the point where the ball hits the ground.

Q17. Consider a circle defined by $x^2 + y^2 - 4x - 6y = 12$.

(i) Write down the coordinates of point A, the centre of the circle.

(ii) Find the radius of the circle.

Points B and D are the points of intersection of the circle and x-axis. Let the x-coordinate of B be smaller than the x-coordinate of D. Point C has coordinates $(5, 1)$.

(iii) Find the coordinates of B and D.

(iv) Is C inside or outside the circle? Justify your answer.

(v) Find an equation of the line k passing through B and C.

(vi) Find an equation of the line m perpendicular to line k and passing through D.

(vii) Find the other point where line m intersects the circle.

Q18. Consider circle $c : x^2 - 0.5x + y^2 - 13.5y + 12.5 = 0$, line $k : 2x + 2y = 17$ and parabola defined by $y = \frac{1}{2}(x - 2)^2 - 1$.

(i) Find the coordinates of A and B, the points of intersection of c and k.

(ii) Knowing that the parabola passes through A and B, too, find all points of intersection of the circle and the parabola.